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A CHARACTERIZATION OF PRIMITIVE PYTHAGOREAN TRIPLES

Roberto Amato

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Abstract. The main aim of this paper is to present an analytic result which characterizes the primitive Pythagorean triples via a cathetus. This way has the convenience to find easily all primitive Pythagorean triples $x, y, z \in \mathbb{N}$ where x is a predetermined integer.

1 Introduction

Let x, y and z be positive integers satisfying

$$x^2 + y^2 = z^2 .$$

Such a triple (x, y, z) is called a Pythagorean triple and if, in addition, x, y and z are coprime, then it is called primitive Pythagorean triple. We recall, the Euclid's formula, that a primitive Pythagorean triple (x, y, z) can be parameterized by

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2 \tag{1.1}$$

if and only if m and n are coprime positive integers of different parities with $m > n > 0$.
We also recall, the variant of the Euclid's formula, that a primitive Pythagorean triple (x, y, z) can be parametrized by

$$x = mn, \quad y = \frac{m^2 - n^2}{2}, \quad z = \frac{m^2 + n^2}{2} \tag{1.2}$$

if and only if m and n are coprime positive odd integers with $m > n > 0$.
The main aim of this paper is to characterize all the primitive Pythagorean triples by a formula which is completely different from (1.1), (1.2) and involves a predetermined integer x.

First, let us recall a recent novel formula that allows to obtain all Pythagorean triples as follows.

Theorem 1.1. *[1] (x, y, z) is a Pythagorean triple if and only if there exists $d \in C(x)$ such that*

$$x = x, \quad y = \frac{x^2}{2d} - \frac{d}{2}, \quad z = \frac{x^2}{2d} + \frac{d}{2}, \tag{1.3}$$

with x positive integer, $x \geq 1$, and where

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd,} \\ D(x) \cap P(x), & \text{if } x \text{ is even,} \end{cases}$$

with

$$D(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d \leq x \text{ and } d \text{ divisor of } x^2 \right\} ,$$

and if x is even with $x = 2^n k$, $n \in \mathbb{N}$ and $k \geq 1$ odd fixed, with

$$P(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d = 2^s l, \text{ with } l \text{ divisor of } x^2 \text{ and } s \in \{1, 2, \ldots, n-1\} \right\}$$

where x is a predetermined integer, which means finding all right triangles whose sides have integer measures and one cathetus is predetermined.
Moreover in [1], based on Theorem 1.1, we have proved the following theorem.

Theorem 1.2. *Each $x \in \mathbb{N}$ can be found as cathetus in at least one Pythagorean triple. Every $x \in \mathbb{N}$ can be represented in the form $x = \sqrt{z^2 - y^2}$ with $y, z, x \in \mathbb{N}$.*

Taking into account Theorem 1.1, in [2] we found relations between the primitive Pythagorean triple (x, y, z) generated by any predetermined positive odd integer x using (1.3) and the primitive Pythagorean triple generated by x^m with $m \in \mathbb{N}$ and $m \geq 2$. In [2] we took care of relations only for the case in which the primitive triple (x, y, z) is generated whith $d \in C(x)$ only with $d = 1$ and the primitive triple (x^m, y', z') is generated with $d_m \in C(x^m)$ only with $d_m = 1$ obtaining formulas that give us y' and z' directly from x, y, z.

Theorem 1.3. *([2]) Let (x, y, z) be the primitive Pythagorean triple generated by any predeterminated positive odd integer $x \geq 1$ using (1.3) with $z - y = d = 1$ and let (x^m, y', z') be the primitive Pythagorean triple generated by x^m, $m \in \mathbb{N}$, $m \geq 2$, using (1.3) with $z' - y' = d_m = 1$, we have the following formulas*

$$\begin{aligned} y' &= y\left[1 + \sum_{p=1}^{m-1} x^{2p}\right], \\ z' &= y\left[1 + \sum_{p=1}^{m-1} x^{2p}\right] + 1, \end{aligned} \tag{1.4}$$

for every $m \in \mathbb{N}$ and $m \geq 2$.
Moreover we have also

$$z\left[(-1)^{m-1} + \sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right] = \begin{cases} y' & \text{if } m \text{ is even,} \\ z' & \text{if } m \text{ is odd,} \end{cases}$$

and

$$z\left[(-1)^{m-1} + \sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right] + (-1)^{m-2} = \begin{cases} z' & \text{if } m \text{ is even,} \\ y' & \text{if } m \text{ is odd.} \end{cases}$$

This was the first step to investigate on other relations between Pythagorean triples.

Afterwards some relations among Pythagorean triples were established in [3]. The main tool was the fundamental characterization of the Pythagorean triples through a cathetus of the Theorem 1.1 that allows to determine the relationships between two Pythagorean triples with an assigned cathetus a and b and the Pythagorean triple with cathetus $a \cdot b$.

Theorem 1.4. *([3]) Let $(a, a_1, a_2), (b, b_1, b_2), (a \cdot b, y, z)$ be the Pythagorean triples generated by $a, b, a \cdot b$ respectively using (1.3) with $a_2 - a_1 = d_1 \in C(a), b_2 - b_1 = d_2 \in C(b), z - y = d_3 \in C(a \cdot b)$. Then*

$$y = a_1b_2 + a_2b_1 , \qquad z = a_1b_2 + a_2b_1 + d_1d_2$$

and moreover

$$y = a_1b_1 + a_2b_2 - d_1d_2 , \qquad z = a_1b_1 + a_2b_2$$

with $d_3 = d_1 \cdot d_2 \in C(a \cdot b)$.

Our main result, that is, Theorem 2.1, is in Section 2. Moreover, in the same section, an immediate consequence (Lemma 2.2) is pointed out, as well as a table of examples. Lemma 2.2 is a new variant of the Euclid's formulas and involves a predetermined integer x.

2 Results

The following theorem holds.

Theorem 2.1. *Let (x, y, z) be all the Pythagorean triples generated by any predetermined positive integer $x \geq 1$ using (1.3), $d \in C(x)$, then (x, y, z) is a primitive Pythagorean triple if and only if following both conditions are verified*

$$\text{if } x \text{ is odd then } \begin{cases} d \text{ is a square} & (2.1) \\ \dfrac{x^2}{d} \text{ with } d \text{ are coprime positive odd integers} & (2.2) \end{cases}$$

$$\text{if } x \text{ is even then } \begin{cases} \dfrac{d}{2} \text{ is a square} & (2.3) \\ \dfrac{x^2}{2d} \text{ with } \dfrac{d}{2} \text{ are coprime positive integers of different parities} & (2.4) \end{cases}$$

Proof. Let x, m, n be positive integers, x odd, $m > n > 0$ such that (x, y, z) is a Pythagorean triple, that is m and n are coprime positive odd integers.

Taking into account (1.2) and Theorem 1.1 it exists $d \in C(x), d \leq x$, d odd such that

$$x = mn$$
$$y = \frac{m^2 - n^2}{2} = \frac{x^2 - d^2}{2d}$$
$$z = \frac{m^2 + n^2}{2} = \frac{x^2 + d^2}{2d}$$

and adding the third relation to the second we have

$$z + y = m^2 = \frac{x^2}{d} \tag{2.5}$$

while subtracting the second relation from the third we have

$$z - y = n^2 = d \tag{2.6}$$

that is d is a square.

Let us observe that since $m, n \in \mathbb{N}$, if m and n are coprime positive odd integers then also m^2 and n^2 are coprime positive odd integers and from (2.5) and (2.6), it must be

$$\frac{x^2}{d} \quad \text{and} \quad d \quad \text{are coprime positive odd integers} \tag{2.7}$$

that is

$$\frac{x}{\sqrt{d}} \quad \text{and} \quad \sqrt{d} \quad \text{are coprime positive odd integers with } \frac{x}{\sqrt{d}} > \sqrt{d} > 0 \tag{2.8}$$

Taking into account (2.6) and (2.7) we have the conditions (2.1) and (2.2).
Therefore Theorem 1.2 holds if $x \in \mathbb{N}$ is odd.

Now, in a similar way, we prove that Theorem 2.1 holds also if $x \in \mathbb{N}$ is even.
Let x, m, n be positive integers, x even, $m > n > 0$ such that (x, y, z) is a Pythagorean triple, that is m and n are coprime positive of different parities.
Taking into account (1.1) and Theorem 1.1 it exists $d \in C(x)$, $d \leq x$, d even such that

$$x = 2mn$$
$$y = m^2 - n^2 = \frac{x^2 - d^2}{2d}$$
$$z = m^2 + n^2 = \frac{x^2 + d^2}{2d}$$

and adding the third relation to the second we have

$$m^2 = \frac{x^2}{2d} \tag{2.9}$$

while subtracting the second relation from the third we have

$$n^2 = \frac{d}{2} \tag{2.10}$$

that is $\frac{d}{2}$ is a square.

With the same observations made before and from (2.5) and (2.6), it must be

$$\frac{x^2}{2d} \quad \text{and} \quad \frac{d}{2} \quad \text{are coprime positive integers of different parities} \tag{2.11}$$

that is

$$\frac{x}{\sqrt{2d}} \text{ and } \sqrt{\frac{d}{2}} \text{ are coprime positive integers of different parities with } \frac{x}{\sqrt{2d}} > \sqrt{\frac{d}{2}} > 0. \tag{2.12}$$

Taking into account (2.10) and (2.11) we have the conditions (2.3) and (2.4).
Therefore Theorem 2.1 holds if $x \in \mathbb{N}$ is even.
Consequently, Theorem 2.1 have thus been proved.

□

Let us observe that Theorem 2.1 gives us a new variant of the Euclid's formulas (1.1) and (1.2). In fact thanks to (2.8) and (2.12) we have the following result in the style of the Euclid's formulas.

Lemma 2.2. *Let (x, y, z) be all the Pythagorean triples generated by any predetermined positive integer $x \geq 1$ using*

$$x = x, \qquad y = \frac{x^2}{2d} - \frac{d}{2}, \qquad z = \frac{x^2}{2d} + \frac{d}{2}$$

and $d \in C(x)$, then (x, y, z) is a primitive Pythagorean triple if and only if

$$\textit{with } x \textit{ odd, then } \frac{x}{\sqrt{d}} \textit{ and } \sqrt{d} \textit{ are coprime positive odd integers with } \frac{x}{\sqrt{d}} > \sqrt{d} > 0$$

$$\textit{with } x \textit{ even, then } \frac{x}{\sqrt{2d}} \textit{ and } \sqrt{\frac{d}{2}} \textit{ are coprime positive odd integers of different parities}$$

$$\textit{with } \frac{x}{\sqrt{2d}} > \sqrt{\frac{d}{2}} > 0$$

To prove the completeness of the Theorem 2.1, we consider the following examples.

Example 2.3. To demonstrate our method, we give the following table with some $x \in \mathbb{N}$. Obviously, the table can be extended for each $x \in \mathbb{N}$.

x odd	**triples**	**primitive triple**	**condition (2.1)**	**condition (2.2)**
x=9, C(x)={1, 3, 9}				
for d=1	(9, 40, 41)	Y	Y	Y
for d=3	(9, 12, 15)	N	N	—-
for d=9	(9, 0, 9)	trivial		
x=15, C(x)={1, 3, 5, 9, 15}				
for d=1	(15, 112, 113)	Y	Y	Y
for d=3	(15, 36, 39)	N	N	—-
for d=5	(15, 20, 25)	N	N	—-
for d=9	(15, 8, 17)	Y	Y	Y
for d=15	(15, 0, 15)	trivial		
x=21, C(x)={1, 3, 7, 9, 21}				
for d=1	(21, 220, 221)	Y	Y	Y
for d=3	(21, 72, 75)	N	N	—-
for d=7	(21, 28, 35)	N	N	—-
for d=9	(21, 20, 29)	Y	Y	Y
for d=21	(21, 0, 21)	trivial		
x=45, C(x)={1, 3, 5, 9, 15, 45}				
for d=1	(45, 1012, 1013)	Y	Y	Y
for d=3	(45, 336, 339)	N	N	—-
for d=5	(45, 200, 205)	N	N	—-
for d=9	(45, 108, 117)	N	Y	N
for d=15	(45, 60, 75)	N	N	—-
for d=45	(45, 0, 45)	trivial		
x even	**triples**	**primitive triple**	**condition (2.3)**	**condition (2.4)**
x=8, C(x)={2, 4, 8}				
for d=2	(8, 15, 17)	Y	Y	Y
for d=4	(8, 6, 10)	N	N	—-
for d=8	(8, 0, 8)	trivial		
x=16, C(x)={2, 4, 8, 16}				
for d=2	(16, 63, 65)	Y	Y	Y
for d=4	(16, 30, 34)	N	N	—-
for d=8	(16, 12, 20)	N	Y	N
for d=16	(16, 0, 16)	trivial		
x=18, C(x)={2, 6, 18}				
for d=2	(18, 80, 82)	N	N	—-
for d=6	(18, 24, 30)	N	N	—-
for d=18	(18, 0, 18)	trivial		
x=20, C(x)={2, 4, 8, 10, 20}				
for d=2	(20, 99, 101)	Y	Y	Y
for d=4	(20, 48, 52)	N	N	—-
for d=8	(20, 21, 29)	Y	Y	Y
for d=10	(20, 15, 25)	N	N	—-
for d=20	(20, 0, 20)	trivial		

3 Conclusion and Remarks

We remember that the Euclid's formulas do not give all Pythagorean triples that involves a predetermined positive integer x, for example the triples $(12, 9, 15)$, $(33, 180, 183)$ and $(33, 44, 55)$, and moreover it can be laborious to find m and n such that $x = m^2 - n^2$ while using Theorem 1.1 it is enough to find all the $d \in C(x)$ to obtain all Pythagorean triples.
In particular if we need to find all primitive Pythagorean triples that involves a predetermined positive integer x now we can use only the $d \in C(x)$ that satisfy the conditions of the Theorem 2.1 or the Lemma 2.2.

The discovery of the parametrization and relations among Pythagorean triples that we have found shows the fundamental role of $d \in C(x)$ and characterizing the results.
This way could be used to study other arithmetic problems, some of which are perhaps still open.

An example could be the Jeśmanowicz' conjecture. Considered the exponential Diophantine equation

$$x^a + y^b = z^c \tag{3.1}$$

with x, y, z, a, b, c positive integers, Jeśmanowicz proposed the following conjecture:

Conjecture 3.1: Equation (3.1) *has only the positive integer solution* $(a, b, c) = (2, 2, 2)$.

Recently, the conjecture has been proved for infinitely many triples (x, y, z) in the paper [7]. Precisely, by using (1.1), Conjecture 3.1 has been proved for the triples $(m^2 - n^2, 2mn, m^2 + n^2)$ with $n = 2$ and without any assumption on m (see [7, Theorem 1]).

One of the next steps will be to study the parametrizations of Pythagorean quadruples, that is quadruples of integers (a, b, c, d) that are solution of the equation

$$a^2 + b^2 + c^2 = d^2$$

looking for a representation similar to (1.3) and finding all Pythagorean quadruples.

Mainly it will be interesting to find other parametrizations, relations and characterizations regard to the Pythagorean n-uples dependent by $d \in C(x)$.

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Author information

Roberto Amato, Department of Engineering , University of Messina, 98166 Messina, Italy.
E-mail: `ramato@unime.it`

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