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New types of Meir-Keeler contractions in metric like spaces

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Abstract In this manuscript, we study a generalized type of Meir-Keeler type contractive mappings, where prove new fixed point theorem for such mappings which will generalize many results in the literature. Moreover, we present an application of our result to coupled α-Meir-Keeler fixed points.

1 Introduction

Metric space concept introduced by Maurice Frechet in 1906 is the milestone for many sciences including mathematics. Generalizations of metric space came into existence just by modifying or reducing the metric axioms by many researchers over the past years. Then the concept of fixed-point theory gave more opportunity by opening new branches of non-linear analysis dealing with finding solutions of various problems of social and natural sciences. One of the pivotal results of analysis which gave scope to many researchers to use the idea of fixed-point theory in several generalized metric space was the introduction of Banach contraction principle in 1922[2]. Billy E. Rhoades in 1977[8] was the first one to classify various definitions of contractive mappings. In 1980 many researchers continued to classify such definitions with fixed-point theorems. In the period 1979- 1993, Billy and many other authors published several papers,generalizing or unifying many of the known fixed point theorems on metric spaces. In 1992, Matthews introduced partial metric space which is a generalized metric space[33]. In partial metric is interesting as the self-distance of any point in such space may not be zero. Banach contraction principle is valid in partial metric spaces which has been proved by Matthews . The concept of dislocated space to generalize the partial metric spaces was introduced by Hitzler and Seda[26]. Then Amini-Harandi introduced the name metric-like space[24]. In the same manner, with some specific characteristics, the generalizations of metric-like space: b-metric-like space and rectangular metric-like spaces were introduced by Alghamdi et al[29].and Mlaiki et al[30].respectively.The Meir-Keeler type contraction mappings attracted many researchers in recent years[19].The fixed-point theorem was established for a generalized Meir-Keeler contraction by Hegedus[20], which deals with the diameter of orbits and the Meir-Keeler type condition. Fixed-point theorems for Meir-Keeler type contraction mappings were introduced by Park and Rhoades[3].

2 Preliminaries

$$\forall \epsilon > 0, \;\; \exists \delta > 0 : \epsilon \leq \mathfrak{L}(\varpi, \vartheta) < \delta + \epsilon \Rightarrow \mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon. \tag{2.1}$$

In 1978 Maiti and Pal [5] generalized a fixed point for maps satisfying the following condition

$$\forall \epsilon > 0, \;\; \exists \delta > 0 : \epsilon \leq \max\{\mathfrak{L}(\varpi, \vartheta), \mathfrak{L}(\varpi, \mathbb{H}\varpi), \mathfrak{L}(\vartheta, \mathbb{H}\vartheta)\} < \delta + \epsilon \Rightarrow \mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon. \tag{2.2}$$

Later in 1981, Park and Rhoades in [3] established fixed point theorems for a pair of mappings $\mathbb{H}, \mathbb{G}$ satisfying the following contractive condition that can be reduced to the following generalization of (2.3) when $\mathbb{H} = \mathbb{G}$.

$\forall \epsilon > 0 \, \exists \delta > 0:$

$$\epsilon \leq \max\{\mathfrak{L}(\varpi, \vartheta), \mathfrak{L}(\varpi, \mathbb{H}\varpi), \mathfrak{L}(\vartheta, \mathbb{H}\vartheta), \frac{\mathfrak{L}(\varpi, \mathbb{H}\vartheta) + \mathfrak{L}(\vartheta, \mathbb{H}\varpi)}{2}\} < \delta + \epsilon \Rightarrow \mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon. \tag{2.3}$$

3 Fixed and common fixed point theorems for generalized Meir-Keeler $\alpha-$contractive maps and pairs

The first part of the following definition was introduced in [6].

Definition 3.1. Let $\mathbb{H}\mathbb{G} : \Upsilon \to \Upsilon$ be self mappings of a set Υ and $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping, then the mapping $\mathbb{H}$ is called $\alpha-$admissible if

$$\varpi, \vartheta \in \Upsilon, \;\; \alpha(\varpi, \vartheta) \geq 1 \Rightarrow \alpha(\mathbb{H}\varpi, \mathbb{H}\vartheta) \geq 1$$

and the pair $(\mathbb{H}\mathbb{G})$ is called $\alpha-$admissible if

$$\varpi, \vartheta \in \Upsilon, \;\; \alpha(\varpi, \vartheta) \geq 1 \Rightarrow \alpha(\mathbb{H}, \varpi\mathbb{G}\vartheta) \geq 1 \texttt{ and } \alpha(\mathbb{G}\varpi, \mathbb{H}\vartheta) \geq 1.$$

Example 3.2.

Let $\Upsilon = \mathbb{R}$ and $\alpha(\varpi, \vartheta) = \left\{ \begin{array}{ll} 1 & \varpi, \vartheta \in [0,1] \\ 0 & otherwise \end{array} \right.$. Then the pair $(\varpi^{1/2}, \varpi^{1/3})$ is $\alpha-$admissible but the pair $(\varpi^{1/2}, \varpi + 1)$ is not $\alpha-$admissible.

Definition 3.3. Let$(\Upsilon, \mathfrak{L})$ be a metric like space and $\mathbb{H} : \Upsilon \to \Upsilon$ be a self mapping, $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping. Then, $\mathbb{H}$ is called Meir-Keeler $\alpha-$contractive if, given an $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leq \mathfrak{L}(\varpi, \vartheta) < \epsilon + \delta \Rightarrow \alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon.$$

Definition 3.4. Let$(\Upsilon, \mathfrak{L})$ be a metric like space and $\mathbb{H} : \Upsilon \to \Upsilon$ be a self mapping, $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping. Then, $\mathbb{H}$ is called generalized Meir-Keeler $\alpha-$contractive if, given an $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leq \mathbb{M}_{\mathbb{H}}(\varpi, \vartheta) < \epsilon + \delta \Rightarrow \alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon,$$

where

$$\mathbb{M}_{\mathbb{H}}(\varpi, \vartheta) = \max\{\mathfrak{L}(\varpi, \vartheta), \mathfrak{L}(\varpi, \mathbb{H}\varpi), \mathfrak{L}(\vartheta, \mathbb{H}\vartheta), \frac{\mathfrak{L}(\varpi, \mathbb{H}\vartheta) + \mathfrak{L}(\vartheta, \mathbb{H}\varpi)}{2}\}$$

Definition 3.5. Let$(\Upsilon, \mathfrak{L})$ be a metric like space and $\mathbb{H}\mathbb{G} : \Upsilon \to \Upsilon$ be self mappings, $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping. Then, the pair $(\mathbb{H}\mathbb{G})$ is called generalized Meir-Keeler $\alpha-$contractive if, given an $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leq M_{(\mathbb{H}\mathbb{G})}(\varpi, \vartheta) < \epsilon + \delta \Rightarrow \alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{H}\varpi\mathbb{G}\vartheta) < \epsilon, \tag{3.1}$$

where

$$M_{(\mathbb{H}\mathbb{G})}(\varpi, \vartheta) = \max\{\mathfrak{L}(\varpi, \vartheta), \mathfrak{L}(\varpi, \mathbb{H}\varpi), \mathfrak{L}(\vartheta\mathbb{G}\vartheta), \frac{\mathfrak{L}(\varpi\mathbb{G}\vartheta) + \mathfrak{L}(\vartheta, \mathbb{H}\varpi)}{2}\}.$$

We write $\mathbb{M}_{\mathbb{H}}(\varpi, \vartheta) = M_{(\mathbb{H},\mathbb{H})}(\varpi, \vartheta)$.

Clearly, $\mathbb{H}$ is generalized Meir-Keeler $\alpha-$contractive if and only if $(\mathbb{H}, \mathbb{H})$ is generalized Meir-Keeler $\alpha-$contractive.

Definition 3.6. Let Υ be any set, $\varpi_0 \in \Upsilon$ and $\mathbb{H}\mathbb{G}$ be self-maps of Υ. Define $\varpi_{2n+1} = \mathbb{H}\varpi_{2n}$ and $\varpi_{2n+2} = \mathbb{G}\varpi_{2n}, \;\; n = 0, 1, 2, ...$ Then $\{\varpi_n\}$ is called the $(\mathbb{H}\mathbb{G})-$orbit of ϖ_0. If $\mathfrak{L}$ is a metric on Υ then $(\Upsilon, \mathfrak{L})$ is called $(\mathbb{H}\mathbb{G})-$orbitally complete if every Cauchy sequence in the $(\mathbb{H}\mathbb{G})-$orbit of ϖ_0 is convergent and the map $\mathbb{H}$ or g is called orbitally continuous if it is continuous on the orbit.

The proof of the following lemma is immediate.

Lemma 3.7. *Let $\mathbb{HG} : \Upsilon \to \Upsilon$ be self mappings of a set Υ, $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping and $\{\varpi_n\}$ be the $(\mathbb{HG})-$orbit of ϖ_0 with $\alpha(\varpi_0, \mathbb{H}\varpi_0) \geq 1$. If the pair $(\mathbb{HG})$ is $\alpha-$admissible, then $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n = 0, 1, 2, ...$*

Theorem 3.8. *Let $(\Upsilon, \mathfrak{L})$ be an $(\mathbb{HG})-$orbitally complete metric like space, where $\mathbb{HG}$ are self-mappings of Υ. Also, let $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping. Assume the following:*

(i) $(\mathbb{HG})$ is $\alpha-$admissible and there exists an $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \mathbb{H}\varpi_0) \geq 1$;

(ii) the pair $(\mathbb{HG})$ is generalized Meir-Keeler $\alpha-$contractive.

Then the sequence $\mathfrak{L}_n = \mathfrak{L}(\varpi_n, \varpi_{n+1})$ is monotone decreasing. If. moreover we assume that

(iii) on the $(\mathbb{HG})-$orbit of ϖ_0, we have $\alpha(\varpi_n, \varpi_j) \geq 1$ for all n even and $j > n$ odd and that $\mathbb{H}$ and g are continuous on the $(\mathbb{HG})-$orbit of ϖ_0.

Then either (1) $\mathbb{H}$ or g has a fixed point in the $(\mathbb{HG})-$ orbit $\{\varpi_n\}$ of ϖ_0 or (2) $\mathbb{H}$ and g have a common fixed point p and $\lim \varpi_n = p$. If moreover, we assume that the following condition (H) holds: If $\{\varpi_n\}$ is a sequence in Υ such that $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all n and $\varpi_n \to x$ implies $\alpha(\varpi_n, \varpi) \geq 1$ for all n, then uniqueness of the fixed point is obtained.

Proof. Define $\mathfrak{L}_n = \mathfrak{L}(\varpi_n, \varpi_{n+1})$, for $n = 0, 1, 2, ...$ If $\mathfrak{L}_n = 0$ for some even integer n, then $\mathbb{H}$ has a fixed point. If $\mathfrak{L}_n = 0$ for some odd integer n, then g has a fixed point. Hence, we may assume that $\mathfrak{L}_n \neq 0$ for each n. That the pair $(\mathbb{HG})$ is generalized Meir-Keeler $\alpha-$contractive implies that,

$$\alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{H}\varpi\mathbb{G}\vartheta) < \mathbb{M}_{\mathbb{H}}(\varpi, \vartheta), \texttt{ for each } \varpi, \vartheta \in \Upsilon, \ \varpi, y \neq 0. \tag{3.2}$$

Note that assumption (3) implies that $\alpha(\varpi_0, \mathbb{H}\varpi_0) \geq 1$. Hence, since $(\mathbb{HG})$ is $\alpha-$admissible then Lemma 3.7 implies that $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n = 0, 1, 2, ...$ and hence by (3.2) we have

$$\begin{aligned} \mathfrak{L}_{2n} &= \mathfrak{L}(\mathbb{H}\varpi_{2n}, \mathbb{G}\varpi_{2n-1}) \\ &\leq \alpha(\varpi_{2n}, \varpi_{2n-1})\mathfrak{L}(\mathbb{H}\varpi_{2n}, \mathbb{G}\varpi_{2n-1}) \\ &< \max\{\mathfrak{L}_{2n-1}, \frac{\mathfrak{L}(\varpi_{2n-1}, \varpi_{2n+1})}{2}\} \\ &\leq \max\{\mathfrak{L}_{2n-1}, \frac{\mathfrak{L}_{2n-1} + \mathfrak{L}_{2n}}{2}\}, \end{aligned} \tag{3.3}$$

whence $\mathfrak{L}_{2n} < \mathfrak{L}_{2n-1}$. □

Similarly, it can be shown that $\mathfrak{L}_{2n+1} < \mathfrak{L}_{2n}$. Thus $\{\mathfrak{L}_n\}$ is monotone decreasing in n and converges to a limit, say ϱ.

Suppose $\varrho > 0$. Then, for each $\delta > 0$ there exists a positive integer $N = N(\delta)$ such that $\varrho \leq \mathfrak{L}_N = \mathfrak{L}(\varpi_N.\varpi_{N+1}) < \varrho + \delta$, where N can be chosen even. Thus from assumption (1) and Lemma 3.7 we have $\mathfrak{L}_{N+1} \leq \alpha(\varpi_N, \varpi_{N+1})\mathfrak{L}(\mathbb{H}\varpi_N, \mathbb{G}\varpi_{N+1}) < \varrho$, a contradiction. Therefore $\varrho = 0$. To show that $\{\varpi_n\}$ is Cauchy we assume the contrary. Thus there exists an $\epsilon' > 0$ such that, for each integer N there exist integers $m > n > N$ such that $\mathfrak{L}(\varpi_m, \varpi_n) \geq \epsilon'$. Define ϵ by $\epsilon' = 2\epsilon$. Choose a number δ, $0 < \delta < \epsilon$ for which (3.1) is satisfied. Since $\varrho = 0$, there exists an integer $N = N(\delta)$ such that $\mathfrak{L}_i < \frac{\delta}{6}$ for $i \geq N$. With this choice of N, pick integers $m > n > N$ such that

$$\mathfrak{L}(\varpi_m, \varpi_n) \geq 2\epsilon > \delta + \epsilon. \tag{3.4}$$

In which it is clear that $m - n > 6$. For, otherwise, $\mathfrak{L}(\varpi_m.\varpi_n)) \leq \sum_{i=0}^{5} \mathfrak{L}_{i+n} < \delta < \delta + \epsilon$, contradicting (3.4). Without loss of generality we may assume that n is even, since, from (3.4), it follows that $\mathfrak{L}(\varpi_m, \varpi_{n+1}) > \epsilon + \frac{\delta}{3}$. From (3.4) there exists a smallest odd integer $j > n$ such that

$$\mathfrak{L}(\varpi_n, \varpi_j) \geq \epsilon + \frac{\delta}{3}. \tag{3.5}$$

Hence, $\mathfrak{L}(\varpi_n, \varpi_{j-2}) < \epsilon + \frac{\delta}{3}$, and so $\mathfrak{L}(\varpi_n.\varpi_j) \leq \mathfrak{L}(\varpi_n, \varpi_{j-2}) + \mathfrak{L}_{j-1} + \mathfrak{L}_j < \epsilon + \frac{\delta}{3} + 2(\frac{\delta}{6}) = \epsilon + \frac{2\delta}{3}$. Therefore, we have

$$\begin{aligned}
\epsilon &< \mathfrak{L}(\varpi_n, \varpi_j) \leq M_{(\mathbb{HG})}(\varpi_n, \varpi_j) \\
&\leq \max\{\mathfrak{L}(\varpi_n.\varpi_j), \frac{\mathfrak{L}(\varpi_n, \varpi_{j+1}) + \mathfrak{L}(\varpi_j, \varpi_{n+1})}{2}\} \\
&\leq \frac{\mathfrak{L}(\varpi_n.\varpi_j) + \mathfrak{L}_j + \mathfrak{L}(\varpi_j, \varpi_n) + \mathfrak{L}_n}{2} \\
&\leq \mathfrak{L}(\varpi_n.\varpi_j) + \frac{\delta}{6} \leq \epsilon + \delta,
\end{aligned}$$

so that, by (3.4) and assumption 3, $\mathfrak{L}(\varpi_{n+1}, \varpi_{j+1}) \leq \alpha(\varpi_n, \varpi_j)\mathfrak{L}(\varpi_{n+1}, \varpi_{j+1}) < \epsilon$. Then, we have

$$\mathfrak{L}(\varpi_n.\varpi_j) \leq \mathfrak{L}_n + \mathfrak{L}(\varpi_{n+1}, \mathfrak{L}_{j+1}) + \mathfrak{L}_j < \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3}.$$

This contradicts the choice of j in (3.5). Therefore, $\{\varpi_n\}$ is Cauchy.

Since Υ is $(\mathbb{HG})-$orbitally complete, $\{\varpi_n\}$ converges to some point $p \in \Upsilon$. since $\mathbb{H}$ and g are orbitally continuous then p is a common fixed point of $\mathbb{H}$ and $\mathbb{G}$. To prove uniqueness, assume p is the common fixed point obtained as $\varpi_n \to p$ and q is another common fixed point. Then, (3.2) and the condition (H) yield to

$$\begin{aligned}
\mathfrak{L}(p, q) &= \mathfrak{L}(\mathbb{H}p, q) \leq \mathfrak{L}(\mathbb{H}p, \mathbb{G}\varpi_n) + \mathfrak{L}(\mathbb{G}\varpi_n, q) \\
&\leq \alpha(\varpi_n, p)\mathfrak{L}(\mathbb{H}p, \mathbb{G}\varpi_n) + \mathfrak{L}(\mathbb{G}\varpi_n, q) \\
&< M_{(\mathbb{HG})}(\varpi_n, p) + \mathfrak{L}(\mathbb{G}\varpi_n, q).
\end{aligned}$$

If we let $n \to \infty$, then we reach at $\mathfrak{L}(p, q) < \mathfrak{L}(p, q)$ which implies that $p = q$.

Corollary 3.9. *Let $(\Upsilon, \mathfrak{L})$ be an $\mathbb{H}-$orbitally complete metric like space, where $\mathbb{H}$ is a self-mappings of Υ. Also, let $\alpha : \Upsilon \times \Upsilon \to \mathbb{R} \cup \{0\}$ be a mapping. Assume the following:*

(i) $\mathbb{H}$ is $\alpha-$admissible and there exists an $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \mathbb{H}\varpi_0) \geq 1$;

(ii) $\mathbb{H}$ is generalized Meir-Keeler $\alpha-$contractive.

Then the sequence $\mathfrak{L}_n = \mathfrak{L}(\varpi_n, \varpi_{n+1})$ is monotone decreasing. If, moreover we assume that

(iii) on the $f-$orbit of ϖ_0, we have $\alpha(\varpi_n, \varpi_j) \geq 1$ for all n even and $j > n$ odd.

Then either (1) $\mathbb{H}$ has a fixed point in the $\mathbb{H}-$ orbit $\{\varpi_n\}$ of ϖ_0 or (2) $\mathbb{H}$ has a fixed point p and $\lim \varpi_n = p$. If moreover, we assume that the following condition (H) holds: If $\{\varpi_n\}$ is a sequence in Υ such that $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all n and $\varpi_n \to x$ then $\alpha(\varpi_n, \varpi) \geq 1$ for all n, then uniqueness of the fixed point is obtained.

Since generalized Meir-Keeler $\alpha-$contractions are Meir-Keeler $\alpha-$contractions, then Corollary 3.9 is valid also for Meir-Keeler $\alpha-$contractions. In the following example the existence and uniqueness of the fixed point can not be proved in the category of Meir-Keeler contractions, but can be proved by means of Corollary 3.9.

Example 3.10. Let $\Upsilon = [0, 2]$ with the absolute value metric $\mathfrak{L}(\varpi, \vartheta) = |\varpi - \vartheta|$. Define $\mathbb{H} : \Upsilon \to \Upsilon$ by

$$\mathbb{H}(\varpi) = \{ \begin{array}{cc} 0 & \varpi = \frac{1}{4} \\ 1 & \varpi \in [0, \frac{1}{2}) - \{\frac{1}{4}\} \\ \frac{3}{2} & \varpi \in [\frac{1}{2}, 2] \end{array} .$$

Then, for $\epsilon = \frac{1}{2}$, $\varpi = \frac{1}{4}$ and any $\delta > 0$ we have $\frac{1}{2} \leq |\frac{1}{4} - \vartheta| < \delta + \frac{1}{2}$ implies $y \in [\frac{1}{2}, 2]$ and hence $\mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) = \mathfrak{L}(0, \frac{3}{2}) = \frac{3}{2} > \epsilon$. Hence, $\mathbb{H}$ is not Meir-Keeler contraction. However, $\mathbb{H}$ is Meir-Keeler $\alpha-$contraction, where

$$\alpha(\varpi, \vartheta) = \{ \begin{array}{ll} 1 & \varpi, \vartheta \in [\frac{1}{2}, 2] \\ 0 & otherwise \end{array} .$$

Indeed, for $0 < \epsilon < 1$ (the case $\epsilon \geq 1$ is trivial, since $|\mathbb{H}\varpi - \mathbb{H}\vartheta| \leq 1$), let $\delta = (1 - \epsilon)$, then $\epsilon \leq \alpha(\varpi, \vartheta)\mathfrak{L}(\varpi, \vartheta) < \delta + \epsilon = 1$ implies that $\varpi, \vartheta \in [\frac{1}{2}, 2]$ and hence $\mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) = |\frac{3}{2} - \frac{3}{2}| = 0 < \epsilon$. Also, notice that $\mathbb{H}$ is continuous on the orbit of $\varpi_0 = 1$ and that $\alpha(\varpi_n, \varpi_j) \geq 1$ for all n, j. Clearly, $p = \frac{3}{2}$ is the unique fixed point.

4 Generalized Meir-Keeler $\alpha - \mathbb{H}-$contractive fixed points

Definition 4.1. Let $\mathbb{H}$ be a continuous self-map of a metric like space $(\Upsilon, \mathfrak{L})$, $C_{\mathbb{H}} = \{g : g : \Upsilon \to \Upsilon, \texttt{ such that } \mathbb{H}g = \mathbb{G}\mathbb{H} \texttt{ and } \mathbb{G}\Upsilon \subseteq \mathbb{H}\Upsilon\}$, the sequence $\{\mathbb{H}\varpi_n\}$ defined by $\mathbb{H}\varpi_{n+1} = \mathbb{G}\varpi_n, \ n = 0, 1, 2, ...$, with the understanding that, if $\mathbb{H}\varpi_n = \mathbb{H}\varpi_{n+1}$ for some n, then $\mathbb{H}\varpi_{n+j} = \mathbb{H}\varpi_n$ for each $j \geq 0$, is called the $\mathbb{H}-$iteration of ϖ_0 under g.

Definition 4.2. Let $\mathbb{H}$ be a self-map of a metric like space $(\Upsilon, \mathfrak{L})$ and $\mathbb{G} \in C_{\mathbb{H}}$. Then g is called a generalized Meir-Keeler $\alpha - \mathbb{H}-$contractive map if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\varpi, \vartheta \in \Upsilon$,

$$\epsilon \leq \mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta) < \epsilon + \delta \Rightarrow \alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{G}\varpi\mathbb{G}\vartheta) < \epsilon. \tag{4.1}$$

Definition 4.3. Let $\mathbb{H}$ be a self-map of a metric like space $(\Upsilon, \mathfrak{L})$ and $\mathbb{G} \in C_{\mathbb{H}}$. Then g is called a generalized Meir-Keeler $\alpha - \mathbb{H}-$contractive map if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\varpi, \vartheta \in \Upsilon$,

$$\epsilon \leq M_g(\mathbb{H})(\varpi, \vartheta) < \epsilon + \delta \Rightarrow \alpha(\varpi, \vartheta)\mathfrak{L}(\mathbb{G}\varpi\mathbb{G}\vartheta) < \epsilon, \tag{4.2}$$

where $M_g(\mathbb{H})(\varpi, \vartheta) = \max\{\mathfrak{L}(\mathbb{H}\varpi, \mathbb{H}\vartheta), \mathfrak{L}(\mathbb{H}\varpi\mathbb{G}\varpi), \mathfrak{L}(\mathbb{H}\vartheta\mathbb{G}\vartheta), \frac{\mathfrak{L}(\mathbb{H}\varpi\mathbb{G}\vartheta)+\mathfrak{L}(\mathbb{H}\vartheta\mathbb{G}\varpi)}{2}\}$.

Lemma 4.4. *Let $\mathbb{H}\mathbb{G}$ be continuous self-maps of a metric like space $(\Upsilon, \mathfrak{L})$ such that $\mathbb{G} \in C_{\mathbb{H}}$. Assume g is a generalized Meir-Keeler $\alpha - \mathbb{H}-$contractive map such that $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all n. Then* $\inf\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) : n = 0, 1, 2, ...\} = 0$.

Proof. Let $\sigma = \inf\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) : n = 0, 1, 2, ...\}$, and $\sigma > 0$. From the definition of the $\mathbb{H}-$iteration of ϖ_0 under g and that g is a generalized Meir-Keeler $\alpha - \mathbb{H}-$contractive map, for each n we have

$$\begin{aligned}
\mathfrak{L}(\mathbb{H}\varpi_{n+1}, \mathbb{H}\varpi_{n+2}) \quad &= \mathfrak{L}(\mathbb{G}\varpi_n, \mathbb{G}\varpi_{n+1}) \leq \alpha(\varpi_n, \varpi_{n+1})\mathfrak{L}(\mathbb{G}\varpi_n, \mathbb{G}\varpi_{n+1}) \\
&< \max\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}), \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{G}\varpi_n), \mathfrak{L}(\mathbb{H}\varpi_{n+1}, \\
&\qquad \mathbb{G}\varpi_{n+1}), \tfrac{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{G}\varpi_{n+1})+\mathfrak{L}(\mathbb{H}\varpi_{n+1}, \mathbb{G}\varpi_n)}{2}\} \\
&= \max\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}), \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}), \mathfrak{L}(\mathbb{H}\varpi_{n+1}, \\
&\qquad \mathbb{H}\varpi_{n+2}), \tfrac{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+2})+0}{2}\} \\
&= \max\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}), \tfrac{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+2})}{2}\} \\
&\leq \max\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}), \\
&\qquad \tfrac{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})+\mathfrak{L}(\mathbb{H}\varpi_{n+1}, \mathbb{H}\varpi_{n+2})}{2}\}.
\end{aligned}$$

Hence, $\mathfrak{L}(\mathbb{H}\varpi_{n+1}, \mathbb{H}\varpi_{n+2}) < \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})$ and $\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})\}$ is monotone decreasing, so that $\sigma = \lim_{n\to\infty} \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})$. From the assumption that g is Meir-Keeler $\alpha - \mathbb{H}-$contractive map, for $\epsilon = \sigma$ find $\delta > 0$ such that (4.2) is satisfied. For the chosen δ pick N so that $\sigma \leq \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) < \sigma + \delta$. Noting that for $\varpi = \varpi_n$ and $y = \varpi_{n+1}$, $M_g(\mathbb{H})(\varpi, \vartheta) = \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})$, we by (4.2), conclude that $\mathfrak{L}(\mathbb{G}\varpi_n, \mathbb{G}\varpi_{n+1}) \leq \alpha(\varpi_n, \varpi_{n+1})\mathfrak{L}(\mathbb{G}\varpi_n, \mathbb{G}\varpi_{n+1}) < \sigma$. But $\mathfrak{L}(\mathbb{G}\varpi_n, \mathbb{G}\varpi_{n+1}) = \mathfrak{L}(\mathbb{H}\varpi_{n+1}, \mathbb{H}\varpi_{n+2}) < \sigma$, a contradiction. $\square$

Theorem 4.5. *Let $\mathbb{H}\mathbb{G}$ be continuous self-maps of a metric like space $(\Upsilon, \mathfrak{L})$ such that $\mathbb{G} \in C_{\mathbb{H}}$. Assume $\alpha(\varpi_n, \varpi_m) \geq 1$ for all $m > n$. If g is a generalized Meir-Keeler $\alpha-\mathbb{H}-$contractive map such that α satisfies the condition (f-H): If $\{\varpi_n\}$ is a sequence in Υ such that $\alpha(\varpi_n, \varpi_m) \geq 1$ for all $m > n$ and $\mathbb{H}\varpi_n \to z$ then $\alpha(\mathbb{H}\varpi_n, z) \geq 1$ and $\alpha(\mathbb{H}\varpi_n, \mathbb{H}z) \geq 1$ for all n. Then $\mathbb{H}$ and g have a unique common fixed point.*

Proof. Let $\varpi_0 \in \Upsilon$ for which its $\mathbb{H}-$iteration under g satisfies the assumptions of the theorem. The proof will be divided into 4 steps.

- Step 1 : By Lemma 4.4, $\inf\{\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) : n = 0, 1, 2, ...\} = 0$.
- Step 2 : We find a coincidence point for $\mathbb{H}$ and g. That is to find a $z \in \Upsilon$ such that $\mathbb{H}z = \mathbb{G}z$. If there exists an n such that $\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) = 0$, then $\mathbb{H}\varpi_{n+1} = \mathbb{G}\varpi_n = \mathbb{H}\varpi_n$ and we are finished. Hence, we may assume that $\mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1}) \neq 0$ for each n. We claim to show that $\{\mathbb{H}\varpi_n\}$ is Cauchy. Suppose not. Then there exists an $\epsilon > 0$ and a subsequence $\{\mathbb{H}\varpi_{n_i}\}$ of $\{\mathbb{H}\varpi_n\}$ such that $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_{n_{i+1}}) > 2\epsilon$. From (4.2), there exists a δ satisfying $0 < \delta < \epsilon$ for which (4.2) is true. Since $\lim_{n\to\infty} \mathfrak{L}(\mathbb{H}\varpi_n, \mathbb{H}\varpi_{n+1})$=0, there exists an N such that

$$\mathfrak{L}(\mathbb{H}\varpi_m, \mathbb{H}\varpi_{m+1}) < \frac{\delta}{6} \texttt{ for all } m > N.$$

Let $n_i \geq N$. We shall show that there exists an integer j satisfying $n_i < j < n_{i+1}$, such that

$$\epsilon + \frac{\delta}{3} \leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) < \epsilon + \frac{2\delta}{3}. \tag{4.3}$$

First of all, there exist values of j such that $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) \geq \epsilon + \frac{\delta}{3}$. For example, choose $j = n_{i+1}$. The inequality is also true for $j = n_{i+1} - 1$. For, if not then $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) < \epsilon + \frac{\delta}{3}$ and hence

$$\begin{aligned} \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_{n_{i+1}}) &\leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_{n_{i+1}} - 1) + \mathfrak{L}(\mathbb{H}\varpi_{n_{i+1}} - 1, \mathbb{H}\varpi_{n_{i+1}}) \\ &< \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < 2\epsilon, \end{aligned}$$

a contradiction. There are also values of j such that $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) < \epsilon + \frac{\delta}{3}$. For example choose $j = n_i + 1$ and $j = n_i + 2$. Pick j to be the smallest integer greater than n_i such that $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) \geq \epsilon + \frac{\delta}{3}$. Then $\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_i - 1) < \epsilon + \frac{\delta}{3}$, and hence

$$\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) \leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j - 1) + \mathfrak{L}(\mathbb{H}\varpi_j - 1, \mathbb{H}\varpi_j) < \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}.$$

Thus (4.3) is established. Now note that

$$\begin{aligned} \epsilon + \frac{\delta}{3} \leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) \leq \max\{&\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j), \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{G}\varpi_{n_i}), \\ &\mathfrak{L}(\mathbb{H}\varpi_j, \mathbb{G}\varpi_j), \frac{\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{G}\varpi_j) + \mathfrak{L}(\mathbb{H}\varpi_j, \mathbb{G}\varpi_{n_i})}{2}\}. \end{aligned}$$

Then from the choice of j and that $\mathbb{H}\varpi_{n_i} + 1 = \mathbb{G}\varpi_{n_i}$, $\mathbb{H}\varpi_j + 1 = \mathbb{G}\varpi_j$ we reach at

$$\epsilon \leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) < \delta + \epsilon.$$

Hence,

$$\mathfrak{L}(\mathbb{H}\varpi_{n_i+1}, \mathbb{H}\varpi_{j+1}) = \mathfrak{L}(\mathbb{G}\varpi_{n_i}, \mathbb{G}\varpi_j) \leq \alpha(\varpi_{n_i}, \varpi_j)\mathfrak{L}(\mathbb{G}\varpi_{n_i}, \mathbb{G}\varpi_j) < \epsilon.$$

On the other hand,

$$\mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}\varpi_j) \leq \mathfrak{L}(\mathbb{H}\varpi_{n_i}, \mathbb{H}_{n_i+1}) + \mathfrak{L}(\mathbb{H}_{n_i+1}, \mathbb{H}\varpi_{j+1}) + \mathfrak{L}(\mathbb{H}\varpi_{j+1}, \mathbb{H}\varpi_j) < \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3},$$

contradicting (4.3). Therefore, $\{\mathbb{H}\varpi_n\}$ is Cauchy, hence convergent to $z \in \Upsilon$. Since $\mathbb{H}\mathbb{H}\varpi_n = \mathbb{H}\mathbb{G}\varpi_{n-1} = \mathbb{G}\mathbb{H}\varpi_{n-1}$, the continuity of $\mathbb{H}$ and g implies that $\mathbb{H}z = \mathbb{G}z$.

- Step 3: We show that $\eta = \mathbb{H}z = \mathbb{G}z$ is a common fixed point for $\mathbb{H}$ and g: Assume $\mathbb{H}\eta \neq \eta$. Then $\mathbb{H}^2 z \neq \mathbb{H}z$ and by the help of the (f-H) condition we have

$$\begin{aligned}
\mathfrak{L}(\eta, \mathbb{H}\eta) &= \mathfrak{L}(\mathbb{G}z, \mathbb{H}\mathbb{G}z) = \mathfrak{L}(\mathbb{G}z, \mathbb{G}\mathbb{H}z) \\
&\leq \mathfrak{L}(\mathbb{G}z, \mathbb{G}\mathbb{H}\varpi_n) + \mathfrak{L}(\mathbb{G}\mathbb{H}\varpi_n \mathbb{G}\mathbb{H}z) \\
&\leq \alpha(\mathbb{H}\varpi_n, z)\mathfrak{L}(\mathbb{G}z, \mathbb{G}\mathbb{H}\varpi_n) + \alpha(\mathbb{H}\varpi_n, \mathbb{H}z)\mathfrak{L}(\mathbb{G}\mathbb{H}\varpi_n \mathbb{G}\mathbb{H}z) \\
&< \max\{\mathfrak{L}(\mathbb{H}z, \mathbb{H}\mathbb{H}\varpi_n), \mathfrak{L}(\mathbb{H}z\mathbb{G}z), \mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\mathbb{H}\varpi_n), \\
&\quad \frac{\mathfrak{L}(\mathbb{H}z, \mathbb{G}\mathbb{H}\varpi_n) + \mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\mathbb{H}z)}{2}\} \\
&+ \max\{\mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{H}\mathbb{H}z), \mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\mathbb{H}\varpi_n), \mathfrak{L}(\mathbb{H}\mathbb{H}z, \mathbb{G}\mathbb{H}z), \\
&\quad \frac{\mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\mathbb{H}z) + \mathfrak{L}(\mathbb{H}\mathbb{H}z, \mathbb{G}\mathbb{H}\varpi_n)}{2}\}
\end{aligned}$$

If we let $n \to \infty$ above and use continuity and commutativity of $\mathbb{H}$ and g then we reach at $\mathfrak{L}(\eta, \mathbb{H}\eta) < \mathfrak{L}(\eta, \mathbb{H}\eta)$ and hence $\mathbb{H}\eta = \eta$. Moreover, $\mathbb{G}\eta = \mathbb{G}\mathbb{H}z = \mathbb{H}\eta = \eta$.

- Step 4: Uniqueness of the common fixed point: Assume $\eta = \mathbb{H}z = \mathbb{G}z$ is our common fixed point for $\mathbb{H}$ and g where $\mathbb{H}\varpi_n \to z$ and ω is another common fixed point. Then by the (f-H) condition we have

$$\begin{aligned}
\mathfrak{L}(\eta, \omega) &= \mathfrak{L}(\mathbb{G}\eta, \omega) \leq \mathfrak{L}(\mathbb{G}\eta, \mathbb{G}\mathbb{H}\varpi_n) + \mathfrak{L}(\mathbb{G}\mathbb{H}\varpi_n, \omega) \\
&\leq \alpha(\eta, \mathbb{H}\varpi_n)\mathfrak{L}(\mathbb{G}\eta, \mathbb{G}\mathbb{H}\varpi_n) + \mathfrak{L}(\mathbb{G}\mathbb{H}\varpi_n, \omega) \\
&< \max\{\mathfrak{L}(\mathbb{H}\eta, \mathbb{H}\mathbb{H}\varpi_n), \mathfrak{L}(\mathbb{H}\eta, \mathbb{G}\eta), \mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\mathbb{H}\varpi_n), \frac{\mathfrak{L}(\mathbb{H}\eta, \mathbb{G}\mathbb{H}\varpi_n) + \mathfrak{L}(\mathbb{H}\mathbb{H}\varpi_n, \mathbb{G}\eta)}{2}\}
\end{aligned}$$

If we let $n \to \infty$ above and use continuity of $\mathbb{H}$ and g, we conclude that $\mathfrak{L}(\eta, \omega) < \mathfrak{L}(\eta, \omega)$ and hence $\eta = \omega$.

□

5 Application to coupled α-Meir-Keeler fixed points

Let $F : \Upsilon \times \Upsilon \to \Upsilon$ be a mapping. We say that $(\varpi, \vartheta) \in \Upsilon \times \Upsilon$ is a coupled fixed point of F if $F(\varpi, \vartheta) = x$ and $F(\vartheta, \varpi) = y$. If we define $T : \Upsilon \times \Upsilon \to \Upsilon \times \Upsilon$ by $T(\varpi, \vartheta) = (F(\varpi, \vartheta), F(\vartheta, \varpi))$, then clearly (ϖ, ϑ) is a coupled fixed point of F if and only if (ϖ, ϑ) is a fixed point of T. If $(\varpi_0, \vartheta_0) \in \Upsilon \times \Upsilon$ then the F-orbit of (ϖ_0, ϑ_0) means the orbit $\{(\varpi_n, \vartheta_n) : n = 0, 1, 2, ...\}$ where $(\varpi_{n+1}, \vartheta_{n+1}) = T(\varpi_n, \vartheta_n)$.

If $(\Upsilon, \mathfrak{L})$ is a metric like space then $\rho : \Upsilon \times \Upsilon \to \mathbb{R}$ defined by $\rho((\varpi, \vartheta), (u, v)) = \mathfrak{L}(\varpi, u) + \mathfrak{L}(\vartheta, v)$ is a metric on $\Upsilon \times \Upsilon$.

Theorem 5.1. *Let $(\Upsilon, \mathfrak{L})$ be a complete metric like space, and $F : \Upsilon \times \Upsilon \to \Upsilon$ be a continuous mapping. Also, let $\alpha : \Upsilon^2 \times \Upsilon^2 \to \mathbb{R} \cup \{0\}$ be a mapping. Assume the following:*

(i) For all $(\varpi, \vartheta), (u, v) \in \Upsilon \times \Upsilon$, we have

$$\alpha((\varpi, \vartheta), (u, v)) \geq 1 \texttt{ implies } \alpha((F(\varpi, \vartheta), (F(\vartheta, \varpi)), (F(u, v), F(v, u))) \geq 1.$$

Also assume there exists $(\varpi_0, \vartheta_0) \in \Upsilon \times \Upsilon$ such that

$$\alpha((F(\vartheta_0, \varpi_0), F(\varpi_0, \vartheta_0)), (\vartheta_0, \varpi_0)) \geq 1$$

and

$$\alpha((\varpi_0, \vartheta_0), (F(\varpi_0, \vartheta_0), F(\vartheta_0, \varpi_0))) \geq 1;$$

(ii) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \frac{1}{2}[\mathfrak{L}(\varpi, u) + \mathfrak{L}(\vartheta, v)] < \delta + \epsilon \Rightarrow \alpha((\varpi, \vartheta), (u, v))\mathfrak{L}(F(\varpi, \vartheta), F(u, v)) < \epsilon.$$

Then the sequence $\rho_n = \rho((\varpi_n, \vartheta_n), (\varpi_{n+1}, \vartheta_{n+1}))$ is monotone decreasing. Moreover, if

(iii) on the $F-$orbit of (ϖ_0, ϑ_0), we have

$$\alpha((\varpi_n, \vartheta_n), (\varpi_j, \vartheta_j)) \geq 1$$

and

$$\alpha((\vartheta_j, \varpi_j), (\vartheta_n, \varpi_n)) \geq 1 \textit{ for all } n, j.$$

*Then either (1) F has a coupled fixed point in the $F-$ orbit $\{(\varpi_n, \vartheta_n)\}$ of (ϖ_0, ϑ_0) or (2) F has a coupled fixed point (p, q) and $\lim \rho(\varpi_n, \vartheta_n) = (p, q)$. If moreover, we assume that the following condition (**H**) holds: If $\{(\varpi_n, \vartheta_n)\}$ is a sequence in $\Upsilon \times \Upsilon$ such that $\alpha((\varpi_n, \vartheta_n), (\varpi_{n+1}, \vartheta_{n+1}) \geq 1$ for all n and $\mathfrak{L}(\varpi_n, \varpi) \to 0, \mathfrak{L}(\vartheta_n, \vartheta) \to 0$ then $\alpha((\varpi_n, \vartheta_n), (\varpi, \vartheta)) \geq 1$ and $\alpha((\vartheta, \varpi), (\vartheta_n, \varpi_n)) \geq 1$ for all n, then uniqueness of the coupled fixed point is obtained.*

Proof. The proof will follow by applying Corollary 3.9 with $\mathbb{H} = T$ as above, on the metric like space $(\Upsilon \times \Upsilon, \rho)$. The controlling function will be $\beta : \Upsilon^2 \times \Upsilon^2 \to \mathbb{R} \cup \{0\}$ given by

$$\beta((\varpi, \vartheta), (u, v)) = \min\{\alpha((\varpi, \vartheta), (u, v)), \alpha((\vartheta, \varpi), (v, u))\}.$$

In fact, if $\epsilon > 0$ is given then by the assumption 2, find $\delta' > 0$ such that

$$\frac{\epsilon}{2} \leq \frac{1}{2}[\mathfrak{L}(\varpi, u) + \mathfrak{L}(\vartheta, v)] < \delta' + \frac{\epsilon}{2} \Rightarrow \alpha((\varpi, \vartheta), (u, v))\mathfrak{L}(F(\varpi, \vartheta), F(u, v)) < \frac{\epsilon}{2}.$$

Let $\delta = 2\delta'$ and assume $\epsilon \leq \rho((\varpi, \vartheta), (u, v)) < \delta + \epsilon$. Then

$$\frac{\epsilon}{2} \leq \frac{1}{2}[\mathfrak{L}(\varpi, u) + \mathfrak{L}(\vartheta, v)] < \delta' + \frac{\epsilon}{2}$$

and

$$\frac{\epsilon}{2} \leq \frac{1}{2}[\mathfrak{L}(v, u) + \mathfrak{L}(u, \varpi)] < \delta' + \frac{\epsilon}{2}.$$

Hence,

$$\alpha((\varpi, \vartheta), (u, v))\mathfrak{L}(F(\varpi, \vartheta), F(u, v)) < \frac{\epsilon}{2}$$

and

$$\alpha((v, u), (\vartheta, \varpi))\mathfrak{L}(F(\vartheta, \varpi), F(v, u)) < \frac{\epsilon}{2}.$$

Which leads to that

$$\beta((\varpi, \vartheta), (u, v))\rho(T(\varpi, \vartheta), T(u, v)) < \epsilon.$$

□

6 Conclusion

In this manuscript, we have proved new fixed point theorem for generalized Meir-Keeler $\alpha-$contractive mappings which is generalization of many results in the literature. Also, we presented an application of our result to coupled α-Meir-Keeler fixed points. In closing, we would like to bring to the reader's attention the following open question;

Question1:

Can we prove Theorem 3.8 and Theoem 4.5, in double controlled metric like spaces?

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