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The norming sets of $\mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$

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Abstract Let $n \in \mathbb{N}$. An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$, where $\mathcal{P}(^nE)$ denotes the space of all continuous n-homogeneous polynomials on E. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P. We classify $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$, where $\mathbb{R}^2_{h(1/2)} = \mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$.

1 Introduction

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a real Banach space E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

$Norm(P)$ is called the *norming set* of P. Notice that $x \in \text{Norm}(P)$ if and only if $-x \in \text{Norm}(P)$. Indeed, if $x \in \text{Norm}(P)$, then

$$|P(-x)| = |(-1)^n P(x)| = |P(x)| = \|P\|,$$

which shows that $-x \in \text{Norm}(P)$. If $-x \in \text{Norm}(P)$, then $x = -(-x) \in \text{Norm}(P)$. The following examples show that it is possible that $\text{Norm}(P)$ be empty, a finite or an infinite set.

Examples. (a) Let

$$P((x_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2c_0).$$

Then, $\text{Norm}(P) = \emptyset$.

(b) Let

$$P((x_i)_{i\in\mathbb{N}}) = x_1^2 - \sum_{i=2}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2c_0).$$

Then,

$$\text{Norm}(P) = \{\pm e_1\}.$$

(c) Let

$$P((x_i)_{i\in\mathbb{N}}) = x_1^2 \in \mathcal{P}(^2c_0).$$

Then,

$$\text{Norm}(P) = \{((\pm 1, x_2, x_3, \ldots) \in c_0 : |x_j| \le 1 \text{ for } j = 2, 3, \ldots\}.$$

If $\text{Norm}(P) \neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* polynomial (see [3]).

It seems to be natural and interesting to study $\text{Norm}(P)$ for $P \in \mathcal{P}(^nE)$. For $m \in \mathbb{N}$, let $l_\infty^m := \mathbb{R}^m$ with the supremum norm. Notice that for every $P \in \mathcal{P}(^nl_\infty^m)$, $\text{Norm}(P) \neq \emptyset$ since $S_{l_\infty^m}$ is compact. Kim [7] classified $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2l_\infty^2)$.

Let $\mathbb{R}^2_{h(1/2)}$ denote the plane with the hexagonal norm

$$\|(x, y)\|_{h(1/2)} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}.$$

In this paper, we classify $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$.

2 Main results

Kim [6] presented an explicit formulae for the norm of $P \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$.

Theorem 2.1. *([6]). Let $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ with $a \ge 0, c \ge 0$ and $a^2 + b^2 + c^2 \neq 0$. Then:*

Case 1: $c < a$

If $a \le 4b$, then

$$\begin{aligned} \|P\| &= \max\Big\{a, b, |\frac{1}{4}a + b| + \frac{1}{2}c, \frac{4ab - c^2}{4a}, \frac{4ab - c^2}{2c + a + 4b}, \frac{4ab - c^2}{|2c - a - 4b|}\Big\} \\ &= \max\Big\{a, b, |\frac{1}{4}a + b| + \frac{1}{2}c\Big\}. \end{aligned}$$

If $a > 4b$, then $\|P\| = \max\Big\{a, |b|, \Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c, \frac{|c^2 - 4ab|}{4a}\Big\}$.

Case 2: $c \ge a$

If $a \le 4b$, then $\|P\| = \max\Big\{a, b, \Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c, \frac{|c^2 - 4ab|}{2c + a + 4b}\Big\}$.

If $a > 4b$, then $\|P\| = \max\Big\{a, |b|, \Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c, \frac{c^2 - 4ab}{2c - a - 4b}\Big\}$.

Notice that if $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a, b, c \in \mathbb{R}$ with $\|P\| = 1$, then $|a| \le 1, |b| \le 1$ and $|c| \le 2$.

In order to classify $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ we need some Lemmas.

Lemma 2.2. *Let $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a \ge 0, c \ge 0$ and $b \in \mathbb{R}$ with $\|P\| = 1$. Let $c < a$ and $4b < a$.*

(1) If $b < 0$, *then*

$$\left(\left|\frac{1}{4}a+b\right|+\frac{1}{2}c<1, \frac{|c^2-4ab|}{4a}<1\right), \left(\left|\frac{1}{4}a+b\right|+\frac{1}{2}c=1, \frac{|c^2-4ab|}{4a}<1\right)$$
$$or \left(\left|\frac{1}{4}a+b\right|+\frac{1}{2}c<1, \frac{|c^2-4ab|}{4a}=1\right).$$

(2) If $b \geq 0$, *then* $\frac{|c^2-4ab|}{4a} < \frac{1}{2}$.

Proof. (1). Assume that

$$(*)\quad \left|\frac{1}{4}a+b\right|+\frac{1}{2}c=1=\frac{|c^2-4ab|}{4a}.$$

Case 1. $|b| \leq \frac{a}{4}$

Adding two equalities in $(*)$, we get $\frac{1}{4}a+\frac{1}{2}c+\frac{c^2}{4a}=2$, so $a^2+2a(c-4)+c^2=0$. Thus $a=4-c-2\sqrt{4-2c}$. Since $c<a=4-c-2\sqrt{4-2c}$, we have $0<c(c-2)<0$, which is a contradiction.

Case 2. $|b| > \frac{a}{4}$

Subtracting two equalities in $(*)$, we get $-\frac{1}{4}a+\frac{1}{2}c-\frac{c^2}{4a}=0$, so $0<(a-c)^2=0$, which is a contradiction.

(2). Since $b < \frac{1}{4}a \leq \frac{1}{4}$, it follows that

$$\begin{aligned}\frac{|c^2-4ab|}{4a} &\leq \frac{c^2+4ab}{4a}=\frac{c^2}{4a}+b\\ &< \frac{c}{4}\left(\frac{c}{a}\right)+\frac{1}{4}<\frac{c}{4}+\frac{1}{4}\\ &< \frac{a}{4}+\frac{1}{4}\leq\frac{1}{2}. \quad \square\end{aligned}$$

Lemma 2.3. *Let* $P(x,y)=ax^2+by^2+cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ *for some* $a \geq 0, c \geq 0$ *and* $b \in \mathbb{R}$ *with* $\|P\|=1$. *If* $c \geq a, 0<a\leq 4b$, *then* $\frac{|c^2-4ab|}{2c+a+4b} < |\frac{1}{4}a+b|+\frac{1}{2}c$.

Proof. Let $l := |\frac{1}{4}a+b|+\frac{1}{2}c$.

Case 1. $c^2-4ab \geq 0$

It follows that

$$\frac{|c^2-4ab|}{2c+a+4b}=\frac{c^2-4ab}{4l}<\frac{c^2}{4l}<\frac{4l^2}{4l}=l.$$

Case 2. $c^2-4ab<0$

By the arithmetic and geometric mean inequality, $\sqrt{ab} \leq \frac{1}{4}a+b \leq l$. Since $c>0$, we have

$$\frac{|c^2-4ab|}{2c+a+4b}=\frac{4ab-c^2}{4l}<\frac{4ab}{4l}=\frac{ab}{l}=\frac{(\sqrt{ab})^2}{l}\leq\frac{l^2}{l}=l.$$

This completes the proof. $\square$

Corollary 2.4. *Let* $P(x,y)=ax^2+by^2+cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ *for some* $a \geq 0, c \geq 0$ *and* $b \in \mathbb{R}$ *with* $\|P\|=1$. *Suppose that* $c \geq a, 0<a\leq 4b$. *If* $|\frac{1}{4}a+b|+\frac{1}{2}c=1$, *then* $\frac{|c^2-4ab|}{2c+a+4b}<1$.

Proof. This follows from Lemma 2.3. $\square$

Lemma 2.5. *Let $P(x,y) = ax^2+by^2+cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a \geq 0, c \geq 0$ and $b \in \mathbb{R}$ with $\|P\| = 1$. Suppose that $a \leq c < 2, 4b < a$ and $\Big(b \geq 0$ or $(b < 0, |b| \leq \frac{a}{4})\Big)$. If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\frac{c^2-4ab}{2c-a-4b} < 1$.*

Proof. Assume that $\frac{c^2-4ab}{2c-a-4b} = 1$.

Case 1. $b \geq 0$

Notice that $c = 2 - \frac{1}{2}a - 2b$ and $0 = \frac{1}{4}a^2 - 2ab + 4b^2 = (\frac{1}{2}a - 2b)^2$, $4b < a = 4b$. This is a contradiction.

Case 2. $b < 0$

Notice that if $|b| \leq \frac{a}{4}$, by an analogous argument as in the case 1, $4b < a = 4b$. This is a contradiction. This completes the proof. □

Let

$$L_1 = \Big\{(t,1) : 0 \leq t \leq \frac{1}{2}\Big\},\ L_2 = \Big\{(t,-2t+2) : \frac{1}{2} < t \leq 1\Big\},$$
$$L_3 = \Big\{(t,2t-2) : \frac{1}{2} < t \leq 1\Big\},\ L_4 = \Big\{(t,-1) : 0 \leq t \leq \frac{1}{2}\Big\}.$$

Lemma 2.6. *Let $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a \geq 0, c \geq 0$ and $b \in \mathbb{R}$ with $\|P\| = 1$. Then $\mathrm{Norm}(P) = \{\pm(x,y) : (x,y) \in \bigcup_{1\leq j\leq 4} L_j,\ |P(x,y)| = 1\}$.*

Proof. This is obvious. □

Lemma 2.7. *Let $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a \geq 0, c \geq 0$ and $b \in \mathbb{R}$ with $\|P\| = 1$ and $abc \neq 0$. Then the following assertions hold:*

(a) If $a = |\frac{1}{4}a + b| + \frac{1}{2}c = 1$ and $(b > 0$ or $\frac{1}{4}a > |b| = -b)$, then

$$\mathrm{Norm}(P) = \Big\{\pm(1,0), \pm(\frac{1}{2}, 1)\Big\}.$$

If $a = |\frac{1}{4}a + b| + \frac{1}{2}c = 1$ and $\frac{1}{4}a = |b| = -b$, then

$$\mathrm{Norm}(P) = \Big\{\pm(1,0), \pm(\frac{1}{2}, \pm 1)\Big\}.$$

If $a = |\frac{1}{4}a + b| + \frac{1}{2}c = 1$ and $\frac{1}{4}a < |b| = -b$, then

$$\mathrm{Norm}(P) = \Big\{\pm(1,0), \pm(\frac{1}{2}, -1)\Big\}.$$

(b) If $c < a, 4b < a$, and $a = \frac{|c^2-4ab|}{4a} = 1 > |\frac{1}{4}a + b| + \frac{1}{2}c$, then

$$\mathrm{Norm}(P) = \Big\{\pm(1,0), \pm\Big(\frac{c}{2a}, -1\Big)\Big\}.$$

(c) If $c \geq a, 4b < a$, and $a = \frac{c^2-4ab}{2c-a-4b} = 1 > |\frac{1}{4}a + b| + \frac{1}{2}c$, then

$$\mathrm{Norm}(P) = \Big\{\pm(1,0)\Big\}.$$

Proof. Notice that

$$\Big|P\Big(\frac{1}{2},\ 1\Big)\Big| = \Big|\frac{1}{4}a + b + \frac{1}{2}c\Big|,\ \Big|P\Big(\frac{c}{2a},\ -1\Big)\Big| = \frac{|c^2 - 4ab|}{4a},$$
$$\Big|P\Big(\frac{c-4b}{2c-a-4b},\ \frac{2(c-a)}{2c-a-4b}\Big)\Big| = \Big|\frac{c^2-4ab}{2c-a-4b}\Big|.$$

Let $(x,y) \in \mathrm{Norm}(P)$. By Lemma 2.6, we may assume that $(x,y) \in \bigcup_{1\leq j\leq 4} L_j$. Notice that $P(x,y) = \pm 1$ are quadratic equations of the variable $0 \leq t \leq 1$. Solving the quadratic equations $P(x,y) = \pm 1$ by the arguments in the proof of Theorem A of [6], we complete the proof. □

Lemma 2.8. *Let $P(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a\geq 0, c\geq 0$ and $b\in\mathbb{R}$ with $\|P\|=1$. Then the following assertions hold:*

(a) *Suppose that*

$$\Big(\frac{1}{4}a+b\Big)+\frac{1}{2}c=\frac{c^2-4ab}{2c-a-4b}=1.$$

If $c\geq a$, then

$$\Big(\frac{c-4b}{2c-a-4b},\ \frac{2(c-a)}{2c-a-4b}\Big)=\Big(\frac{1}{2},1\Big).$$

(b) *Suppose that*

$$\frac{|c^2-4ab|}{4a}=-\Big(\frac{1}{4}a+b\Big)+\frac{1}{2}c=1.$$

If $c\leq a$, then

$$\Big(\frac{c}{2a},-1\Big)=\Big(\frac{1}{2},-1\Big).$$

Proof. (a). By Theorem 2.1, $c\geq a=4b$. Thus,

$$\Big(\frac{c-4b}{2c-a-4b},\ \frac{2(c-a)}{2c-a-4b}\Big)=\Big(\frac{1}{2},1\Big).$$

(b). We claim that $c=a$. Suppose not. By Theorem 2.1, $c<a=4b$. Since

$$1=\Big|\frac{1}{4}a+b\Big|+\frac{1}{2}c=\frac{1}{4}a+b+\frac{1}{2}c,$$

$2-a=c<a$, so $a>1$. This is a contradiction. Hence, $c=a$. This shows (b). □

We are in position to prove the main result of this paper.

Theorem 2.9. *Let $P(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2\mathbb{R}^2_{h(1/2)})$ for some $a\geq 0, c\geq 0$ and $b\in\mathbb{R}$ with $\|P\|=1$. Then the following assertions hold:*

Case 1. $abc=0$

If $a=1, b=c=0$ then $\mathrm{Norm}(P)=\Big\{\pm(1,0)\Big\}$.

If $b=1, a=c=0$, then $\mathrm{Norm}(P)=\Big\{\pm(t,\pm1): 0\leq t\leq\frac{1}{2}\Big\}$.

If $c=2, a=b=0$, then $\mathrm{Norm}(P)=\Big\{\pm(\frac{1}{2},\pm1)\Big\}$.

Let $ab\neq 0, c=0$.

Suppose that $b>0$.

Let $a\leq\frac{4}{3}b$.

If $a=1$, then $\mathrm{Norm}(P)=\Big\{\pm(1,0),\pm(\frac{1}{2},\pm1)\Big\}$.

If $0<a<1$, then $\mathrm{Norm}(P)=\Big\{\pm(\frac{1}{2},\pm1)\Big\}$.

If $a>\frac{4}{3}b$, then $\mathrm{Norm}(P)=\Big\{\pm(1,0)\Big\}$.

Suppose that $b<0$.

If $a>|b|$, then $\mathrm{Norm}(P)=\Big\{\pm(1,0)\Big\}$.

If $a<|b|$, then $\mathrm{Norm}(P)=\Big\{\pm(0,1)\Big\}$.

If $a=|b|$, then $\mathrm{Norm}(P)=\Big\{\pm(1,0),\pm(0,1)\Big\}$.

Let $ac\neq 0, b=0$.

If $c\leq a$, then $\mathrm{Norm}(P)=\Big\{\pm(1,0)\Big\}$.

If $c>a$, then $\mathrm{Norm}(P)=\Big\{\pm\Big(\frac{c}{2c-a},\frac{2(c-a)}{2c-a}\Big)\Big\}$.

If $bc\neq 0, b>0, a=0$ then $\mathrm{Norm}(P)=\Big\{\pm(\frac{1}{2},1)\Big\}$.

If $bc \neq 0, b < 0, a = 0$ *then* $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, -1)\Big\}$.

Case 2. $abc \neq 0$

Subcase 1. $0 < c < a$ *and* $0 < a \leq 4b$

Suppose that $a = 1$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm(\frac{1}{2}, 1)\Big\}$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, *then* $\text{Norm}(P) = \{\pm(1,0)\}$.
Suppose that $a < 1$. *Then* $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, 1)\Big\}$.

Subcase 2. $0 < c < a$ *and* $a > 4b$

Let $b > 0$.

Suppose that $a = 1$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm(\frac{1}{2}, 1)\Big\}$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, *then* $\text{Norm}(P) = \{\pm(1,0)\}$.
Suppose that $a < 1$. *Then* $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, 1)\Big\}$.
Let $b < 0$.

Suppose that $a = 1$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm(\frac{1}{2}, 1)\Big\}$.
If $\frac{|c^2-4ab|}{4a} = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm\Big(\frac{c}{2a}, \ -1\Big)\Big\}$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$ *and* $\frac{|c^2-4ab|}{4a} < 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0)\Big\}$.

Subcase 3. $c \geq a > 0$ *and* $0 < a \leq 4b$

If $a = 1, b < 1$ *and* $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm(\frac{1}{2}, 1)\Big\}$.
If $a = 1, b < 1$ *and* $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, *then* $\text{Norm}(P) = \{\pm(1,0)\}$.
If $a < 1, b < 1$ *and* $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, 1)\Big\}$.

Subcase 4. $c \geq a$ *and* $a > 4b$

Suppose that $a = 1$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm(\frac{1}{2}, 1)\Big\}$.
If $\frac{c^2-4ab}{2c-a-4b} = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (1,0), \pm\Big(\frac{c-4b}{2c-a-4b}, \ \frac{2(c-a)}{2c-a-4b}\Big)\Big\}$.
If $\frac{c^2-4ab}{2c-a-4b} < 1, |\frac{1}{4}a + b| + \frac{1}{2}c < 1$, *then* $\text{Norm}(P) = \{\pm(1,0)\}$.

Suppose that $0 < a < 1$.
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, *then* $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, 1)\Big\}$.
If $\frac{c^2-4ab}{2c-a-4b} = 1$, *then* $\text{Norm}(P) = \Big\{ \pm \Big(\frac{c-4b}{2c-a-4b}, \ \frac{2(c-a)}{2c-a-4b}\Big)\Big\}$.

Proof. We use Theorem 2.1 and Lemmas 2.2, 2.3, 2.5–2.8.
Let $(x, y) \in \text{Norm}(P)$. By Lemma 2.6, we may assume that $(x, y) \in \bigcup_{1\leq j\leq 4} L_j$.

Case 1. $abc = 0$

Let $P(x, y) = x^2$. It is obvious that $\text{Norm}(P) = \Big\{ \pm (1,0)\Big\}$.

Let $P(x, y) = \pm y^2$, It is obvious that $\text{Norm}(P) = \Big\{ \pm (t, \pm 1) : 0 \leq t \leq \frac{1}{2}\Big\}$.

Let $P(x, y) = 2xy$. It is obvious that $\text{Norm}(P) = \Big\{ \pm (\frac{1}{2}, \pm 1)\Big\}$.
Let $P(x, y) = ax^2 + by^2$ for $ab \neq 0$.
Suppose that $b > 0$.

Let $a \leq \frac{4}{3}b$. Notice that $P(x,y) = ax^2 + (1-\frac{1}{4}a)y^2$ for $0 < a \leq 1$. If $a = 1$, then $P(x,y) = x^2 + \frac{3}{4}y^2$ and $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, \pm 1)\Big\}$. If $0 < a < 1$, then $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, \pm 1)\Big\}$.

Suppose that $b < 0$. Notice that if $a > |b|$, then $\text{Norm}(P) = \Big\{ \pm(1,0)\Big\}$, if $a < |b|$, then $\text{Norm}(P) = \Big\{ \pm(0,1)\Big\}$, and that if $a = |b|$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(0,1)\Big\}$.

Let $P(x,y) = ax^2 + cxy$ for $ac \neq 0$. If $c < a$, then $P(x,y) = x^2 + cxy$ for $0 < c < 1$. Thus, $\text{Norm}(P) = \Big\{ \pm(1,0)\Big\}$. If $c = a$, then $P(x,y) = x^2 + xy$ and $\text{Norm}(P) = \Big\{ \pm(1,0)\Big\}$. Notice that if $c > a$, then $\text{Norm}(P) = \Big\{ \pm\Big(\frac{c}{2c-a}, \frac{2(c-a)}{2c-a}\Big)\Big\}$.

Let $P(x,y) = by^2 + cxy$ for $bc \neq 0, b > 0$. Notice that $P(x,y) = by^2 + 2(1-b)xy$ for $0 < b < 1$. Hence, $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, 1)\Big\}$.

Let $P(x,y) = by^2 + cxy$ for $bc \neq 0, b < 0$. Notice that $P(x,y) = by^2 + 2(1+b)xy$ for $-1 < b < 0$. Thus, $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, -1)\Big\}$.

Case 2. $abc \neq 0$

We claim that $|b| < 1$. Indeed, note that if $b = 1$, then $P(x,y) = y^2$. Thus $a = c = 0$. This is a contradiction. Note also that if $b = -1$, then $c = 0$. This is a contradiction.

Subcase 1. $0 < c < a$ and $0 < a \leq 4b$

Suppose that $a = 1$. By Lemma 2.7, the following hold:
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, 1)\Big\}$;
If $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, then $\text{Norm}(P) = \{\pm(1,0)\}$;
If $a < 1$, then $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, 1)\Big\}$.

Subcase 2. $0 < c < a$ and $a > 4b$

Let $b > 0$.

Suppose that $a = 1$.
By Lemma 2.7, the following hold:
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, 1)\Big\}$;
If $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, then $\text{Norm}(P) = \{\pm(1,0)\}$;
If $a < 1$, then $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, 1)\Big\}$.

Let $b < 0$.

Suppose that $a = 1$. By Lemma 2.2(1),

$$\Big(\Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c < 1, \frac{|c^2 - 4ab|}{4a} < 1\Big), \ \Big(\Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c = 1, \frac{|c^2 - 4ab|}{4a} < 1\Big)$$
$$\text{or } \Big(\Big|\frac{1}{4}a + b\Big| + \frac{1}{2}c < 1, \frac{|c^2 - 4ab|}{4a} = 1\Big).$$

By Lemma 2.7, the following hold:
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, 1)\Big\}$;
If $\frac{|c^2-4ab|}{4a} = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm\Big(\frac{c}{2a}, \ -1\Big)\Big\}$.

Subcase 3. $c \geq a > 0$ and $0 < a \leq 4b$

Note that $c < 2$.
By Lemma 2.3, $\frac{|c^2-4ab|}{2c+a+4b} < |\frac{1}{4}a + b| + \frac{1}{2}c \leq 1$. By Lemma 2.7, the following hold:
If $a = 1, b < 1$ and $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, 1)\Big\}$;
If $a = 1, b < 1$ and $|\frac{1}{4}a + b| + \frac{1}{2}c < 1$, then $\text{Norm}(P) = \{\pm(1,0)\}$;
If $a < 1, b < 1$ and $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, 1)\Big\}$.

Subcase 4. $c \geq a$ and $a > 4b$

Suppose that $a = 1$. The following hold:
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm(\frac{1}{2}, 1)\Big\}$;
If $\frac{c^2-4ab}{2c-a-4b} = 1$, then $\text{Norm}(P) = \Big\{ \pm(1,0), \pm\Big(\frac{c-4b}{2c-a-4b}, \frac{2(c-a)}{2c-a-4b}\Big)\Big\}$;
If $\frac{c^2-4ab}{2c-a-4b} < 1, |\frac{1}{4}a + b| + \frac{1}{2}c < 1$, then $\text{Norm}(P) = \{\pm(1,0)\}$.

Suppose that $0 < a < 1$.
By Lemma 2.7, the following hold:
If $|\frac{1}{4}a + b| + \frac{1}{2}c = 1$, then $\text{Norm}(P) = \Big\{ \pm(\frac{1}{2}, 1)\Big\}$;
If $\frac{c^2-4ab}{2c-a-4b} = 1$, then $\text{Norm}(P) = \Big\{ \pm\Big(\frac{c-4b}{2c-a-4b}, \frac{2(c-a)}{2c-a-4b}\Big)\Big\}$.
This completes the proof. □

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