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On Generalized Conharmonically Recurrent Manifolds

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Abstract In this paper, we have studied in detail about Generalized Conharmonically Recurrent Manifolds in which we have shown that in a generalized Ricci recurrent manifold of a non-zero constant scalar curvature r, the 1-form α is closed iff the 1-form β is closed. We have also shown that in a generalized Ricci recurrent manifold of a non constant scalar curvature r, the 1-form α and β both are closed iff the 1-form α is collinear with the 1-form β. Further we have shown that if the scalar curvature tensor r is constant in a generalised conharmonically recurrent manifold that satisfies the condition of Einstein manifold, then the 1-forms α and β are proportional to each other. Some other interesting and fruitful results on generalized conharmonically recurrent manifolds are also obtained.

1 Introduction

The generalized recurrent manifold was introduced and studied by De and Guha[10] and then studied by so many authors such as Singh and Khan[4], De, Guha and Kamilya[11] and De and Kamilya[9]. In Reimannian manifold (M^n,g) of dimension n, the set of differentiable vector fileds is denoted by χ(M). The curvature tensor K(X,Y,Z) of type (1,3) for non-flat Reimannian manifold (M^n,g) satisfied the condition ([10], [4]) :

$$(\nabla_U K)(X, Y, Z) = \alpha(U)K(X, Y, Z) + \beta(U)[g(Y, Z)X - g(X, Z)Y] \tag{1.1}$$

where α and β are 1-forms. β is non-zero and ∇ refer as the operator of covariant differentiation with respect to the metric g. Such a manifold has been called a generalized recurrent manifold and is represented with $(GK)_n$. Here β is called the associated 1-form of $(GK)_n$. The $(GK)_n$ reduces to a recurrent manifold, in case the 1-form $\beta(U)$ becomes zero in the above expression, by Ruse [5], walker[2] and khan[8].

The 1-forms α and β are expressed as

$$g(X, P) = \alpha(X) \tag{1.2}$$

and

$$g(X, Q) = \beta(X) \tag{1.3}$$

In the above equations (1.2) and (1.3), P and Q are shown as vector fields.

A non-flat Riemannian manifold (M^n, g), ($n \geq 2$) is called a generalized Recci-recurrent manifold [9] in case the Ricci tensor S of type (0,2) satisfies the condition.

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(X)g(Y, Z) \tag{1.4}$$

where α and β are two non-zero 1-forms, which are already defined by (1.2) and (1.3). Such a manifold has been denoted by $(GR)_n$. If the 1-form $\beta(X)$ becomes zero in (1.4), then the generalised Ricci-recurrent manifold reduces to a Ricci-recurrent manifold ([1] & [8]).

In this paper, we have considered a non-flat n-dimensional Riemannian manifold in which the conharmonic curvature tensor N satisfies the condition

$$(\nabla_U N)(X,Y,Z) = \alpha(U)N(X,Y,Z) + \beta(U)[g(Y,Z)X - g(X,Z)Y] \tag{1.5}$$

where α and β are two non-zero 1-forms and the conharmonic curvature N is defined by (see [6] & [7])

$$\begin{aligned} N(X,Y,Z) =& K(X,Y,Z) + \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)R(X) - g(X,Z)R(Y)] \end{aligned} \tag{1.6}$$

where R is the Ricci tensor of type (1,1), defined by

$$S(X,Y) = g(R(X),Y). \tag{1.7}$$

Such an n-dimensional Riemannian manifold shall be called a generalized Conharmonically recurrent Riemannian manifold and is denoted by $(GC)_n$.

If a Riemannian manifold is an Einstein manifold (See [7] and [3]), then

$$S(X,Y) = \lambda g(X,Y) \tag{1.8}$$

where λ is constant. From (1.8), we have

$$R(X) = \lambda X. \tag{1.9}$$

Contarcating (1.9) with respect to X, we get

$$r = n\lambda \tag{1.10}$$

where *r* is a scalar curvature.

The above results will be used in next section. In section 2, we have discussed the nature of the 1-forms α and β and in section 3, we have studied about generalized conharmonically recurrent manifold.

2 Nature of the 1-forms α and β on generalized Ricci-recurrent manifold

Taking covariant derivative of (1.7) with respect to U, we have

$$g((\nabla_U R)(X),Y) = (\nabla_U S)(X,Y). \tag{2.1}$$

Using (1.4) in (2.1), we have

$$g((\nabla_U R)(X),Y) = \alpha(U)S(X,Y) + \beta(U)g(X,Y).$$

which yields

$$(\nabla_U R)(X) = \alpha(U)R(X) + \beta(U)X. \tag{2.2}$$

Contacting (2.2) with respect to 'X', we have

$$Ur = \alpha(U)r + n\beta(U). \tag{2.3}$$

where r is the scalar curvature which may be or may not be constant.

If r is a constant and is different from zero. Then from (2.3), we have

$$\alpha(U)r + n\beta(U) = 0. \tag{2.4}$$

Taking covariant derivative of (2.4) with respect to 'V', we have

$$(\nabla_V \alpha)(U)r + n(\nabla_V \beta)(U) = 0. \tag{2.5}$$

Interchanging U and V in (2.5), and then subtracting them, we have

$$[(\nabla_V \alpha)(U) - (\nabla_U \alpha)(V)]r + n[(\nabla_V \beta)(U) - (\nabla_U \beta)(V)] = 0.$$

Thus, we have the following result:

Theorem 2.1. *In a $(GR)_n$ of a non-zero constant scalar curvature r, the 1-form α is closed iff the 1-form β is closed.*

Next we consider the case when the scalar curvature r is not constant, From (2.3), it follows that

$$VUr = (\nabla_V \alpha)(U)r + \alpha(U)(Vr) + n(\nabla_V \beta)(U) = 0. \tag{2.6}$$

Interchanging U and V in (2.6), and then subtracting them, we have

$$\begin{aligned}[(\nabla_V \alpha)(U) - (\nabla_U \alpha)(V)]r + \alpha(U)(Vr) - \alpha(V)(Ur) \\ +n[(\nabla_V \beta)(U) - (\nabla_U \beta)(V)] = 0.\end{aligned} \tag{2.7}$$

Using (2.3) in (2.7), we have

$$\begin{aligned}[(\nabla_V \alpha)(U) - (\nabla_U \alpha)(V)]r + n[\alpha(U)\beta(V) - \alpha(V)\beta(U)] \\ +n[(\nabla_V \beta)(U) - (\nabla_U \beta)(V) = 0]\end{aligned}$$

Thus, we have the following result:

Theorem 2.2. *In a $(GR)_n$ of a non constant scalar curvature r, the 1-form α and β both are closed iff the 1-form α is collinear with the 1-form β.*

3 Generalized Conharmonically recurrent Riemannian manifolds

Taking covariant derivative of (1.6) with respect to 'U', we have

$$\begin{aligned}(\nabla_U N)(X,Y,Z) = (\nabla_U K)(X,Y,Z) - \frac{1}{n-1}[(\nabla_U S)(Y,Z)X - \\ (\nabla_U S)(X,Z)Y + g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y)]\end{aligned} \tag{3.1}$$

which in view of (1.5) gives

$$\begin{aligned}&\alpha(U)N(X,Y,Z) + \beta(U)[g(Y,Z)X - g(X,Z)Y] \\ &= (\nabla_U K)(X,Y,Z) - \frac{1}{n-2}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y + \\ &g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y)].\end{aligned} \tag{3.2}$$

Using (1.6) in (3.2) and rearranging the terms, we have

$$\begin{aligned}&(\nabla_U K)(X,Y,Z) - (\alpha(U)K)(X,Y,Z) - \beta(U)[g(Y,Z)X - g(X,Z)Y] \\ &= \frac{1}{n-2}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y + \\ &g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y) - \\ &\alpha(U)\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)\}].\end{aligned} \tag{3.3}$$

Permutting equation (3.3) twice with respect to U, X, Y; adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}
&\alpha(U)K(X,Y,Z)+\alpha(X)K(Y,U,Z)+\alpha(Y)K(U,X,Z)\\
&\quad+\beta(U)[g(Y,Z)X-g(X,Z)Y]+\beta(X)[g(U,Z)Y-g(Y,Z)U]\\
&\quad+\beta(Y)[g(X,Z)U-g(U,Z)X]\\
&\quad+\frac{1}{n-2}[(\nabla_U S)(Y,Z)X-(\nabla_U S)(X,Z)Y\\
&\quad+g(Y,Z)(\nabla_U R)(X)-g(X,Z)(\nabla_U R)(Y)\\
&\quad+(\nabla_X S)(U,Z)Y-(\nabla_X S)(Y,Z)U+g(U,Z)(\nabla_X R)(Y)-g(Y,Z)(\nabla_X R)(U)\\
&\quad+(\nabla_Y S)(X,Z)U-(\nabla_Y S)(U,Z)X+g(X,Z)(\nabla_Y R)(U)-g(U,Z)(\nabla_Y R)(X)\\
&\quad-\alpha(U)\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)R(X)-g(X,Z)R(Y)\}\\
&\quad-\alpha(X)\{S(U,Z)Y-S(Y,Z)U+g(U,Z)R(Y)-g(Y,Z)R(U)\}\\
&\quad-\alpha(Y)\{S(X,Z)U-S(U,Z)X+g(X,Z)R(U)-g(U,Z)R(X)\}]=0.
\end{aligned}\tag{3.4}$$

Contracting (3.4) with respect to 'X', we have

$$\begin{aligned}
&\alpha(U)S(Y,Z)+\alpha(K(Y,U,Z))-\alpha(Y)S(U,Z)\\
&+(n-1)\beta(U)g(Y,Z)+\beta(Y)[g(U,Z)-\beta(U)g(Y,Z)+(1-n)\beta(Y)g(U,Z)\\
&+\frac{1}{n-2}[(n-1)(\nabla_U S)(Y,Z)+g(Y,Z)(Ur)-g((\nabla_U R)(Y),Z)+\\
&(\nabla_Y S)(U,Z)-(\nabla_U S)(Y,Z)+\frac{1}{2}g(U,Z)(Yr)-\frac{1}{2}g(Y,Z)(Ur)\\
&+(1-n)(\nabla_Y S)(U,Z)-g((\nabla_Y R)(U),Z)-g(U,Z)(Yr)\\
&-(n-1)\alpha(U)S(Y,Z)-\alpha(U)g(Y,Z)r+\alpha(U)g(R(Y),Z)\\
&-\alpha(Y)S(U,Z)+\alpha(U)S(Y,Z)-\alpha(R(Y)g(U,Z)+\alpha(R(U))g(Y,Z)\\
&+(n-1)\alpha(Y)S(U,Z)-\alpha(Y)g(R(U),Z)+\alpha(Y)g(U,Z)r]=0
\end{aligned}\tag{3.5}$$

which in view of (1.3), (1.7), $g(K(X,Y,Z),U)=\acute{K}(X,Y,Z,U)$ and $\acute{K}(X,Y,Z,U)$ $=-\acute{K}(X,Y,U,Z)$ gives

$$\begin{aligned}
&\alpha(U)g(R(Y),Z)-g(K(Y,U,P),Z)-\alpha(Y)g(R(U),Z)\\
&\quad+(n-1)\beta(U)g(Y,Z)+\beta(Y)g(U,Z)-\beta(U)g(Y,Z)+(1-n)\beta(Y)g(U,Z)\\
&\quad+\frac{1}{n-2}[(n-1)g((\nabla_U R)(Y),Z)+g(Y,Z)(Ur)-g((\nabla_U R)(Y,Z)+\\
&\quad+g((\nabla_Y R)(Y),Z)-g((\nabla_U R)(Y),Z)+\frac{1}{2}g(U,Z)(Yr)-\frac{1}{2}g(Y,Z)(Ur)\\
&\quad+(1-n)g((\nabla_Y R)(U),Z)+g((\nabla_Y,R)(U),Z)-g(U,Z)(Yr)\\
&\quad-(n-1)\alpha(U)g(R(Y),Z)-\alpha(U)g(Y,Z)r+\alpha(U)g(R(Y),Z)\\
&\quad-\alpha(Y)g(R(U),Z)+\alpha(U)g(R(Y),Z)-g(R(Y),P)g(U,Z)\\
&\quad-g(R(U),P)g(Y,Z)+(n-1)\alpha(Y)g(R(U),Z)\\
&\quad-\alpha(Y)g(R(U),Z)+\alpha(Y)g(U,Z)r]
\end{aligned}\tag{3.6}$$

which yields

$$\begin{aligned}
&\alpha(U)R(Y) - (K(Y,U,P) - \alpha(Y)R(U) \\
&\quad + (n-2)[\beta(U)Y - \beta(Y)U] \\
&\quad + \frac{1}{n-2}[(n-1)(\nabla_U R)(Y) + Y(Ur) - (\nabla_U R)(Y) + \\
&\quad + (\nabla_Y R)(U) - (\nabla_U R)(Y) + \frac{1}{2}U(Yr) - \frac{1}{2}Y(Ur) \\
&\quad + (1-n)(\nabla_Y R)(U) + (\nabla_Y, R)(U) - U(Yr) - (n-1)\alpha(U)R(Y) \\
&\quad - \alpha(U)(Y)r + \alpha(U)R(Y) - \alpha(Y)R(U) + \alpha(U)R(Y) \\
&\quad - g(R(Y),P)U - g(R(U),P)Y + (n-1)\alpha(Y)R(U) \\
&\quad - \alpha(Y)R(U) + \alpha(Y)U(r)] = 0
\end{aligned}$$

or

$$\begin{aligned}
&(K(Y,U,P) - (n-2)[\beta(U)Y - \beta(Y)U] \\
&= \frac{1}{n-2}[(n-3)(\nabla_U R)(Y) - (n-3)(\nabla_Y R)(U) \\
&+ \alpha(U)R(Y) - \alpha(Y)R(U) + g(R(U),P)Y - g(R(Y),P)U \\
&+ \frac{1}{2}Y(Ur) - \frac{1}{2}U(Yr) - \alpha(U)Y(r) + \alpha(Y)U(r)].
\end{aligned} \tag{3.7}$$

Contracting (3.7) w.r.t. 'Y', we have

$$\begin{aligned}
S(U,P) - (n-1)(n-2)\beta(U) = \frac{1}{(n-2)}&[(n-3)(Ur) - \frac{1}{2}(n-3)(Ur) \\
&+\alpha(U)r - \alpha(R(U)) + (n-1)\alpha(R(U)) - (n-1)\alpha(U)r].
\end{aligned}$$

or

$$-(n-1)(n-2)\beta(U) = Ur - \alpha(U)r.$$

Thus, we have the follwoing results:

Theorem 3.1. *The necessary and sufficient conditions that the scalar curvature r of a generalized conharmonically recurrent Riemannian manifolds be constant is that*

$$\alpha(U)r - (n-1)(n-2)\beta(U) = 0.$$

Let the Riemannian manifold be an Einstein manifold. Then in view of (1.8) and (1.9), the relation (1.6) becomes

$$N(X,Y,Z) = K(X,Y,Z) - \frac{2\lambda}{n-2}[g(Y,Z)X - g(X,Z)Y] \tag{3.8}$$

which on taking covariant derivative with respect to 'U' gives

$$(\nabla_U N)(X.Y,Z) = (\nabla_U K)(X,Y,Z).$$

Using (1.5) in above relation, we have

$$\alpha(U)N(X,Y,Z) + \beta(U)[g(Y,Z)X - g(X,Z)Y] = (\nabla_U K)(X,Y,Z). \tag{3.9}$$

Using (3.8) in (3.9), we have

$$\begin{aligned}
\alpha(U)[K(X,Y,Z) - \frac{2\lambda}{n-2}&\{g(Y,Z)X - g(X,Z)Y\}] \\
&+ \beta(U)[g(Y,Z)X - g(X,Z)Y] - (\nabla_U K)(X,Y,Z) = 0.
\end{aligned} \tag{3.10}$$

Contracting (3.10) with respect to 'X', we have

$$\begin{aligned}\alpha(U)[S(Y,Z) - \frac{2(n-1)\lambda}{n-2}g(Y,Z)] + (n-1)\beta(U)g(Y,Z) \\ - (\nabla_U S)(Y,Z) = 0.\end{aligned} \tag{3.11}$$

In view of (1.7) and (2.1), the relation (3.11) gives

$$\begin{aligned}\alpha(U)[g(R(Y),Z) - \frac{2(n-1)\lambda}{n-2}g(Y,Z)] + (n-1)\beta(U)g(Y,Z) \\ - g((\nabla_U R)(Y),Z) = 0.\end{aligned} \tag{3.12}$$

Factoring off Z in (3.12), we have

$$\begin{aligned}\alpha(U)[R(Y) - \frac{2(n-1)\lambda}{n-2}Y] + (n-1)\beta(U)Y \\ - (\nabla_U R)(Y) = 0.\end{aligned} \tag{3.13}$$

Contracting (3.13) with respect to 'Y', we have

$$\alpha(U)[r - \frac{2n(n-1)\lambda}{n-2}] + n(n-1)\beta(U) - Ur = 0. \tag{3.14}$$

If the scalar curvature tensor r is constant, then Ur = 0 and in this case α(U) is propoetional to β(U).

Hence we have the following result:

Theorem 3.2. *If the scalar curvature tensor r is constant in a generalised conharmonically recurrent manifold which satisfying the condition of Einstein manifold, then the 1-forms α and β are proportional to each other.*

Let (M^n, g) be conharmonically flat, then (1.6) reduces to

$$\begin{aligned}K(X,Y,Z) = \frac{1}{n-2}[S(Y,Z)X - \\ S(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)].\end{aligned} \tag{3.15}$$

Taking covariant derivative of (3.15) with respect to 'U', we have

$$\begin{aligned}(\nabla_U K)(X,Y,Z) = \frac{1}{n-2}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\ + g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y)].\end{aligned} \tag{3.16}$$

Using (1.1) in (3.16), we have

$$\begin{aligned}\alpha(U)K(X,Y,Z) + \beta(U)[g(Y,Z)X - g(X,Z)Y] = \\ \frac{1}{n-2}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\ + g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y)].\end{aligned} \tag{3.17}$$

Contracting w.r.t. 'X' in (3.17), we have

$$\begin{aligned}\alpha(U)S(Y,Z) + (n-1)\beta(U)g(Y,Z) \\ = \frac{1}{n-2}[(n-1)(\nabla_U S)(Y,Z) - g(Y,Z)(Ur) - g((\nabla_U R)(Y),Z)].\end{aligned} \tag{3.18}$$

Factoring off Z in (3.18), taking in mind $S(Y,Z) = g(R(Y), Z$ and ($\nabla_U S$)(Y,Z) = g(($\nabla_U R$)(Y),Z), we have

$$\alpha(U)R(Y) + (n-1)\beta(U)Y = (\nabla_U R)(Y) + Y(Ur)$$

which on contracting with respect to 'Y' gives

$$\alpha(U)r + n(n-1)\beta(U) = (n+1)(Ur).$$

Thus we have the following result:

Theorem 3.3. *The necessary and sufficient condition that the scalar curvature tensor r of a conharmonically flat generalized recurrent manifold be constant is that $\alpha(U)r$ + n(n-1)$\beta(U)$ = 0.*

References

[1] A.A. Shaikh, H. Kundu, M. Ali and Z. Ahsan; Curvature properties of a special type of pure radiation metrics, Journal of Geometry and Physics, 2019.

[2] A.G. Walker; On Ruse's space of recurrent curvature, Proc. London Math. Soc. 52, 36-64, 1951.

[3] A.U. Khan and Q. Khan; On Special Weakly Projective Symmetric manifolds, Ganita Vol. 71(2), 73-79, 2021.

[4] H. Singh and Q. Khan; On generalized recurrent manifolds, Publ. Math. Debrecen, 56/1-2, 87-95, 2000.

[5] H.S. Ruse; On simply harmonic space, J. London Math Soc. 21, 1946.

[6] M. Ali, M. Salman and M. Bilal; Conharmonic Curvature Inheritance in Spacetime of General Relativity, Universe, 2021.

[7] Q. Khan; Differential Geometry of Manifolds, PHI Learning Private Limited, Delhi -110019, 2012.

[8] Q. Khan; On recurrent Riemannian manifolds, Kyungpook Math. J. 44, 269-276, 2004.

[9] U.C. De and D. Kamilya; On generalised Conharmonically recurrent Riemannian manifolds, Indian Journal of Mathematics Vol. 36, No.1, 49-54, April 1994.

[10] U.C. De and N. Guha; On generalized recurrent manifold, J. National Academy of Math. India, 9, 1991.

[11] U.C. De, N. Guha and D. Kamilya; On generalized Ricci recurrent manifold, Tensor (N.S.), 56/3, 312-317, 1995.

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