General Dissection Formula for Ramanujan Quantity
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Communicated by V. Lokesha

MSC 2020 Classifications: Primary 33D15, 40A15, 11A55, 30B70.

Keywords and phrases: Continued fraction, Quantities, Theta functions.

Abstract. In this paper, we study the several General Dissection Formula of degree 4, 8, 12 and 16 for Ramanujan Quantity \( R(a, b; p, q) \) when \( b \) is a positive even integer. Also, we obtain some applications to theory of partitions of general dissection that we obtain in this paper itself.

1 Introduction

In Chapter 16 of his second notebook [2, 8], Ramanujan develops the theory of theta-function and is defined by

\[
f(a, b) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty}, \quad (1.1)
\]

where \((a; q)_0 = 1\) and \((a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2)\ldots\).

Also, we define

\[
(q^{r\pm}; q^s)_{\infty} := (q^r, q^{s-r}; q^s)_{\infty},
\]

where \(r\) and \(s\) are positive integers with \(r < s\). We use this particular notation in the last section of this paper.

Following Ramanujan, we defined

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q^2; -q^2)_\infty}{(q^2; -q^2)_\infty},
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n^2} = \frac{(q^2; q^2)_\infty}{(q^3; q^3)_\infty},
\]

\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(-q^2; -q^2)_\infty}{(q^2; -q^2)_\infty},
\]

and

\[
\chi(q) := \frac{(-q; -q^2)_\infty}{(q^2; -q^2)_\infty}.
\]

In [6],[7] Nikos Bagis define Ramanujan Quantities \( R(a, b; p, q) \) as

\[
R(a, b; p, q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \prod_{n=0}^{\infty} (1 - q^{np})(1 - q^{p-np}) \prod_{n=0}^{\infty} (1 - q^{pnp})(1 - q^{np^2}),
\]

where \(a, b,\) and \(p\) are positive rationales such that \(a + b < p\). General Theorem such

\[
\frac{q^{B-A} \left( a-bq_1 \right)(b_1-a_1q_1) \left( a_1-b_1q_1^2 \right)(b_1-a_1q_1^2) \ldots}{1-a_1b_1 (1-a_1b_1)(q_1^2 + 1)}
\]
\[
\prod_{n=0}^{\infty} \frac{(1 - q^n q^{2n}) (1 - q^{2n+1})}{(1 - q^{2n+1}) (1 - q^{2n+2})} = \prod_{n=0}^{\infty} \frac{(1 - q^n a q^{2n}) (1 - q^{2n+1})}{(1 - q^{2n+1}) (1 - q^{2n+2})}
\]

(1.7)

where \( a_1 = q^A, b_1 = q^B, q_1 = q^{A+B}, a = 2A + 3p/4, 2B + p/4, \) and \( p = 4(A + B), |q| < 1, \)
are proved.

In this paper, we study the several General Dissection Formula for Ramanujan Quantity \( R(a, b, p; q) \).

The 2-dissections, 4-dissections and 8-dissections are studied by several mathematicians including
S. Ramanujan. For details one may refer [9, p. 50], [1] and [4].

In the last section, we extract partition theoretic interpretations of some identities that we
obtain in the other sections.

Definition. “A positive integer \( n \) has \( k \) colors if there are \( k \) copies of \( n \) available and all of
them are viewed as distinct objects. Partitions of positive integer into parts with colors are called
colored partitions.”

For example, if 1 is allowed to have two colors, say \( r \) (red) and \( g \) (green), then all the colored
partitions of 2 are 2, 1\(_r\) + 1\(_r\), 1\(_r\) + 1\(_g\) and 1\(_g\) + 1\(_g\). Note that

\[
\frac{1}{(q^u, q^v)^k}\]

is the generating function for the number of partitions of \( n \) where all the parts are congruent to
\( u \) (mod \( v \)) and have \( k \) colors. Also rewriting the same modular relation in different ways results
in different partition theoretic interpretations.

2 Preliminary results

[7] [a, p; q] = (q^{p-a}; q^p)_{\infty}(q^a; q^p)_{\infty}

(2.1)

where \( q = e^{-\pi \sqrt{r}} \) and \( a, p, r > 0. \)

[7] \( R(a, b, p; q) := q^{-(a-b)/2 + (a^2-b^2)/(2p)} [a, p; q] [b, p; q]. \)

(2.2)

Lemma 2.1. [2, Entry 29(i), (ii), p. 45] If \( p = ab = cd, \) then

\[
f(a, b) f(c, d) + f(-a, -b) f(-c, -d) = 2f(ac, bd) f(ad, bc),
\]

(2.3)

\[
f(a, b) f(c, d) - f(-a, -b) f(-c, -d) = 2af\left(\frac{b}{c}, \frac{c}{b}, abcd\right) f\left(\frac{b}{d}, \frac{d}{b}, abcd\right),
\]

(2.4)

Lemma 2.2. [2, Entry 30(ii), (iii), (v), (vi), p. 46]

\[
f(a, b) + f(-a, -b) = 2f(a^2 b, ab^3),
\]

(2.5)

\[
f(a, b) - f(-a, -b) = 2af\left(\frac{b}{a}, \frac{a}{b}, a^4 b^4\right),
\]

(2.6)

\[
f^2(a, b) + f^2(-a, -b) = 2f(a^2, b^2) \varphi(ab),
\]

(2.7)

\[
f^2(a, b) - f^2(-a, -b) = 4af\left(\frac{b}{a}, \frac{a}{b}, a^2 b^2\right) \psi(a^2 b^2).
\]

(2.8)
Lemma 2.3. [2, Entry 25(iii), p.40] We have,
\[ \varphi^2(-q^2) = \varphi(q)\varphi(-q), \quad \varphi(-q^2)\psi(q^2) = \psi(q)\psi(-q). \quad (2.9) \]

Lemma 2.4. [2, Entry 25(iv), p.40] We have,
\[ \psi^2(q^2) = \psi(q^2)\varphi(q), \quad (2.10) \]

Lemma 2.5. [5]
\[ f(-q, -q^{11}) = \frac{\varphi(q) - \varphi(q^2)}{2q\chi(q)} \quad (2.11) \]

3 Ramanujan Quantities of \( R(a, b, p; q) \)

In this section, we obtain certain Ramanujan Quantities of \( R(a, b, p; q) \) in terms of Ramanujan’s theta–function.

Theorem 3.1. We have
\[ R(q) := R(1, k, 2k; q) = q^{(k-1)^2/4k} \frac{f(-q, -q^{2k-1})}{f(-q^k, -q^k)}. \quad (3.1) \]

Proof. Employing equations (2) and (2) with \( a = 1, b = k, \) a positive even integer and \( p = 2k, \) we get
\[ R(1, k, 2k; q) = q^{-(1-k)/2+(1-k^2)/(4k)} \frac{(q; q^{2k})_\infty(q^{2k-1}; q^{2k})_\infty}{(q^{2k-1}; q^{2k})_\infty(q^k; q^{2k})_\infty}. \quad (3.2) \]

Using Ramanujan’s Theat–function definition in above equation (3.2), we obtain (3.1). \( \Box \)

4 General \( n \)-Dissections of \( R^*(q) \)

4.1 The 2-dissection of \( R^*(q) \)

Theorem 4.1. If \( R^*(q) := \frac{f(-q, -q^{2k-1})}{f(-q^k, -q^k)} = \sum_{n=0}^{\infty} a_n q^n, \) then
\[ \sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^{k+1}, q^{3k-1})}{\varphi(-q^{k/2})}, \quad (4.1) \]
\[ \sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f(q^{k-1}, q^{3k+1})}{\varphi(-q^{k/2})}. \quad (4.2) \]

Proof. We have
\[ \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q, -q^{2k-1})}{f(-q^k, -q^k)}, \]
\[ = \frac{f(-q, -q^{2k-1})f(q^k, q^k)}{f(-q^k, -q^k)f(q^k, q^k)}. \quad (4.3) \]

Putting \( a = -q, b = -q^{2k-1} \) and \( c = d = q^k \) in Lemma 2.1, we obtain
\[ f(-q, -q^{2k-1})f(q^k, r^k) = f^2(-q^{k+1}, -q^{3k-1}) - qf^2(-q^{k-1}, -q^{3k+1}). \quad (4.4) \]

Putting \( a = q^k \) and \( b = q^k \) in Lemma 2.2, we get
\[ f(q^k, q^k)f(-q^k, -q^k) = f^2(-q^{2k}, -q^{2k}). \quad (4.5) \]
Employing (4.4) and (4.5) in (4.3), we deduce that
\[ R^*(q) = \sum_{n=0}^{\infty} a_n q^n = \frac{f^2(-q^{k+1}, -q^{3k-1}) - qf^2(-q^{k-1}, -q^{3k+1})}{\varphi^2(-q^{2k})}. \] (4.6)

Hence
\[ R^*(q) + R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n} q^{2n} = \frac{f^2(-q^{k+1}, -q^{3k-1}) + f^2(q^{k+1}, q^{3k-1}) - q [f^2(-q^{k-1}, -q^{3k+1}) - f^2(q^{k-1}, q^{3k+1})]}{\varphi^2(-q^{2k})}. \] (4.7)

Putting \( a = q^{k+1} \) and \( b = q^{3k-1} \) in (2.7), \( a = -q^{k-1} \) and \( b = -q^{3k+1} \) in (2.8), we get
\[ f^2(q^{k+1}, q^{3k+1}) + f^2(-q^{k+1}, -q^{3k-1}) = 2f(q^{2(k+1)}, q^{2(3k-1)}) \varphi(q^{4k}), \] (4.8)
\[ f^2(-q^{k-1}, -q^{3k+1}) - f^2(q^{k-1}, q^{3k+1}) = -4q^{k-1} f(q^{2(k+1)}, q^{2(3k-1)}) \psi(q^{8k}). \] (4.9)

Employing the above equations (4.8) and (4.9) in (4.7), we deduce that
\[ R^*(q) + R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n} q^{2n} = 2f(q^{2(k+1)}, q^{2(3k-1)}) \left[ \varphi(q^{4k}) + 2q^k \psi(q^{8k}) \right]. \] (4.10)

Putting \( a = q^k \) in Lemma 2.3, we get
\[ \varphi(q^k) = \varphi(q^{4k}) + 2q^k \psi(q^{8k}). \] (4.11)

Substituting equation (4.11) in the equation (4.10), we obtain
\[ R^*(q) + R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n} q^{2n} = 2f(q^{2(k+1)}, q^{2(3k-1)}) \varphi(q^k). \] (4.12)

Using (2.9) in the above equation, we get
\[ 2 \sum_{n=0}^{\infty} a_{2n} q^{2n} = 2f(q^{2(k+1)}, q^{2(3k-1)}) \varphi(q^k). \] (4.13)

Replace \( q \) by \( q^{1/2} \) in (4.13), we obtain (4.1).

Similarly
\[ R^*(q) - R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} \]
\[ = \frac{f^2(-q^{k+1}, -q^{3k-1}) - f^2(q^{k+1}, q^{3k-1}) - q [f^2(-q^{k-1}, -q^{3k+1}) + f^2(q^{k-1}, q^{3k+1})]}{\varphi^2(-q^{2k})}. \] (4.14)

Putting \( a = -q^{k+1} \) and \( b = -q^{3k-1} \) in (2.8), \( a = q^{k-1} \) and \( b = q^{3k+1} \) in (2.7), we get
\[ f^2(-q^{k+1}, -q^{3k-1}) - f^2(q^{k+1}, q^{3k-1}) = -4q^{k+1} f(q^{2(k+1)}, q^{2(3k+1)}) \psi(q^{8k}), \] (4.15)
\[ f^2(-q^{k-1}, -q^{3k+1}) + f^2(q^{k-1}, q^{3k+1}) = 2f(q^{2(k-1)}, q^{2(3k+1)}) \varphi(q^{4k}). \] (4.16)
Employing the above equations (4.15) and (4.16) in (4.14), we deduce that

$$R^*(q) - R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} = -2q \left( q^{2(2k-1)} + q^{2(3k+1)} \right) \frac{\varphi(q^{4k}) + 2q^{k}\psi(q^{4k})}{\varphi^2(-q^{2k})}.$$  

(4.17)

Substituting equation (4.11) in the above equation, we get

$$R^*(q) - R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n} = -2q \left( q^{2(2k-1)} + q^{2(3k+1)} \right) \frac{\varphi(q^{k})}{\varphi^2(-q^{k})}.$$  

(4.18)

Using (2.9) in the above equation, we get

$$R^*(q) - R^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n} = -2q \left( q^{2(2k-1)} + q^{2(3k+1)} \right) \frac{\varphi(q^{k})}{\varphi(-q^{k})}.$$  

(4.19)

Replace $q$ by $q^{1/2}$ in (4.19), we obtain (4.2).

\[ \square \]

### 4.2 The 4-dissection of $R^*(q)$

**Theorem 4.2.** We have

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f(q^{3k+1}, q^{5k-1})}{\varphi(-q^{k/2})},$$  

(4.20)

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f(q^{3k-1}, q^{5k+1})}{\varphi(-q^{k/2})},$$  

(4.21)

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^{k/2} \frac{f(q^{k-1}, q^{5k+1})}{\varphi(-q^{k/2})},$$  

(4.22)

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -q^{(k-2)/2} \frac{f(q^{k+1}, q^{7k-1})}{\varphi(-q^{k/2})}.$$  

(4.23)

**Proof.** From (4.1), We have

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^{k+1}, q^{3k-1})}{\varphi(-q^{k/2})}.$$  

(4.24)

Putting $a = q^{k+1}$ and $b = q^{3k-1}$ in (2.5) and (2.6), we obtain

$$f(q^{k+1}, q^{3k-1}) = f(q^{2(3k+1)}, q^{2(5k-1)}) + q^{k+1} f(q^{2(k-1)}, q^{2(7k-1)}).$$  

(4.25)

Employing the above equation (4.25) in (4.24), we have

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^{2(3k+1)}, q^{2(5k-1)}) + q^{k+1} f(q^{2(k-1)}, q^{2(7k+1)})}{\varphi(-q^{k/2})}.$$  

(4.26)

This implies

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = \frac{f(q^{2(3k+1)}, q^{2(5k-1)})}{\varphi(-q^{k/2})},$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^{2n+1} = q^{k+1} \frac{f(q^{2(k-1)}, q^{2(7k+1)})}{\varphi(-q^{k/2})}.$$  

Changing $q$ by $q^{1/2}$ in the above equations, we obtain (4.20) and (4.22).

Proofs of (4.21) and (4.23) are similar.  

\[ \square \]
4.3 The 8-dissection of $R^*(q)$

Theorem 4.3. We have

\[ \sum_{n=0}^{\infty} a_{8n} q^n = \frac{f(q^{7k+1}, q^{9k-1})}{\varphi(-q^{k/8})}, \]  
(4.27)

\[ \sum_{n=0}^{\infty} a_{8n+1} q^n = \frac{f(q^{7k-1}, q^{9k+1})}{\varphi(-q^{k/8})}, \]  
(4.28)

\[ \sum_{n=0}^{\infty} a_{8n+2} q^n = q^{k/4} \frac{f(q^{3k-1}, q^{11k+1})}{\varphi(-q^{k/8})}, \]  
(4.29)

\[ \sum_{n=0}^{\infty} a_{8n+3} q^n = -q^{(k-2)/4} \frac{f(q^{5k+1}, q^{11k-1})}{\varphi(-q^{k/8})}, \]  
(4.30)

\[ \sum_{n=0}^{\infty} a_{8n+4} q^n = q^{3k/2} \frac{f(q^{-1}, q^{15k+1})}{\varphi(-q^{k/8})}, \]  
(4.31)

\[ \sum_{n=0}^{\infty} a_{8n+5} q^n = -q^{(3k-2)/2} \frac{f(q^{k+1}, q^{15k-1})}{\varphi(-q^{k/8})}, \]  
(4.32)

\[ \sum_{n=0}^{\infty} a_{8n+6} q^n = q^{(3k-4)/4} \frac{f(q^{3k+1}, q^{15k-1})}{\varphi(-q^{k/8})}, \]  
(4.33)

\[ \sum_{n=0}^{\infty} a_{8n+7} q^n = -q^{(3k-2)/4} \frac{f(q^{3k-1}, q^{13k+1})}{\varphi(-q^{k/8})}. \]  
(4.34)

Proof. From (4.20), we have

\[ \sum_{n=0}^{\infty} a_{4n} q^n = \frac{f(q^{3k+1}, q^{5k-1})}{\varphi(-q^{k/4})}. \]  
(4.35)

Putting $a = q^{3k+1}$ and $b = q^{5k-1}$ in (2.5) and (2.6), we obtain

\[ f(q^{3k+1}, q^{5k-1}) = f(q^{2(7k+1)}, q^{2(9k-1)}) + q^{3k+1} f(q^{2(k-1)}, q^{2(15k+1)}). \]  
(4.36)

Employing the above equation (4.36) in (4.35), we have

\[ \sum_{n=0}^{\infty} a_{8n} q^n = \frac{f(q^{2(7k+1)}, q^{2(9k-1)}) + q^{3k+1} f(q^{2(k-1)}, q^{2(15k+1)})}{\varphi(-q^{k/4})}. \]  
(4.37)

This implies

\[ \sum_{n=0}^{\infty} a_{8n} q^{2n} = \frac{f(q^{2(7k+1)}, q^{2(9k-1)})}{\varphi(-q^{k/4})}, \]

\[ \sum_{n=0}^{\infty} a_{8n+4} q^{2n+1} = q^{3k+1} \frac{f(q^{2(k-1)}, q^{2(15k+1)})}{\varphi(-q^{k/4})}. \]

Changing $q$ by $q^{1/2}$ in the above equations, we obtain (4.27) and (4.31). Proofs of (4.28), (4.29), (4.30), (4.32), (4.33) and (4.34) are similar. \hfill \square

5 Dissections of $R^*_4(q)$ of degree 4

In this section, we obtain certain Ramanujan Quantities of $R(1, 2, 4; q)$ in terms of Ramanujan’s theta–function.

Theorem 5.1. We have

\[ R_4(q) := R(1, 2, 4; q) = q^{1/8} \frac{f(-q, -q^3)}{f(-q^2, -q^2)}. \]  
(5.1)

Proof. Employing (3.1) with $k = 2$, we get (5.1). \hfill \square
5.1 The 2-dissection of $R_{8}^{*}(q)$ of degree 4

Theorem 5.2. If $R_{8}^{*}(q) := \frac{f(-q, -q^{3})}{f(-q^{2}, -q^{2})} = \sum_{n=0}^{\infty} a_{n}q^{n}$, then

\begin{align*}
\sum_{n=0}^{\infty} a_{2n}q^{n} & = \frac{f(q^{3}, q^{5})}{\varphi(-q)}. \quad (5.2) \\
\sum_{n=0}^{\infty} a_{2n+1}q^{n} & = -\frac{f(q, q^{7})}{\varphi(-q)}. \quad (5.3)
\end{align*}

Proof. Employing equations (4.20)–(4.22) with $k = 2$, we get (5.2) and (5.3). \hfill \Box

5.2 The 4-dissection of $R_{8}^{*}(q)$ of degree 4

Theorem 5.3. We have

\begin{align*}
\sum_{n=0}^{\infty} a_{4n}q^{n} & = \frac{f(q^{7}, q^{9})}{\varphi(-q^{1/2})}, \quad (5.4) \\
\sum_{n=0}^{\infty} a_{4n+1}q^{n} & = -\frac{f(q^{5}, q^{11})}{\varphi(-q^{1/2})}, \quad (5.5) \\
\sum_{n=0}^{\infty} a_{4n+2}q^{n} & = q^{1/2}\frac{f(q^{9}, q^{23})}{\varphi(-q^{1/4})}, \quad (5.6) \\
\sum_{n=0}^{\infty} a_{4n+3}q^{n} & = -\frac{f(q^{11}, q^{21})}{\varphi(-q^{1/4})}. \quad (5.7)
\end{align*}

Proof. Employing equations (4.20)–(4.22) with $k = 2$, we get (5.4) – (5.7). \hfill \Box

5.3 The 8-dissection of $R_{8}^{*}(q)$ of degree 4

Theorem 5.4. We have

\begin{align*}
\sum_{n=0}^{\infty} a_{8n}q^{n} & = \frac{f(q^{15}, q^{17})}{\varphi(-q^{1/4})}, \quad (5.8) \\
\sum_{n=0}^{\infty} a_{8n+1}q^{n} & = -\frac{f(q^{13}, q^{19})}{\varphi(-q^{1/4})}, \quad (5.9) \\
\sum_{n=0}^{\infty} a_{8n+2}q^{n} & = q^{1/2}\frac{f(q^{9}, q^{23})}{\varphi(-q^{1/4})}, \quad (5.10) \\
\sum_{n=0}^{\infty} a_{8n+3}q^{n} & = -\frac{f(q^{11}, q^{21})}{\varphi(-q^{1/4})}, \quad (5.11) \\
\sum_{n=0}^{\infty} a_{8n+4}q^{n} & = q^{3}\frac{f(q, q^{31})}{\varphi(-q^{1/4})}, \quad (5.12) \\
\sum_{n=0}^{\infty} a_{8n+5}q^{n} & = -q^{2}\frac{f(q^{7}, q^{29})}{\varphi(-q^{1/4})}, \quad (5.13) \\
\sum_{n=0}^{\infty} a_{8n+6}q^{n} & = q^{1/2}\frac{f(q^{7}, q^{25})}{\varphi(-q^{1/4})}, \quad (5.14) \\
\sum_{n=0}^{\infty} a_{8n+7}q^{n} & = -q\frac{f(q^{5}, q^{27})}{\varphi(-q^{1/4})}. \quad (5.15)
\end{align*}

Proof. Employing equations (4.27)–(4.34) with $k = 2$, we get (7.14) and (5.15). \hfill \Box
6 Dissections of \( R_8^*(q) \) of degree 8

In this section, we obtain certain Ramanujan Quantities of \( R(1, 4, 8; q) \) in terms of Ramanujan’s theta–function.

Theorem 6.1. If

\[
R(q) := R(1, 4, 8; q) = q^{9/16} f\left(\frac{-q, -q^7}{f(-q^4, -q^8)}\right)
\]  

(6.1)

then

\[
R(q) = \left\{ \frac{\varphi^{1/8}(q^2)}{2\varphi(q)} \left[ \frac{\varphi(q)}{\varphi(q^2)} - 1 \right] \right\}^{1/2}
\]  

(6.2)

Proof. Employing (7.1) with \( k = 4 \), we get

\[
R(1, 4, 8; q) = q^{9/16} \frac{(q; q^8)_\infty(q^7; q^8)_\infty}{(q^4; q^8)_\infty(q^8; q^8)_\infty} = q^{9/16} \frac{f(-q, -q^7)}{f(-q^4, -q^8)}
\]  

(6.3)

Employing the Lemma (2.1) with \( a = q, b = q^3 \) and \( c = d = q^2 \), we obtain

\[
f(q, q^3)f(q^2, q^7) - f(-q, -q^3)f(-q^2, -q^7) = 2qf(q, q^7)f(q, q^7)
\]  

(6.4)

using Theta-function definitions (1.1), (1.2) and (1.3) in (6.4), we obtain

\[
\psi(q)\varphi(q^2) - \psi(-q)\varphi(-q^2) = 2qf^2(q, q^7)
\]  

(6.5)

then above equation can be written with replace \( q \) by \(-q\) as

\[
\psi(q)\varphi(-q^2) - \psi(-q)\varphi(q^2) = 2qf^2(-q, -q^7)
\]  

(6.6)

Consider

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi(q)\varphi(-q^2) - \psi(-q)\varphi(q^2)}{2q\varphi(-q^4)} \right\}^{1/2}
\]  

(6.7)

Using (2.9) with replace \( q \) by \( q^2 \) then (6.7) can be written as

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi(q)\varphi(-q^2) - \psi(-q)\varphi(q^2)}{2q\varphi(-q^4)} \right\}^{1/2}
\]  

(6.8)

Employing the equation (2.9) in (6.9), we obtain

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi(q)}{2q\varphi(q^2)} - \frac{\psi(q^2)}{2q\varphi(q)} \right\}^{1/2}
\]  

(6.10)

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi^2(q) - \varphi(q^7)\psi(q^2)}{2q\psi(q)\varphi(q^2)} \right\}^{1/2}
\]  

(6.11)

Using the Lemma (2.4) in equation (6.12), we get

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi(q^2)\varphi(q) - \varphi(q^2)\psi(q)}{2q\psi(q)\varphi(q^2)} \right\}^{1/2}
\]  

(6.12)

\[
\frac{f(-q, -q^7)}{f(-q^4, -q^8)} = \left\{ \frac{\psi(q^2)\varphi(q) - \varphi(q^2)\psi(q)}{2q\psi(q)\varphi(q^2)} \right\}^{1/2}
\]  

(6.13)

Using the above identity (6.13) in (6.3) we obtain required result.
6.1 The 2-dissection of $R^*_2(q)$ of degree 8

**Theorem 6.2.** If $R(q) := \frac{f(-q, -q^7)}{f(-q^4, -q^4)} = \sum_{n=0}^{\infty} a_n q^n$, then

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^5, q^{11})}{\varphi(-q^2)}.$$  \hspace{1cm} (6.14)

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f(q^3, q^{13})}{\varphi(-q^2)}. \hspace{1cm} (6.15)$$

**Proof.** Employing equations (4.1) and (4.2) with $k = 4$, we get (6.14) and (6.15). \hfill \Box

6.2 The 4-dissection of $R^*_4(q)$ of degree 8

**Theorem 6.3.** We have

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f(q^{13}, q^{19})}{\varphi(-q)}.$$ \hspace{1cm} (6.16)

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f(q^{11}, q^{21})}{\varphi(-q)}.$$ \hspace{1cm} (6.17)

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^2 \frac{f(q^3, q^{29})}{\varphi(-q)}.$$ \hspace{1cm} (6.18)

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -q^2 \frac{f(q^5, q^{27})}{\varphi(-q)}.$$ \hspace{1cm} (6.19)

**Proof.** Employing equations (4.20)–(4.22) with $k = 4$, we get (6.16) – (6.19). \hfill \Box

6.3 The 8-dissection of $R^*_8(q)$ of degree 8

**Theorem 6.4.** We have

$$\sum_{n=0}^{\infty} a_{8n} q^n = \frac{f(q^{39}, q^{33})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.20)

$$\sum_{n=0}^{\infty} a_{8n+1} q^n = -\frac{f(q^{27}, q^{37})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.21)

$$\sum_{n=0}^{\infty} a_{8n+2} q^n = q \frac{f(q^{19}, q^{45})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.22)

$$\sum_{n=0}^{\infty} a_{8n+3} q^n = -q^{1/2} \frac{f(q^{21}, q^{43})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.23)

$$\sum_{n=0}^{\infty} a_{8n+4} q^n = q^6 \frac{f(q^3, q^{61})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.24)

$$\sum_{n=0}^{\infty} a_{8n+5} q^n = -q^5 \frac{f(q^5, q^{59})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.25)

$$\sum_{n=0}^{\infty} a_{8n+6} q^n = q^2 \frac{f(q^{13}, q^{51})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.26)

$$\sum_{n=0}^{\infty} a_{8n+7} q^n = -q^{5/2} \frac{f(q^{11}, q^{53})}{\varphi(-q^{1/2})}.$$ \hspace{1cm} (6.27)

**Proof.** Employing equations (4.27)–(4.34) with $k = 2$, we get (6.20) – (7.7). \hfill \Box
7 Dissections of $R_{12}^*(q)$ of degree 12

Theorem 7.1. If

$$R_{12}(q) := R(1, 6, 12; q) = q^{25/24} \frac{f(-q, -q^{11})}{f(-q^6, -q^6)},$$

then

$$R_{12}(q) = q^{1/24} \frac{\varphi(q) - \varphi(q^3)}{2\chi(q)\varphi(-q^6)}.$$  \hspace{1cm} (7.2)

Proof. Employing (3.1) with $k = 8$, we get

$$R(1, 6, 12; q) = q^{25/24} \frac{(q; q^{12})_{\infty}(q^{11}; q^{12})_{\infty}}{(q^6; q^{12})_{\infty}(q^6; q^{12})_{\infty}} = q^{25/24} \frac{f(-q, -q^{11})}{f(-q^6, -q^6)}.$$  \hspace{1cm} (7.3)

Employing equation (2.11) in the above equation (7.3), we obtain (7.2).

7.1 The 2-dissection of $R_{12}^*(q)$ of degree 12

Theorem 7.2. If $R_{12}^*(q) := \frac{f(-q, -q^{11})}{f(-q^6, -q^6)} = \sum_{n=0}^{\infty} a_n q^n$, then

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^7, q^{17})}{\varphi(-q^3)}.$$  \hspace{1cm} (7.4)

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f(q^5, q^{19})}{\varphi(-q^3)}.$$  \hspace{1cm} (7.5)

Proof. Employing equations (4.1) and (4.2) with $k = 4$, we get (7.4) and (7.5).

7.2 The 4-dissection of $R_{12}^*(q)$ of degree 12

Theorem 7.3. We have

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f(q^{19}, q^{29})}{\varphi(-q^{1/2})},$$  \hspace{1cm} (7.6)

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f(q^7, q^{41})}{\varphi(-q^{3/2})},$$  \hspace{1cm} (7.7)

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^3 \frac{f(q^5, q^{43})}{\varphi(-q^{3/2})},$$  \hspace{1cm} (7.8)

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -q^3 \frac{f(q^{17}, q^{31})}{\varphi(-q^{3/2})}.$$  \hspace{1cm} (7.9)

Proof. Employing equations (4.20)–(4.22) with $k = 4$, we get (7.6) – (7.9).

7.3 The 8-dissection of $R_{12}^*(q)$ of degree 12

Theorem 7.4. We have

$$\sum_{n=0}^{\infty} a_{8n} q^n = \frac{f(q^{43}, q^{53})}{\varphi(-q^{3/4})},$$  \hspace{1cm} (7.10)

$$\sum_{n=0}^{\infty} a_{8n+1} q^n = -\frac{f(q^{41}, q^{55})}{\varphi(-q^{3/4})},$$  \hspace{1cm} (7.11)

$$\sum_{n=0}^{\infty} a_{8n+2} q^n = q^{3/2} \frac{f(q^{29}, q^{67})}{\varphi(-q^{3/4})}.$$  \hspace{1cm} (7.12)
\[
\sum_{n=0}^{\infty} a_{8n+3} q^n = -q \frac{f(q^{31}, q^{65})}{\varphi(-q^{3/4})}, \tag{7.13}
\]
\[
\sum_{n=0}^{\infty} a_{8n+4} q^n = q^9 \frac{f(q^5, q^{91})}{\varphi(-q^{3/4})}, \tag{7.14}
\]
\[
\sum_{n=0}^{\infty} a_{8n+5} q^n = -q^8 \frac{f(q^7, q^{89})}{\varphi(-q^{3/4})}, \tag{7.15}
\]
\[
\sum_{n=0}^{\infty} a_{8n+6} q^n = q^{7/2} \frac{f(q^{19}, q^{77})}{\varphi(-q^{3/4})}, \tag{7.16}
\]
\[
\sum_{n=0}^{\infty} a_{8n+7} q^n = -q^4 \frac{f(q^{17}, q^{79})}{\varphi(-q^{3/4})}. \tag{7.17}
\]

**Proof.** Employing equations (4.27)–(4.34) with \(k = 2\), we get (7.10) – (7.17).

8 The dissection of \(R_{16}^*(q)\) of degree 16

In this section, we obtain certain Ramanujan Quantities of \(R(1, 8, 16; q)\) in terms of Ramanujan’s theta–function.

**Theorem 8.1.** We have

\[
R_{16}(q) := R(1, 8, 16; q) = q^{49/32} \frac{f(-q, -q^{15})}{f(-q^8, -q^8)}, \tag{8.1}
\]

**Proof.** Employing (3.1) with \(k = 2\), we get (8.1).

8.1 The 2-dissection of \(R_{16}^*(q)\) of degree 16

**Theorem 8.2.** If \(R_{16}^*(q) := \frac{f(-q, -q^{15})}{f(-q^8, -q^8)} = \sum_{n=0}^{\infty} a_n q^n\), then

\[
\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^9, q^{23})}{\varphi(-q^4)}. \tag{8.2}
\]
\[
\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f(q^7, q^{25})}{\varphi(-q^4)}. \tag{8.3}
\]

**Proof.** Employing equations (4.1) and (4.2) with \(k = 4\), we get (8.2) and (8.3).

8.2 The 4-dissection of \(R_{16}^*(q)\) of degree 16

**Theorem 8.3.** We have

\[
\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f(q^{25}, q^{59})}{\varphi(-q^2)}, \tag{8.4}
\]
\[
\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f(q^3, q^{41})}{\varphi(-q^2)}, \tag{8.5}
\]
\[
\sum_{n=0}^{\infty} a_{4n+2} q^n = q^4 \frac{f(q^7, q^{57})}{\varphi(-q^2)}, \tag{8.6}
\]
\[
\sum_{n=0}^{\infty} a_{4n+3} q^n = -q^3 \frac{f(q^9, q^{55})}{\varphi(-q^2)}. \tag{8.7}
\]

**Proof.** Employing equations (4.20)–(4.22) with \(k = 4\), we get (8.4) – (8.7).
8.3 The 8-dissection of $R_{16}^*(q)$ of degree 16

Theorem 8.4. We have

\[ \sum_{n=0}^{\infty} a_{8n} q^n = \frac{f(q^{37}, q^{71})}{\varphi(-q)}, \]  
\[ (8.8) \]

\[ \sum_{n=0}^{\infty} a_{8n+1} q^n = -\frac{f(q^{55}, q^{73})}{\varphi(-q)}, \]  
\[ (8.9) \]

\[ \sum_{n=0}^{\infty} a_{8n+2} q^n = q^2 \frac{f(q^{39}, q^{89})}{\varphi(-q)}, \]  
\[ (8.10) \]

\[ \sum_{n=0}^{\infty} a_{8n+3} q^n = -q^{3/2} \frac{f(q^{41}, q^{87})}{\varphi(-q)}, \]  
\[ (8.11) \]

\[ \sum_{n=0}^{\infty} a_{8n+4} q^n = q^{12} \frac{f(q^{7}, q^{121})}{\varphi(-q)}, \]  
\[ (8.12) \]

\[ \sum_{n=0}^{\infty} a_{8n+5} q^n = -q^{11/2} \frac{f(q^9, q^{119})}{\varphi(-q)}, \]  
\[ (8.13) \]

\[ \sum_{n=0}^{\infty} a_{8n+6} q^n = q^5 \frac{f(q^{25}, q^{101})}{\varphi(-q)}, \]  
\[ (8.14) \]

\[ \sum_{n=0}^{\infty} a_{8n+7} q^n = -q^{11/2} \frac{f(q^{33}, q^{105})}{\varphi(-q)}. \]  
\[ (8.15) \]

Proof. Employing equations (4.27)–(4.34) with $k = 2$, we get (8.8) – (8.15).

\[ \square \]

9 Applications to Partitions

Clearly equation (4.1) can be written as

\[ \sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^{k+1}, q^{3k-1})}{\varphi(-q^{k/2})} = \frac{(-q^{k+1}, -q^{3k-1}, q^{4k}; q^{4k})}{(q^{k/2}; q^{k/2})_{\infty} (q^{k/2}; q^{k})_{\infty}}. \]  
\[ (9.1) \]

\[ = \frac{(q^{2(k+1)}, q^{2(3k-1)}, q^{4k}; q^{4k})_{\infty} (q^{k/2}; q^{k/2})_{\infty} (q^{k/2}; q^{k})_{\infty}}{(q^{2(k+1)}, q^{2(3k-1)}, q^{4k}; q^{4k})_{\infty} (q^{k/2}; q^{k/2})_{\infty} (q^{k/2}; q^{k})_{\infty}}, \]

where $0 \leq x \leq 7$, $1 \leq y \leq 16$.

\[ \sum_{n=0}^{\infty} a_{2n} q^n = \frac{(q^{2(k+1)}, q^{2(3k-1)}, q^{8k})_{\infty} (q^{2k+1}; q^{2k+1})_{\infty} (q^{2k+1}; q^{2k})_{\infty}}{(q^{2(k+1)}, q^{2(3k-1)}, q^{8k})_{\infty} (q^{2k+1}; q^{2k+1})_{\infty} (q^{2k+1}; q^{2k})_{\infty}}, \]  
\[ (9.2) \]

where $0 \leq x \leq 7$, $1 \leq y \leq 7$, $9 \leq y \leq 15$.

If we put $k = 2$ in (9.1), we obtain equation (5.2) and that can be expressed as below using (9.2)

\[ \sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^3, q^5)}{\varphi(-q)} = \frac{1}{(q^{3\pm}, q^{5\pm}; q^{16})_{\infty} (q^{1\pm}, q^{7\pm}; q^{16})_{\infty} (q^{2\pm}, q^{4\pm}; q^{16})_{\infty}}. \]  
\[ (9.3) \]
Then $a_{2n}$ denote the number of partitions of $n$ into parts congruent to $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 7$ (mod 16) with $\pm 3, \pm 5$ (mod 16) having 3 colors and $\pm 1, \pm 7$ (mod 16) having 2 colors.

If we put $k = 4$ in (9.1), we obtain equation (6.14) and that can be expressed as below using (9.2)

$$
\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f(q^5, q^{11})}{\varphi(-q^2)} = \frac{1}{(q^{2\pm}, q^{6\pm}, q^{14\pm}; q^{32})^2} (q^{4\pm}, q^{8\pm}, q^{10\pm}, q^{11\pm}, q^{12\pm}; q^{32})_{\infty}.
$$

(9.4)

Then $a_{2n}$ denote the number of partitions of $n$ into parts congruent to $\pm 2, \pm 4, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12, \pm 14$ (mod 32) with $\pm 2, \pm 6, \pm 14$ (mod 32) having 2 colors.

Similarly equation (4.2) can be written as

$$
\sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{f(q^{k-1}, q^{3k+1})}{\varphi(-q^{k/2})} = \frac{(-q^{k-1}, -q^{3k+1}, q^{4k}; q^{4k})_{\infty}}{(q^{k/2}; q^{k/2})_{\infty}} (q^{k/2}; q^{k})_{\infty}.
$$

(9.5)

$$
\sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{f(q^{k-1}, q^{3k+1}, q^{8k})}{(q^{k-1}, q^{3k-1}, q^{3k+1}, q^{7k+1}; q^{8k})_{\infty}} (q^{xk+k/2}; q^{8k})_{\infty}.
$$

(9.6)

where $1 \leq y \leq 7$.

If we put $k = 2$ in (9.5), we obtain equation (5.3) and that can be expressed as below using (9.6)

$$
\sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{f(q, q^7)}{\varphi(-q)} = \frac{1}{(q^{1\pm}, q^{7\pm}; q^{16})^3} (q^{3\pm}, q^{5\pm}; q^{16})^2 (q^{4\pm}, q^{6\pm}; q^{16})_{\infty}.
$$

(9.7)

Then $a_{2n+1}$ denote the number of partitions of $n$ into parts congruent to $\pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7$ (mod 16) with $\pm 1, \pm 7$ (mod 16) having 3 colors and $\pm 3, \pm 5$ (mod 16) having 2 colors.

If we put $k = 4$ in (9.5), we obtain equation (6.15) and that can be expressed as below using (9.6)

$$
\sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{f(q^3, q^{13})}{\varphi(-q^2)} = \frac{1}{(q^{2\pm}, q^{10\pm}, q^{14\pm}; q^{32})^2} (q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{12\pm}, q^{13\pm}; q^{32})_{\infty}.
$$

(9.8)

Then $a_{2n+1}$ denote the number of partitions of $n$ into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 13, \pm 14$ (mod 32) with $\pm 2, \pm 10, \pm 14$ (mod 32) having 2 colors.

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Received: 2022-01-10
Accepted: 2022-03-15