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General Dissection Formula for Ramanujan Quantity

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Abstract. In this paper, We study the several General Dissection Formula of degree 4, 8, 12 and 16 for Ramanujan Quantity $R(a, b, p; q)$ when b is a positive even integer. Also, we obtain some applications to theory of partitions of general dissection that we obtain in this paper itself.

1 Introduction

In Chapter 16 of his second notebook [2, 8], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \mid ab \mid < 1, \tag{1.1}$$

$$= (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty$$

where $(a; q)_0 = 1$ and $(a; q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.
Also, we define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty \ ,$$

where r and s are positive integers with $r < s$. We use this particular notation in the last section of this paper.
Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.4}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{1.5}$$

In [6],[7] Nikos Bagis define Ramanujan Quantities $R(a, b, p; q)$ as

$$R(a,b,p;q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{\prod_{n=0}^{\infty}(1-q^a q^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^b q^{np})(1-q^{p-b}q^{np})}, \tag{1.6}$$

where a, b, and p are positive rationales such that $a + b < p$. General Theorem such

$$\frac{q^{B-A}}{1-a_1 b_1} \frac{(a_1 - b_1 q_1)(b_1 - a_1 q_1)}{(1-a_1 b_1)(q_1^2+1)} \frac{(a_1 - b_1 q_1^3)(b_1 - a_1 q_1^3)}{(1-a_1 b_1)(q_1^4+1)} \cdots$$

$$= \frac{\prod_{n=0}^{\infty}(1-q^a q^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^b q^{np})(1-q^{p-b}q^{np})} \tag{1.7}$$

where $a_1 = q^A$, $b_1 = q^B$, $q_1 = q^{A+B}$, $a = 2A + 3p/4$, $2B + p/4$, and $p = 4(A+B)$, $|q| < 1$, are proved.

In this paper, we study the several General Dissection Formula for Ramanujan Quantity $R(a, b, p; q)$.

The 2-dissections, 4-dissections and 8-dissections are studied by several mathematicians including S. Ramanujan. For details one may refer [9, p. 50] , [1] and [4].

In the last section, we extract partition theoretic interpretations of some identities that we obtain in the other sections.

Definition. “A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called *colored partitions*."
For example, if 1 is allowed to have two colors, say r (red) and g (green), then all the colored partitions of 2 are 2, $1_r + 1_r$, $1_r + 1_g$ and $1_g + 1_g$. Note that

$$\frac{1}{(q^u, q^v)_\infty^k}$$

is the generating function for the number of partitions of n where all the parts are congruent to $u \pmod v$ and have k colors. Also rewriting the same modular relation in different ways results in different partition theoretic interpretations.

2 Preliminary results

[7]

$$[a, p; q] = (q^{p-a}; q^p)_\infty (q^a; q^p)_\infty \tag{2.1}$$

where $q = e^{-\pi\sqrt{r}}$ and $a, p, r > 0$.

[7]

$$R(a, b, p; q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a, p; q]}{[b, p; q]}. \tag{2.2}$$

Lemma 2.1. *[2, Entry 29(i),(ii), p.45] If $p = ab = cd$, then*

$$f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)\, f(ad, bc)\,, \tag{2.3}$$

$$f(a, b)f(c, d) - f(-a, -b)f(-c, -d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.4}$$

Lemma 2.2. *[2, Entry 30(ii),(iii),(v),(vi), p.46]*

$$f(a, b) + f(-a, -b) = 2f\left(a^3b, ab^3\right), \tag{2.5}$$

$$f(a, b) - f(-a, -b) = 2af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right), \tag{2.6}$$

$$f^2(a, b) + f^2(-a, -b) = 2f\left(a^2, b^2\right)\varphi(ab)\,, \tag{2.7}$$

$$f^2(a, b) - f^2(-a, -b) = 4af\left(\frac{b}{a}, \frac{a}{b}a^2b^2\right)\psi\left(a^2b^2\right). \tag{2.8}$$

Lemma 2.3. *[2, Entry 25(iii), p.40] We have,*

$$\varphi^2\left(-q^2\right) = \varphi(q)\varphi(-q), \quad \varphi\left(-q^2\right)\psi\left(q^2\right) = \psi(q)\psi(-q). \tag{2.9}$$

Lemma 2.4. *[2, Entry 25(iv), p.40] We have,*

$$\psi^2\left(q^2\right) = \psi(q^2)\varphi(q), \tag{2.10}$$

Lemma 2.5. *[5]*

$$f\left(-q, -q^{11}\right) = \frac{\varphi(q) - \varphi(q^3)}{2q\chi(q)} \tag{2.11}$$

3 Ramanujan Quantities of $R(a, b, p; q)$

In this section, we obtain certain Ramanujan Quantities of $R(a, b, p; q)$ in terms of Ramanujan's theta–function.

Theorem 3.1. *We have*

$$R(q) := R(1, k, 2k; q) = q^{(k-1)^2/4k} \frac{f\left(-q, -q^{2k-1}\right)}{f\left(-q^k, -q^k\right)}. \tag{3.1}$$

Proof. Employing equations (2) and (2) with $a = 1, b = k$, a positive even integer and $p = 2k$, we get

$$R(1, k, 2k; q) = q^{-(1-k)/2+(1-k^2)/(4k)} \frac{(q; q^{2k})_\infty (q^{2k-1}; q^{2k})_\infty}{(q^{2k-k}; q^{2k})_\infty (q^k; q^{2k})_\infty}. \tag{3.2}$$

Using Ramanujan's Theat–function definition in above equation (3.2), we obtain (3.1). □

4 General n-Dissections of $R^*(q)$

4.1 The 2-dissection of $R^*(q)$

Theorem 4.1. *If* $R^*(q) := \dfrac{f(-q, -q^{2k-1})}{f(-q^k, -q^k)} = \sum_{n=0}^{\infty} a_n\, q^n,$ *then*

$$\sum_{n=0}^{\infty} a_{2n}\, q^n = \frac{f\left(q^{k+1}, q^{3k-1}\right)}{\varphi(-q^{k/2})}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} a_{2n+1}\, q^n = -\frac{f\left(q^{k-1}, q^{3k+1}\right)}{\varphi(-q^{k/2})}. \tag{4.2}$$

Proof. We have

$$\sum_{n=0}^{\infty} a_n\, q^n = \frac{f(-q, -q^{2k-1})}{f(-q^k, -q^k)},$$

$$= \frac{f(-q, -q^{2k-1}) f(q^k, q^k)}{f(-q^k, -q^k) f(q^k, q^k)}. \tag{4.3}$$

Putting $a = -q$, $b = -q^{2k-1}$ and $c = d = q^k$ in Lemma **2.1**, we obtain

$$f(-q, -q^{2k-1}) f(q^k, r^k) = f^2(-q^{k+1}, -q^{3k-1}) - q f^2(-q^{k-1}, -q^{3k+1}). \tag{4.4}$$

Putting $a = q^k$ and $b = q^k$ in Lemma **2.2**, we get

$$f(q^k, q^k) f(-q^k, -q^k) = f^2\left(-q^{2k}, -q^{2k}\right). \tag{4.5}$$

Employing (4.4) and (4.5) in (4.3), we deduce that

$$R^*(q) = \sum_{n=0}^{\infty} a_n\ q^n = \frac{f^2(-q^{k+1}, -q^{3k-1}) - q f^2(-q^{k-1}, -q^{3k+1})}{\varphi^2\left(-q^{2k}\right)}. \tag{4.6}$$

Hence

$$\begin{aligned} R^*(q) + R^*(-q) &= 2\sum_{n=0}^{\infty} a_{2n}\ q^{2n} \\ &= \frac{\left[f^2(-q^{k+1}, -q^{3k-1}) + f^2(q^{k+1}, q^{3k-1})\right] - q\left[f^2(-q^{k-1}, -q^{3k+1}) - f^2(q^{k-1}, q^{3k+1})\right]}{\varphi^2\left(-q^{2k}\right)}. \end{aligned} \tag{4.7}$$

Putting $a = q^{k+1}$ and $b = q^{3k-1}$ in (2.7), $a = -q^{k-1}$ and $b = -q^{3k+1}$ in (2.8), we get

$$f^2(q^{k+1}, q^{3k+1}) + f^2(-q^{k+1}, -q^{3k-1}) = 2f\left(q^{2(k+1)}, q^{2(3k-1)}\right)\varphi\left(q^{4k}\right), \tag{4.8}$$

$$f^2(-q^{k-1}, -q^{3k+1}) - f^2(q^{k-1}, q^{3k+1}) = -4q^{k-1} f\left(q^{2(k+1)}, q^{2(3k-1)}\right)\psi\left(q^{8k}\right). \tag{4.9}$$

Employing the above equations (4.8) and (4.9) in (4.7), we deduce that

$$R^*(q) + R^*(-q) = 2\sum_{n=0}^{\infty} a_{2n}\ q^{2n} = 2\frac{f\left(q^{2(k+1)}, q^{2(3k-1)}\right)\left[\varphi\left(q^{4k}\right) + 2q^k\psi\left(q^{8k}\right)\right]}{\varphi^2\left(-q^{2k}\right)}. \tag{4.10}$$

Putting $a = q^k$ in Lemma **2.3**, we get

$$\varphi(q^k) = \varphi(q^{4k}) + 2q^k\psi\left(q^{8k}\right). \tag{4.11}$$

Substituting equation (4.11) in the equation (4.10), we obtain

$$R^*(q) + R^*(-q) = 2\sum_{n=0}^{\infty} a_{2n}\ q^{2n} = 2\frac{f\left(q^{2(k+1)}, q^{2(3k-1)}\right)\varphi(q^k)}{\varphi^2\left(-q^{2k}\right)}. \tag{4.12}$$

Using (2.9) in the above equation, we get

$$2\sum_{n=0}^{\infty} a_{2n}\ q^{2n} = 2\frac{f\left(q^{2(k+1)}, q^{2(3k-1)}\right)}{\varphi(-q^k)}. \tag{4.13}$$

Replace q by $q^{1/2}$ in (4.13), we obtain (4.1).

Similarly

$$\begin{aligned} R^*(q) - R^*(-q) &= 2\sum_{n=0}^{\infty} a_{2n+1}\ q^{2n+1} \\ &= \frac{\left[f^2(-q^{k+1}, -q^{3k-1}) - f^2(q^{k+1}, q^{3k-1})\right] - q\left[f^2(-q^{k-1}, -q^{3k+1}) + f^2(q^{k-1}, q^{3k+1})\right]}{\varphi^2\left(-q^{2k}\right)}. \end{aligned} \tag{4.14}$$

Putting $a = -q^{k+1}$ and $b = -q^{3k-1}$ in (2.8), $a = q^{k-1}$ and $b = q^{3k+1}$ in (2.7), we get

$$f^2(-q^{k+1}, -q^{3k-1}) - f^2(q^{k+1}, q^{3k-1}) = -4q^{k+1} f\left(q^{2(k-1)}, q^{2(3k+1)}\right)\psi\left(q^{8k}\right), \tag{4.15}$$

$$f^2(-q^{k-1}, -q^{3k+1}) + f^2(q^{k-1}, q^{3k+1}) = 2f\left(q^{2(k-1)}, q^{2(3k+1)}\right)\varphi\left(q^{4k}\right). \tag{4.16}$$

Employing the above equations (4.15) and (4.16) in (4.14), we deduce that

$$R^*(q) - R^*(-q) = 2\sum_{n=0}^{\infty} a_{2n+1}\ q^{2n+1} = -\frac{2qf\left(q^{2(k-1)}, q^{2(3k+1)}\right)\left[\varphi\left(q^{4k}\right) + 2q^k\psi\left(q^{8k}\right)\right]}{\varphi^2\left(-q^{2k}\right)}. \tag{4.17}$$

Substituting equation (4.11) in the equation (4.17), we obtain

$$R^*(q) - R^*(-q) = 2\sum_{n=0}^{\infty} a_{2n+1}\ q^{2n} = -2\frac{f\left(q^{2(k-1)}, q^{2(3k+1)}\right)\varphi(q^k)}{\varphi^2\left(-q^{2k}\right)}. \tag{4.18}$$

Using (2.9) in the above equation, we get

$$R^*(q) - R^*(-q) = 2\sum_{n=0}^{\infty} a_{2n+1}\ q^{2n} = -2\frac{f\left(q^{2(k-1)}, q^{2(3k+1)}\right)}{\varphi(-q^k)}. \tag{4.19}$$

Replace q by $q^{1/2}$ in (4.19), we obtain (4.2). □

4.2 The 4-dissection of $R^*(q)$

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty} a_{4n}\ q^n = \frac{f\left(q^{3k+1}, q^{5k-1}\right)}{\varphi(-q^{k/4})}, \tag{4.20}$$

$$\sum_{n=0}^{\infty} a_{4n+1}\ q^n = -\frac{f\left(q^{3k-1}, q^{5k+1}\right)}{\varphi(-q^{k/4})}, \tag{4.21}$$

$$\sum_{n=0}^{\infty} a_{4n+2}\ q^n = q^{k/2}\ \frac{f\left(q^{k-1}, q^{7k+1}\right)}{\varphi(-q^{k/4})}, \tag{4.22}$$

$$\sum_{n=0}^{\infty} a_{4n+3}\ q^n = -q^{(k-2)/2}\ \frac{f\left(q^{k+1}, q^{7k-1}\right)}{\varphi(-q^{k/4})}. \tag{4.23}$$

Proof. From (4.1), We have

$$\sum_{n=0}^{\infty} a_{2n}\ q^n = \frac{f\left(q^{k+1}, q^{3k-1}\right)}{\varphi(-q^{k/2})}. \tag{4.24}$$

Putting $a = q^{k+1}$ and $b = q^{3k-1}$ in (2.5) and (2.6), we obtain

$$f(q^{k+1}, q^{3k-1}) = f\left(q^{2(3k+1)}, q^{2(5k-1)}\right) + q^{k+1} f\left(q^{2(k-1)}, q^{2(7k+1)}\right). \tag{4.25}$$

Employing the above equation (4.25) in (4.24), we have

$$\sum_{n=0}^{\infty} a_{2n}\ q^n = \frac{f\left(q^{2(3k+1)}, q^{2(5k-1)}\right) + q^{k+1} f\left(q^{2(k-1)}, q^{2(7k+1)}\right)}{\varphi(-q^{k/2})}. \tag{4.26}$$

This implies

$$\sum_{n=0}^{\infty} a_{4n}\ q^{2n} = \frac{f\left(q^{2(3k+1)}, q^{2(5k-1)}\right)}{\varphi(-q^{k/2})},$$

$$\sum_{n=0}^{\infty} a_{4n+2}\ q^{2n+1} = q^{k+1}\frac{f\left(q^{2(k-1)}, q^{2(7k+1)}\right)}{\varphi(-q^{k/2})}.$$

Changing q by $q^{1/2}$ in the above equations, we obtain (4.20) and (4.22).
Proofs of (4.21) and (4.23) are similar.. □

4.3 The 8-dissection of $R^*(q)$

Theorem 4.3. *We have*

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{7k+1}, q^{9k-1}\right)}{\varphi(-q^{k/8})}, \tag{4.27}$$

$$\sum_{n=0}^{\infty} a_{8n+1}\ q^n = -\frac{f\left(q^{7k-1}, q^{9k+1}\right)}{\varphi(-q^{k/8})}, \tag{4.28}$$

$$\sum_{n=0}^{\infty} a_{8n+2}\ q^n = q^{k/4}\ \frac{f\left(q^{5k-1}, q^{11k+1}\right)}{\varphi(-q^{k/8})}, \tag{4.29}$$

$$\sum_{n=0}^{\infty} a_{8n+3}\ q^n = -q^{(k-2)/4}\ \frac{f\left(q^{5k+1}, q^{11k-1}\right)}{\varphi(-q^{k/8})}, \tag{4.30}$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^n = q^{3k/2}\frac{f\left(q^{k-1}, q^{15k+1}\right)}{\varphi(-q^{k/8})}, \tag{4.31}$$

$$\sum_{n=0}^{\infty} a_{8n+5}\ q^n = -q^{(3k-2)/2}\frac{f\left(q^{k+1}, q^{15k-1}\right)}{\varphi(-q^{k/8})}, \tag{4.32}$$

$$\sum_{n=0}^{\infty} a_{8n+6}\ q^n = q^{(3k-4)/4}\ \frac{f\left(q^{3k+1}, q^{13k-1}\right)}{\varphi(-q^{k/8})}, \tag{4.33}$$

$$\sum_{n=0}^{\infty} a_{8n+7}\ q^n = -q^{(3k-2)/4}\ \frac{f\left(q^{3k-1}, q^{13k+1}\right)}{\varphi(-q^{k/8})}. \tag{4.34}$$

Proof. From (4.20), We have

$$\sum_{n=0}^{\infty} a_{4n}\ q^n = \frac{f\left(q^{3k+1}, q^{5k-1}\right)}{\varphi(-q^{k/4})}. \tag{4.35}$$

Putting $a = q^{3k+1}$ and $b = q^{5k-1}$ in (2.5) and (2.6), we obtain

$$f(q^{3k+1}, q^{5k-1}) = f\left(q^{2(7k+1)}, q^{2(9k-1)}\right) + q^{3k+1} f\left(q^{2(k-1)}, q^{2(15k+1)}\right). \tag{4.36}$$

Employing the above equation (4.36) in (4.35), we have

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{2(7k+1)}, q^{2(9k-1)}\right) + q^{3k+1} f\left(q^{2(k-1)}, q^{2(15k+1)}\right)}{\varphi(-q^{k/4})}. \tag{4.37}$$

This implies

$$\sum_{n=0}^{\infty} a_{8n}\ q^{2n} = \frac{f\left(q^{2(7k+1)}, q^{2(9k-1)}\right)}{\varphi(-q^{k/4})},$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^{2n+1} = q^{3k+1}\frac{f\left(q^{2(k-1)}, q^{2(15k+1)}\right)}{\varphi(-q^{k/4})}.$$

Changing q by $q^{1/2}$ in the above equations, we obtain (4.27) and (4.31).
Proofs of (4.28), (4.29), (4.30), (4.32), (4.33) and (4.34) are similar. □

5 Dissections of $R_4^*(q)$ of degree 4

In this section, we obtain certain Ramanujan Quantities of $R(1,2,4;q)$ in terms of Ramanujan's theta–function.

Theorem 5.1. *We have*

$$R_4(q) := R(1,2,4;q) = q^{1/8}\ \frac{f\left(-q, -q^3\right)}{f\left(-q^2, -q^2\right)}. \tag{5.1}$$

Proof. Employing (3.1) with $k = 2$, we get (5.1). □

5.1 The 2-dissection of $R_4^*(q)$ of degree 4

Theorem 5.2. *If* $R_4^*(q) := \dfrac{f(-q,-q^3)}{f(-q^2,-q^2)} = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^3,q^5\right)}{\varphi(-q)}. \tag{5.2}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f\left(q,q^7\right)}{\varphi(-q)}. \tag{5.3}$$

Proof. Employing equations (4.1) and (4.2) with $k=2$, we get (5.2) and (5.3). □

5.2 The 4-dissection of $R_4^*(q)$ of degree 4

Theorem 5.3. *We have*

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f\left(q^7,q^9\right)}{\varphi(-q^{1/2})}, \tag{5.4}$$

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f\left(q^5,q^{11}\right)}{\varphi(-q^{1/2})}, \tag{5.5}$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q\frac{f\left(q,q^{15}\right)}{\varphi(-q^{1/2})}. \tag{5.6}$$

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -\frac{f\left(q^3,q^{13}\right)}{\varphi(-q^{1/2})}, \tag{5.7}$$

Proof. Employing equations (4.20)–(4.22) with $k=2$, we get (5.4) – (5.7). □

5.3 The 8-dissection of $R_4^*(q)$ of degree 4

Theorem 5.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{15},q^{17}\right)}{\varphi(-q^{1/4})}, \tag{5.8}$$

$$\sum_{n=0}^{\infty} a_{8n+1}\ q^n = -\frac{f\left(q^{13},q^{19}\right)}{\varphi(-q^{1/4})}, \tag{5.9}$$

$$\sum_{n=0}^{\infty} a_{8n+2}\ q^n = q^{1/2}\ \frac{f\left(q^9,q^{23}\right)}{\varphi(-q^{1/4})}, \tag{5.10}$$

$$\sum_{n=0}^{\infty} a_{8n+3}\ q^n = -\ \frac{f\left(q^{11},q^{21}\right)}{\varphi(-q^{1/4})}, \tag{5.11}$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^n = q^3\frac{f\left(q,q^{31}\right)}{\varphi(-q^{1/4})}, \tag{5.12}$$

$$\sum_{n=0}^{\infty} a_{8n+5}\ q^n = -q^2\frac{f\left(q^3,q^{29}\right)}{\varphi(-q^{1/4})}, \tag{5.13}$$

$$\sum_{n=0}^{\infty} a_{8n+6}\ q^n = q^{1/2}\ \frac{f\left(q^7,q^{25}\right)}{\varphi(-q^{1/4})}, \tag{5.14}$$

$$\sum_{n=0}^{\infty} a_{8n+7}\ q^n = -q\ \frac{f\left(q^5,q^{27}\right)}{\varphi(-q^{1/4})}. \tag{5.15}$$

Proof. Employing equations (4.27)–(4.34) with $k=2$, we get (7.14) and (5.15). □

6 Dissections of $R_8^*(q)$ of degree 8

In this section, we obtain certain Ramanujan Quantities of $R(1,4,8;q)$ in terms of Ramanujan's theta–function.

Theorem 6.1. *If*

$$R(q) := R(1,4,8;q) = q^{9/16}\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)}, \tag{6.1}$$

then

$$R(q) = \left\{\frac{q^{1/8}\psi(q^2)}{2\psi(q)}\left[\frac{\varphi(q)}{\varphi(q^2)}-1\right]\right\}^{1/2}. \tag{6.2}$$

Proof. Employing (7.1) with $k=4$, we get

$$R(1,4,8;q) = q^{9/16}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^4;q^8)_\infty(q^4;q^8)_\infty} = q^{9/16}\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} \tag{6.3}$$

Employing the Lemma (2.1) with $a=q$, $b=q^3$ and $c=d=q^2$, we obtain

$$f(q,q^3)f(q^2,q^2) - f(-q,-q^3)f(-q^2,-q^2) = 2qf\left(q,q^7\right)f\left(q,q^7\right) \tag{6.4}$$

using Theta-function definitions (1.1), (1.2) and (1.3) in (6.4),we obtain

$$\psi(q)\varphi(q^2) - \psi(-q)\varphi(-q^2) = 2qf^2\left(q,q^7\right) \tag{6.5}$$

then above equation can be written with replace q by $-q$ as

$$\psi(q)\varphi(-q^2) - \psi(-q)\varphi(q^2) = 2qf^2\left(-q,-q^7\right) \tag{6.6}$$

Consider

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \frac{\left\{\psi(q)\varphi(-q^2)-\psi(-q)\varphi(q^2)\right\}^{1/2}}{\sqrt{2q}\varphi\left(-q^4\right)} \tag{6.7}$$

Using (2.9) with replace q by q^2 then (6.7) can be written as

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi(q)\varphi(-q^2)-\psi(-q)\varphi(q^2)}{2q\varphi\left(-q^2\right)\varphi\left(q^2\right)}\right\}^{1/2}. \tag{6.8}$$

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi(q)}{2q\varphi(q^2)} - \frac{\psi(-q)}{2q\varphi(-q^2)}\right\}^{1/2}. \tag{6.9}$$

Employing the equation (2.9) in (6.9), we obtain

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi(q)}{2q\varphi(q^2)} - \frac{\psi(q^2)}{2q\psi(q)}\right\}^{1/2}. \tag{6.10}$$

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi^2(q)-\varphi(q^2)\psi(q^2)}{2q\psi(q)\varphi(q^2)}\right\}^{1/2}. \tag{6.11}$$

Using the Lemma (2.4) in equation (6.12), we get

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi(q^2)\varphi(q)-\varphi(q^2)\psi(q^2)}{2q\psi(q)\varphi(q^2)}\right\}^{1/2}. \tag{6.12}$$

$$\frac{f\left(-q,-q^7\right)}{f\left(-q^4,-q^4\right)} = \left\{\frac{\psi(q^2)}{2q\psi(q)}\left[\frac{\varphi(q)}{\varphi(q^2)}-1\right]\right\}^{1/2}. \tag{6.13}$$

Using the above identity (6.13) in (6.3) we obtain required result. □

6.1 The 2-dissection of $R_8^*(q)$ of degree 8

Theorem 6.2. *If* $R(q) := \dfrac{f(-q,-q^7)}{f(-q^4,-q^4)} = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^5, q^{11}\right)}{\varphi(-q^2)}. \tag{6.14}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f\left(q^3, q^{13}\right)}{\varphi(-q^2)}. \tag{6.15}$$

Proof. Employing equations (4.1) and (4.2) with $k = 4$, we get (6.14) and (6.15). □

6.2 The 4-dissection of $R_8^*(q)$ of degree 8

Theorem 6.3. *We have*

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f\left(q^{13}, q^{19}\right)}{\varphi(-q)}, \tag{6.16}$$

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f\left(q^{11}, q^{21}\right)}{\varphi(-q)}, \tag{6.17}$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^2 \frac{f\left(q^3, q^{29}\right)}{\varphi(-q)}. \tag{6.18}$$

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -q \frac{f\left(q^5, q^{27}\right)}{\varphi(-q)}, \tag{6.19}$$

Proof. Employing equations (4.20)–(4.22) with $k = 4$, we get (6.16) – (6.19). □

6.3 The 8-dissection of $R_8^*(q)$ of degree 8

Theorem 6.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{29}, q^{35}\right)}{\varphi(-q^{1/2})}, \tag{6.20}$$

$$\sum_{n=0}^{\infty} a_{8n+1}\ q^n = -\frac{f\left(q^{27}, q^{37}\right)}{\varphi(-q^{1/2})}, \tag{6.21}$$

$$\sum_{n=0}^{\infty} a_{8n+2}\ q^n = q\ \frac{f\left(q^{19}, q^{45}\right)}{\varphi(-q^{1/2})}, \tag{6.22}$$

$$\sum_{n=0}^{\infty} a_{8n+3}\ q^n = -q^{1/2}\ \frac{f\left(q^{21}, q^{43}\right)}{\varphi(-q^{1/2})}, \tag{6.23}$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^n = q^6 \frac{f\left(q^3, q^{61}\right)}{\varphi(-q^{1/2})}, \tag{6.24}$$

$$\sum_{n=0}^{\infty} a_{8n+5}\ q^n = -q^5 \frac{f\left(q^5, q^{59}\right)}{\varphi(-q^{1/2})}, \tag{6.25}$$

$$\sum_{n=0}^{\infty} a_{8n+6}\ q^n = q^2\ \frac{f\left(q^{13}, q^{51}\right)}{\varphi(-q^{1/2})}, \tag{6.26}$$

$$\sum_{n=0}^{\infty} a_{8n+7}\ q^n = -q^{5/2}\ \frac{f\left(q^{11}, q^{53}\right)}{\varphi(-q^{1/2})}. \tag{6.27}$$

Proof. Employing equations (4.27)–(4.34) with $k = 2$, we get (6.20) – (7.7). □

7 Dissections of $R_{12}^{*}(q)$ of degree 12

Theorem 7.1. *If*

$$R_{12}(q) := R(1,6,12;q) = q^{25/24}\frac{f\left(-q,-q^{11}\right)}{f\left(-q^{6},-q^{6}\right)}, \tag{7.1}$$

then

$$R_{12}(q) = q^{1/24}\frac{\varphi(q)-\varphi(q^{3})}{2\chi(q)\varphi(-q^{6})}. \tag{7.2}$$

Proof. Employing (3.1) with $k=8$, we get, we get

$$R(1,6,12;q) = q^{25/24}\frac{(q;q^{12})_{\infty}(q^{11};q^{12})_{\infty}}{(q^{6};q^{12})_{\infty}(q^{6};q^{12})_{\infty}} = q^{25/24}\frac{f\left(-q,-q^{11}\right)}{f\left(-q^{6},-q^{6}\right)}. \tag{7.3}$$

Employing equation (2.11) in the above equation (7.3), we obtain (7.2). □

7.1 The 2-dissection of $R_{12}^{*}(q)$ of degree 12

Theorem 7.2. *If* $R_{12}^{*}(q) := \dfrac{f(-q,-q^{11})}{f(-q^{6},-q^{6})} = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^{7},q^{17}\right)}{\varphi(-q^{3})}. \tag{7.4}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f\left(q^{5},q^{19}\right)}{\varphi(-q^{3})}. \tag{7.5}$$

Proof. Employing equations (4.1) and (4.2) with $k=4$, we get (7.4) and (7.5). □

7.2 The 4-dissection of $R_{12}^{*}(q)$ of degree 12

Theorem 7.3. *We have*

$$\sum_{n=0}^{\infty} a_{4n}\ q^n = \frac{f\left(q^{19},q^{29}\right)}{\varphi(-q^{3/2})}, \tag{7.6}$$

$$\sum_{n=0}^{\infty} a_{4n+1}\ q^n = -\frac{f\left(q^{7},q^{41}\right)}{\varphi(-q^{3/2})}, \tag{7.7}$$

$$\sum_{n=0}^{\infty} a_{4n+2}\ q^n = q^{3}\frac{f\left(q^{5},q^{43}\right)}{\varphi(-q^{3/2})}, \tag{7.8}$$

$$\sum_{n=0}^{\infty} a_{4n+3}\ q^n = -q\frac{f\left(q^{17},q^{31}\right)}{\varphi(-q^{3/2})}. \tag{7.9}$$

Proof. Employing equations (4.20)–(4.22) with $k=4$, we get (7.6) – (7.9). □

7.3 The 8-dissection of $R_{12}^{*}(q)$ of degree 12

Theorem 7.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{43},q^{53}\right)}{\varphi(-q^{3/4})}, \tag{7.10}$$

$$\sum_{n=0}^{\infty} a_{8n+1}\ q^n = -\frac{f\left(q^{41},q^{55}\right)}{\varphi(-q^{3/4})}, \tag{7.11}$$

$$\sum_{n=0}^{\infty} a_{8n+2}\ q^n = q^{3/2}\ \frac{f\left(q^{29},q^{67}\right)}{\varphi(-q^{3/4})}, \tag{7.12}$$

$$\sum_{n=0}^{\infty} a_{8n+3}\ q^n = -q\ \frac{f\left(q^{31}, q^{65}\right)}{\varphi(-q^{3/4})}, \tag{7.13}$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^n = q^9\ \frac{f\left(q^{5}, q^{91}\right)}{\varphi(-q^{3/4})}, \tag{7.14}$$

$$\sum_{n=0}^{\infty} a_{8n+5}\ q^n = -q^8\ \frac{f\left(q^{7}, q^{89}\right)}{\varphi(-q^{3/4})}, \tag{7.15}$$

$$\sum_{n=0}^{\infty} a_{8n+6}\ q^n = q^{7/2}\ \frac{f\left(q^{19}, q^{77}\right)}{\varphi(-q^{3/4})}, \tag{7.16}$$

$$\sum_{n=0}^{\infty} a_{8n+7}\ q^n = -q^4\ \frac{f\left(q^{17}, q^{79}\right)}{\varphi(-q^{3/4})}. \tag{7.17}$$

Proof. Employing equations (4.27)–(4.34) with $k = 2$, we get (7.10) – (7.17). □

8 The dissection of $R^*_{16}(q)$ of degree 16

In this section, we obtain certain Ramanujan Quantities of $R(1, 8, 16; q)$ in terms of Ramanujan's theta–function.

Theorem 8.1. *We have*

$$R_{16}(q) := R(1, 8, 16; q) = q^{49/32}\ \frac{f\left(-q, -q^{15}\right)}{f\left(-q^8, -q^8\right)}. \tag{8.1}$$

Proof. Employing (3.1) with $k = 2$, we get (8.1). □

8.1 The 2-dissection of $R^*_{16}(q)$ of degree 16

Theorem 8.2. *If* $R^*_{16}(q) := \dfrac{f(-q, -q^{15})}{f(-q^8, -q^8)} = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^{9}, q^{23}\right)}{\varphi(-q^4)}. \tag{8.2}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{f\left(q^{7}, q^{25}\right)}{\varphi(-q^4)}. \tag{8.3}$$

Proof. Employing equations (4.1) and (4.2) with $k = 4$, we get (8.2) and (8.3). □

8.2 The 4-dissection of $R^*_{16}(q)$ of degree 16

Theorem 8.3. *We have*

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{f\left(q^{25}, q^{39}\right)}{\varphi(-q^2)}, \tag{8.4}$$

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{f\left(q^{23}, q^{41}\right)}{\varphi(-q^2)}, \tag{8.5}$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^4\frac{f\left(q^{7}, q^{57}\right)}{\varphi(-q^2)}, \tag{8.6}$$

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -q^3\frac{f\left(q^{9}, q^{55}\right)}{\varphi(-q^2)}. \tag{8.7}$$

Proof. Employing equations (4.20)–(4.22) with $k = 4$, we get (8.4) – (8.7). □

8.3 The 8-dissection of $R^*_{16}(q)$ of degree 16

Theorem 8.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n}\ q^n = \frac{f\left(q^{57}, q^{71}\right)}{\varphi(-q)}, \tag{8.8}$$

$$\sum_{n=0}^{\infty} a_{8n+1}\ q^n = -\frac{f\left(q^{55}, q^{73}\right)}{\varphi(-q)}, \tag{8.9}$$

$$\sum_{n=0}^{\infty} a_{8n+2}\ q^n = q^2\ \frac{f\left(q^{39}, q^{89}\right)}{\varphi(-q)}, \tag{8.10}$$

$$\sum_{n=0}^{\infty} a_{8n+3}\ q^n = -q^{3/2}\ \frac{f\left(q^{41}, q^{87}\right)}{\varphi(-q)}, \tag{8.11}$$

$$\sum_{n=0}^{\infty} a_{8n+4}\ q^n = q^{12}\frac{f\left(q^{7}, q^{121}\right)}{\varphi(-q)}, \tag{8.12}$$

$$\sum_{n=0}^{\infty} a_{8n+5}\ q^n = -q^{11/2}\frac{f\left(q^{9}, q^{119}\right)}{\varphi(-q)}, \tag{8.13}$$

$$\sum_{n=0}^{\infty} a_{8n+6}\ q^n = q^5\ \frac{f\left(q^{25}, q^{103}\right)}{\varphi(-q)}, \tag{8.14}$$

$$\sum_{n=0}^{\infty} a_{8n+7}\ q^n = -q^{11/2}\ \frac{f\left(q^{23}, q^{105}\right)}{\varphi(-q)}. \tag{8.15}$$

Proof. Employing equations (4.27)–(4.34) with $k = 2$, we get (8.8) – (8.15). □

9 Applications to Partitions

Clearly equation (4.1) can be written as

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^{k+1}, q^{3k-1}\right)}{\varphi\left(-q^{k/2}\right)} = \frac{\left(-q^{k+1}, -q^{3k-1}, q^{4k}; q^{4k}\right)_\infty}{\left(q^{k/2}; q^{k/2}\right)_\infty \left(q^{k/2}; q^{k}\right)_\infty} \tag{9.1}$$

$$= \frac{\left(q^{2(k+1)}, q^{2(3k-1)}, q^{4k}, q^{8k}; q^{8k}\right)_\infty}{\left(q^{k+1}, q^{3k-1}; q^{4k}\right)_\infty \left(q^{k/2}; q^{k/2}\right)_\infty \left(q^{k/2}; q^{k}\right)_\infty}$$

$$= \frac{\left(q^{2(k+1)}, q^{2(3k-1)}, q^{4k}, q^{8k}; q^{8k}\right)_\infty}{\left(q^{k+1}, q^{5k+1}, q^{3k-1}, q^{7k-1}; q^{8k}\right)_\infty \left(q^{xk+k/2}; q^{8k}\right)_\infty \left(q^{yk/2}; q^{8k}\right)_\infty}, \text{ where } \begin{array}{c} 0 \le x \le 7, \\ 1 \le y \le 16. \end{array}$$

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{\left(q^{2(k+1)}, q^{2(3k-1)}; q^{8k}\right)_\infty}{\left(q^{k+1}, q^{5k+1}, q^{3k-1}, q^{7k-1}; q^{8k}\right)_\infty \left(q^{xk+k/2}; q^{8k}\right)_\infty \left(q^{yk/2}; q^{8k}\right)_\infty}, \tag{9.2}$$

$$\text{where} \quad \begin{array}{c} 0 \le x \le 7, \\ 1 \le y \le 7, \\ 9 \le y \le 15. \end{array}$$

If we put $k = 2$ in (9.1), we obtain equation (5.2) and that can be expressed as below using (9.2)

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{f\left(q^3, q^5\right)}{\varphi\left(-q\right)} = \frac{1}{\left(q^{3\pm}, q^{5\pm}; q^{16}\right)_\infty^3 \left(q^{1\pm}, q^{7\pm}; q^{16}\right)_\infty^2 \left(q^{2\pm}, q^{4\pm}; q^{16}\right)_\infty}. \tag{9.3}$$

Then a_{2n} denote the number of partitions of n into parts congruent to $\pm1, \pm2, \pm3, \pm4, \pm5, \pm7$ (mod 16) with $\pm3, \pm5$ (mod 16) having 3 colors and $\pm1, \pm7$ (mod 16) having 2 colors.

If we put $k = 4$ in (9.1), we obtain equation (6.14) and that can be expressed as below using (9.2)

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{f\left(q^5, q^{11}\right)}{\varphi\left(-q^2\right)} = \frac{1}{\left(q^{2\pm}, q^{6\pm}, q^{14\pm}; q^{32}\right)_\infty^2 \left(q^{4\pm}, q^{5\pm}, q^{8\pm}, q^{10\pm}, q^{11\pm}, q^{12\pm}; q^{32}\right)_\infty}. \tag{9.4}$$

Then a_{2n} denote the number of partitions of n into parts congruent to $\pm2, \pm4, \pm5, \pm6, \pm8, \pm10, \pm11, \pm12, \pm14$ (mod 32) with $\pm2, \pm6, \pm14$ (mod 32) having 2 colors.

Similarly equation (4.2) can be written as

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{f\left(q^{k-1}, q^{3k+1}\right)}{\varphi\left(-q^{k/2}\right)} = \frac{\left(-q^{k-1}, -q^{3k+1}, q^{4k}; q^{4k}\right)_\infty}{\left(q^{k/2}; q^{k/2}\right)_\infty \left(q^{k/2}; q^{k}\right)_\infty} \tag{9.5}$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{\left(q^{2(k-1)}, q^{2(3k+1)}; q^{8k}\right)_\infty}{\left(q^{k-1}, q^{5k-1}, q^{3k+1}, q^{7k+1}; q^{8k}\right)_\infty \left(q^{xk+k/2}; q^{8k}\right)_\infty \left(q^{yk/2}; q^{8k}\right)_\infty}, \tag{9.6}$$

$$\text{where} \quad \begin{aligned} 0 &\le x \le 7, \\ 1 &\le y \le 7, \\ 9 &\le y \le 15. \end{aligned}$$

If we put $k = 2$ in (9.5), we obtain equation (5.3) and that can be expressed as below using (9.6)

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{f\left(q, q^7\right)}{\varphi\left(-q\right)} = \frac{1}{\left(q^{1\pm}, q^{7\pm}; q^{16}\right)_\infty^3 \left(q^{3\pm}, q^{5\pm}; q^{16}\right)_\infty^2 \left(q^{4\pm}, q^{6\pm}; q^{16}\right)_\infty}. \tag{9.7}$$

Then a_{2n+1} denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm4, \pm5, \pm6, \pm7$ (mod 16) with $\pm1, \pm7$ (mod 16) having 3 colors and $\pm3, \pm5$ (mod 16) having 2 colors.

If we put $k = 4$ in (9.5), we obtain equation (6.15) and that can be expressed as below using (9.6)

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{f\left(q^3, q^{13}\right)}{\varphi\left(-q^2\right)} = \frac{1}{\left(q^{2\pm}, q^{10\pm}, q^{14\pm}; q^{32}\right)_\infty^2 \left(q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{12\pm}, q^{13\pm}; q^{32}\right)_\infty}. \tag{9.8}$$

Then a_{2n+1} denote the number of partitions of n into parts congruent to $\pm2, \pm3, \pm4, \pm6, \pm8, \pm10, \pm12, \pm13, \pm14$ (mod 32) with $\pm2, \pm10, \pm14$ (mod 32) having 2 colors.

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