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On strongly $(*)$-rings

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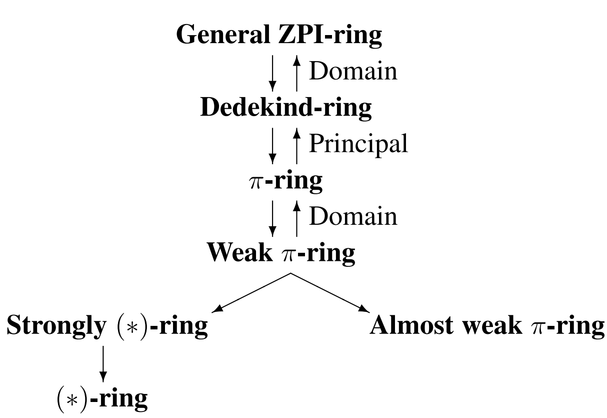
Abstract In this paper, we investigate strongly $(*)$-ring and we conjecture that this class of rings and the class of $(*)$-rings are equivalent. Also, we study the transfer of weak π-ring property, $(*)$-ring property and strongly $(*)$-ring property to some extensions of a ring.

1 Introduction

Throughout this paper, all rings considered are assumed to be commutative with an identity; in particular, R denotes such a ring, and all modules are assumed to be unitary modules. Dedekind domains are integral domains in which every ideal is a finite product of prime ideals and have the property that every non-zero prime ideal is maximal. Dedekind rings with zero divisors are defined by the same way for regular ideals [16, 17]. A ring R with zero divisors is said to be general ZPI-ring if every ideal is a finite product of prime ideals. These rings has the property that every prime ideal (or equivalently, every ideal) is finitely generated and locally principal. General ZPI-rings are also characterized by the property that R is a finite direct product of Dedekind domains and special principal ideal rings (SPIRs), that is, a local principal ideal ring, not a field, whose maximal ideal is nilpotent. A ring R has the property that every principal ideal is a finite product of prime ideals if and only if R is a finite direct product of (1) π-domains, (2) SPIRs, and (3) fields. In this case, R called a π-ring [13, p. 572]. A weak π-rings is a ring in which every regular principal ideal is a finite product of prime ideals [18]. R is called a $(*)$-ring if each regular principal ideal has a primary decomposition, that is, a finite intersection of primary ideals [18]. R is said to be an almost weak π-ring if for each regular principal ideal I, I_M is a finite product of prime ideals in R_M for all maximal ideals M containing I [18]. A ring R is called arithmetical ring if every finitely generated ideal of R is locally principal.

We summarize the relationship that exists between the above-mentioned notions by the following implications noted in the following figure.



We denote respectively by $qf(R)$, $Z(R)$, $Jac(R)$, $Reg(R)$, $Max(R)$ and $Ann(I)$ the quotien field of R, the set of zero-divisor of R, the Jacobson radical of R, the set of all regular elements of R, the set of all maximal ideals of R and the annihilator of an ideal I. I is said to be an invertible ideal if $II^{-1} = R$, where $I^{-1} = (R : I) = \{x \in qf(R) | xI \subseteq R\}$.

We recall the Nagata's idealization: let R be a ring, E be an R-module and $R \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(r, e)(s, f) = (rs, rf + se)$. $R \propto E$ is called the trivial ring extension of R by E (also called the idealization of E over R). For instance, the reader may consult [4, 5, 19].

Let R and S be two rings, let J be an ideal of S and let $f : R \to S$ be a ring homomorphism. The subring of $R \times S$, $R \bowtie^f J := \{(r, f(r) + j) | r \in R, j \in J\}$ is called the amalgamation of R with S along J with respect to f.
This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [7]), denoted by $R \bowtie I$, which is the subring of $R \times R$ given by $R \bowtie I := \{(r, r + i) | r \in R, i \in I\}$. For instance, the reader may consult [6, 7, 8, 9, 10, 11, 21].

The purpose of this article is to introduce the strongly $(*)$-ring property and study its transfer to these extension of rings and the transfer of the two notions weak π-ring and $(*)$-ring to the amalgamated duplication along an ideal and the amalgamed algebra along an ideal.
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2 General results

We start with the following definition:

Definition 2.1. A ring R is called a strongly $(*)$-ring if every invertible ideal has a primary decomposition.

Note that Laskerian rings [14] and Noetherian rings [12] are examples of strongly $(*)$-rings. Obviously, a strongly $(*)$-ring is a $(*)$-ring. We conjecture that these class of rings are equivalent. Easily, we can see that are equivalent when R is principal or local.

By the next proposition, we prove that a weak π-ring is a strongly $(*)$-ring.

Proposition 2.2. *If R is a weak π-ring, then R is a strongly $(*)$-ring.*

Proof. Let I be an invertible ideal of R. Thanks to [15, Lemma 18.1, p. 110], I contains a regular element, say r. Since R is a weak π-ring, $< r >= P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}$ where P_k's are prime ideals of R. Then $I = P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}$. This can be seen by applying [18, Lemma 12], as I is a multiplication ideal and P_k's are distinct invertible prime ideals by [2, Lemma 2.3]. As it now suffices to show that $P_k^{\alpha_k}$'s are invertible P_k-primary ideals, an appeal to [3, Theorem 3] completes the proof. □

We note that the converse need not be true, as the following example shows.

Example 2.3. Let $R = \mathbb{Z} \propto \mathbb{Z}$, we claim that R is not a weak π-ring.
Deny, let (a, b) be a regular element of R, we get $< (a, b) >= (P_1 \propto \mathbb{Z})...(P_n \propto \mathbb{Z})$, so each regular principal ideal is homogenous and hence $\mathbb{Z}$ is divisible as a $\mathbb{Z}$-module by [4, Theorem 3.9], which is a contradiction. Finally, R is not a weak π-ring, while R is a strongly $(*)$-ring since it is Noetherian.

Proposition 2.4. *Let $R = \prod_{i=1}^{n} R_i$, then R is a strongly $(*)$-ring if and only if so are R_i's.*

Proof. It suffices to show that the result hold for a pairs of rings R and S. Suppose that $R \times S$ is a strongly $(*)$-ring. Let I and J be two invertible ideals respectively of R and S. By our assumption, $I \times J = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$ where P_l's and Q_l's are respectively primary ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$.

Conversely, let K be an invertible ideal of $R \times S$. Then $K = I \times J$, clearly I(resp. J) is an invertible ideal of R(resp. S). By hypothesis, $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$ where P_l's and Q_l's are respectively primary ideals of R and S, thus $K = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$, as desired. □

For a multiplicative set S of a ring R, we obtain the next result:

Proposition 2.5. *Let R be a ring and S a multiplicative set of R such that $S \subseteq Reg(R)$. If R is a strongly $(*)$-ring , then so is $S^{-1}R$.*

Proof. Let J be an invertible ideal of $S^{-1}R$. By [15, Lemma 18.1], $J =< \frac{a_1}{s_1}, ..., \frac{a_n}{s_n} >= S^{-1} < a_1, ..., a_n >= S^{-1}I$ where $I =< a_1, ..., a_n >$. Since $J_{MS^{-1}R} = (S^{-1}I)_{MS^{-1}R} = I_M$ for all $M \in Max(R, I)$, it follows that I is locally principal. Also I is regular since J is regular and $S \subseteq Reg(R)$, then I is invertible. As R is a strongly $(*)$-ring, we get $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R. Therefore, $J = S^{-1}I = S^{-1}(P_1 \cap ... \cap P_n) = S^{-1}P_1 \cap ... \cap S^{-1}P_n$. As P_l is a primary ideal, then so is $S^{-1}P_l$ for all l, as desired. □

3 The transfer to the trivial extension ring

Now, we provide a result which translates the strongly $(*)$-ring property of $A \propto E$ in terms of A and E.

Theorem 3.1. *Let A be a ring, E an A-module and $R = A \propto E$ such that $E = aE$ for all $a \in S = A - (Z(A) \cup Z(E))$. Then R is a strongly $(*)$-ring if and only if every invertible ideal of A not disjoint from S has a primary decomposition.*

Proof. Suppose that R is a strongly $(*)$-ring and let I be an invertible ideal of A such that $I \cap S \neq \emptyset$. By hypothesis and [1, Theorem 7(1),(2)], $I \propto IE = I \propto E$ is an invertible ideal of R. As R is a strongly $(*)$-ring, then $I \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E) = (P_1 \cap ... \cap P_n) \propto E$ where P_k's are primary ideals of A, therefore $I = P_1 \cap ... \cap P_n$, as desired.
Conversely, let J be an invertible ideal of R. By hypothesis, $J = I \propto E$ where I is an invertible ideal not disjoint from S and $I = P_1 \cap ... \cap P_n$ such that P_k's are primary ideals of A. So, $J = (P_1 \cap ... \cap P_n) \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E)$. which completes the proof. □

As an immediate consequence of the previous theorem, we provide the following corollary:

Corollary 3.2. *Let A be a ring, E an A-module and $R = A \propto E$ such that $Z(E) \subseteq Z(A)$ and $E = aE$ for all $a \in S = A - (Z(A) \cup Z(E))$. Then R is a strongly $(*)$-ring if and only if so is A.*

As mentioned above, Noetherian rings are strongly $(*)$-rings. By the next corollary we provide an example of a non-Noetherian ring which is a strongly $(*)$-ring.

Corollary 3.3. *Let D be a domain and E a divisible R-module. Then:*

(i) D is a strongly $(*)$-ring if and only if so is $D \propto E$.

(ii) If E is a non-finitely generated D-module, then $D \propto E$ is a non-Noetherian ring and D is a strongly $(*)$-ring if and only if so is $D \propto E$.

4 The transfer to the amalgamation of rings and amalgamated duplication along an ideal

The proof of the first major result of this section (Theorem 4.3) relies on the following lemmas which are of independent interest. The next lemma investigate the form of regular ideals of $R \bowtie^f J$.

Lemma 4.1. *Let R and S be two rings, J an ideal of S and $f : R \to S$ be a ring homomorphism. Suppose that $Ann(f^{-1}(J)) = 0$, then the following statements are equivalent:*

(i) *Every regular ideal has the form* $I \bowtie^f J$ *where* I *is regular.*

(ii) $J = (f(a)+j)J$ *for all* $(a, f(a)+j) \in Reg(R \bowtie^f J)$.

Proof. (1) $\Rightarrow$ (2) Assume (1) holds and let $H =< (r, f(r)+j) >$, by hypothesis $H = I \bowtie^f J$ where $I =< r >$ is regular. Now let $k \in J$, we have $(0,k) = (0,k')(r, f(r)+j)$ for some $k' \in J$ since r is regular. Hence $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$.

(2) $\Rightarrow$ (1) Let H be a regular ideal of $R \bowtie^f J$, so there exists $(r, f(r)+j) \in H$. Now let $K =< (r, f(r)+j) >$. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$, so for all $k \in J$ there exists $k' \in J$ such that $k = (f(r)+j)k'$, thus $(0,k) = (0,k')(r, f(r)+j) \in K$, we get then $0 \times J \subseteq K \subseteq H$ and hence $H = I \bowtie^f J$ where $I =< r >$. We claim that I is regular. Deny, let $x \in R$ such that $rx = 0$, since $Ann(f^{-1}(J)) = 0$, there exists $y \in f^{-1}(J)$ such that $xy \neq 0$, then $(r, f(r)+j)(xy, 0) = (0,0)$, contradiction. □

Lemma 4.2. *Let* R *and* S *be two rings,* J *an ideal of* S*,* I *an ideal of* R *and* $f : R \to S$ *be a ring homomorphism.*

(i) I *is finitely generated (resp. a principal) if and only* $I \bowtie^f f(I)J$ *is finitely generated (resp. a principal) ideal of* $R \bowtie^f J$.

(ii) *If* $I \bowtie^f f(I)J$ *is a locally principal ideal of* $R \bowtie^f J$*, then* I *is a locally principal ideal of* R.

(iii) *Suppose that* $J \subseteq Jac(S)$*. If* I *is locally principal of* R *then* $I \bowtie^f f(I)J$ *is a locally principal ideal of* $R \bowtie^f J$.

(iv) *Suppose that* $I \bowtie^f f(I)J$ *is regular and* $J = (f(a)+j)J$ *for all* $(a, f(a)+j) \in Reg(R \bowtie^f J)$*. If* I *is a locally principal ideal of* R *then* $I \bowtie^f f(I)J$ *is a locally principal ideal of* $R \bowtie^f J$.

(v) *Suppose that* J *is regular and* $I = aI$ *for all* $a \in Reg(R)$*. If* J *is a locally principal ideal of* R *then* $J \bowtie I$ *is a locally principal ideal of* $R \bowtie I$.

Proof. (i) Let $I =< r_1, ..., r_n >$ be a finitely generated ideal of R. We claim that $I \bowtie^f f(I)J =< (r_1, f(r_1)), ..., (r_n, f(r_n)) >$.
For that, let $(a, f(a) + \Sigma_{k=1}^m f(b_k)j_k) \in I \bowtie^f f(I)J$, then:

$$\begin{aligned}(a, f(a) + \Sigma_{k=1}^m f(b_k)j_k) &= (\Sigma_{i=1}^n \alpha_i r_i, f(\Sigma_{i=1}^n \alpha_i r_i) + \Sigma_{k=1}^m f(\Sigma_{i=1}^n \beta_{k,i} r_i)j_k) \\ &= (\Sigma_{i=1}^n \alpha_i r_i, \Sigma_{i=1}^n f(\alpha_i)f(r_i)) + (0, \Sigma_{k=1}^m \Sigma_{i=1}^n f(\beta_{k,i})f(r_i)j_k) \\ &= \Sigma_{i=1}^n (\alpha_i, f(\alpha_i))(r_i, f(r_i)) + \Sigma_{i=1}^n (0, \Sigma_{k=1}^m f(\beta_k)j_k)(r_i, f(r_i)) \\ &= \Sigma_{i=1}^n [(\alpha_i, f(\alpha_i)) + (0, \Sigma_{k=1}^m f(\beta_k)j_k)](r_i, f(r_i))\end{aligned}$$

As desired.
Conversely, suppose that $I \bowtie^f f(I)J =< (r_1, f(r_1)+j_1), ..., (r_1, f(r_n)+j_n) >$, we show that $I =< r_1, ..., r_n >$. So let $a \in I$, $(a, f(a)) \in I \bowtie^f f(I)J$, we get then:

$$\begin{aligned}(a, f(a)) &= \Sigma_{i=1}^n (\alpha_i, f(\alpha_i) + \beta_i)(r_i, f(r_i) + j_i) \\ &= (\Sigma_{i=1}^n \alpha_i r_i, \Sigma_{i=1}^n (f(\alpha_i) + \beta_i)(f(r_i) + j_i))\end{aligned}$$

Hence, $a = \Sigma_{i=1}^n \alpha_i r_i$, which completes the proof. Similarly, we prove that I is a principal ideal of R if and only if so is $I \bowtie^f f(I)J$.

(ii) Suppose that $I \bowtie^f f(I)J$ is a locally principal ideal of $R \bowtie^f J$ and let $M \in Max(R, I)$. Using the ring homomorphism induced by φ defined in [20, Lemma 2.6.], we prove that:
If $M \in Max(R, I) \backslash V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ where $M'^f = M \bowtie^f J$, as desired.
If $M \in Max(R, I) \cap V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ where $T_M = f(R - M) + J$ is a multiplicative set of S and f_M is the ring homomorphism induced by f for all $M \in Max(R)$. We get the result by (1).

(iii) If $J \subseteq Jac(B)$, hence all maximal ideals contining $I \bowtie^f f(I)J$ have the form $M \bowtie^f J$ where $M \in Max(R, I)$. The remaining proof is straightforward since $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ if $M \in Max(R, I) \backslash V(f^{-1}(J))$ and $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ if $M \in Max(R, I) \cap V(f^{-1}(J))$.

(iv) It's obvious since $I \bowtie^f f(I)J = I \bowtie^f J$ by Lemma 4.1. The remaining proof is similar to (2).

(v) The result holds by (1) and by the fact that $H_{M'} = (J \bowtie I)_{M'} \cong J_M$ if $M \in Max(R) \backslash V(I)$ where $M' = M \bowtie I$ and $H_{M'} = (J \bowtie I)_{M'} \cong J_M \bowtie I_M$ if $M \in Max(R) \cap V(I)$.

□

At present, we study the possible transfer of the properties of being a weak π-ring, a $(*)$-ring and a strongly $(*)$-ring between a commutative ring R and $R \bowtie^f J$.

Theorem 4.3. *Let R and S be two rings, J an ideal of S, $f : R \to S$ be a ring homomorphism and $R \bowtie^f J$.*

(i) Suppose $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$. If $R \bowtie^f J$ is a weak π-ring, so is R.

(ii) Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is a weak π-ring, then so is $R \bowtie^f J$.

(iii) Suppose that $J = (f(a) + j)J$ for all $(a, f(a) + j) \in Reg(R \bowtie^f J)$. If $R \bowtie^f J$ is a $(*)$-ring, then every regular principal ideal $I =< r >$ of R such that $(r, f(r)+j)$ is regular for some $j \in J$ has a primary decomposition. Moreover, if $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$ then R is a $(*)$-ring.

(iv) Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is a $(*)$-ring, then so is $R \bowtie^f J$.

(v) Suppose that $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If $R \bowtie^f J$ is a strongly $(*)$-ring, then every invertible ideal I of R such that $I \bowtie^f f(I)J$ is regular has a primary decomposition. In addition to that, if $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$ then R is a strongly $(*)$-ring.

(vi) Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is a strongly $(*)$-ring, then so is $R \bowtie^f J$.

(vii) If $f^{-1}(J) = 0$, then $R \bowtie^f J$ is a weak π-ring (resp. $(*)$-ring, strongly $(*)$-ring) if and only if so is $f(R) + J$.

Proof. (i) Let $I =< r >$ be a regular ideal of R. As $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$, $H =< (r, f(r)) >$ is a regular ideal of $R \bowtie^f J$, and so $H = P_1'^f ... P_n'^f \overline{Q}_1^f ... \overline{Q}_m^f$ where $P_k'^f$'s and $\overline{Q}_l^f$'s are prime ideals of $R \bowtie^f J$. We show that $I = P_1 ... P_n f^{-1}(Q_1) ... f^{-1}(Q_m)$. For that, let $\alpha \in R$:
$(\alpha, f(\alpha))(r, f(r)) = \Sigma_{l=1}^d (p_1^l, f(p_1^l))...(p_n^l, f(p_n^l))(q_1^l, f(q_1^l))...(q_m^l, f(q_m^l))$ where for all $l = 1, ..., d$ we have $p_t^l \in P_t$ for all $t = 1, ..., n$ and $q_h^l \in Q_h$ for all $h = 1, ..., m$. Then:
$(\alpha, f(\alpha))(r, f(r)) = \Sigma_{l=1}^d (p_1^l ... p_n^l q_1^l ... q_m^l, f(p_1^l ... p_n^l q_1^l ... q_m^l))$, so $\alpha r = \Sigma_{l=1}^d p_1^l ... p_n^l q_1^l ... q_m^l$, and thus $I \subseteq P_1 ... P_n f^{-1}(Q_1) ... f^{-1}(Q_m)$.
Now, let $p_1 ... p_n q_1 ... q_m \in P_1 ... P_n f^{-1}(Q_1) ... f^{-1}(Q_m)$, then;
$(p_1 ... p_n q_1 ... q_m, f(p_1 ... p_n q_1 ... q_m)) = (p_1, f(p_1)) ... (p_n, f(p_n))(q_1, f(q_1))$
$...(q_m, f(q_m)) \in P_1'^f ... P_n'^f \overline{Q}_1^f ... \overline{Q}_m^f =< (r, f(r)) >$, so there exists $\alpha \in R$ such that $p_1 ... p_n q_1 ... q_m = \alpha r$, consequently $I = P_1 ... P_n f^{-1}(Q_1) ... f^{-1}(Q_m)$.

(ii) Let $H =< (r, f(r) + j) >$ be a regular ideal. As $J = (f(a) + j)J$ for all $(a, f(a) + j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 4.1 we get $H = I \bowtie^f J$ where $I =< r >$ is a regular ideal of R. As R is a weak π-ring, we get $I = P_1 ... P_n$ where P_l's are prime ideals of R, so $H = I \bowtie^f J = (P_1 ... P_n) \bowtie^f J \supseteq (P_1 \bowtie^f J)...(P_n \bowtie^f J)$. Now let $(a, f(a) + k) \in (P_1 ... P_n) \bowtie^f J$, so $(a, f(a) + k) = (\Sigma_{l=1}^m p_1^l ... p_n^l, f(\Sigma_{l=1}^m p_1^l ... p_n^l) + k) = \Sigma_{l=1}^m (p_1^l, f(p_1^l))...(p_n^l, f(p_n^l)) + (0, k) = \Sigma_{l=1}^m (p_1^l, f(p_1^l))...(p_n^l, f(p_n^l) + (a, f(a) + k)...(a, f(a) + k)(0, k')$ since $(a^{n-1}, (f(a) + k)^{n-1})$ is regular and $k = (f(a) + k)^{n-1} k'$ for some $k' \in J$, hence $(P_1 ... P_n) \bowtie^f J \subseteq (P_1 \bowtie^f J)...(P_n \bowtie^f J)$. Consequently, $R \bowtie^f J$ is a weak π-ring.

(iii) Let $I =< r >$ be a regular ideal of R. By hypothesis, $< (r, f(r) + j) >= I \bowtie^f J$ is regular for some $j \in J$. Thus $I \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J) = (P_1 \cap ... \cap P_n) \bowtie^f J$ where

P_l's are primary ideals, since $R \bowtie^f J$ is a $(*)$-ring. Hence I has a primary decomposition. The "moreover" statement is clear since $(r, f(r))$ is regular for each $r \in Reg(R)$, and the remaining proof is similar to the last one.

(iv) Suppose R is a $(*)$-ring, and let $H =< (r, f(r)+j) >$ be a regular ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 4.1 we get $H = I \bowtie^f J$ where $I =< r >$ is a regular ideal of R. Since R is a $(*)$-ring, it follows that $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = I \bowtie^f J = (P_1 \cap ... \cap P_n) \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J)$ where $P_l \bowtie^f J$ is a primary ideal for all $l = 1, ..., n$. Consequently, $R \bowtie^f J$ is a $(*)$-ring.

(v) Let I be an invertible ideal of R such that $I \bowtie^f f(I)J$ is regular. By hypothesis and lemma 4.2, we get $I \bowtie^f f(I)J = I \bowtie^f J$ an invertible ideal of $R \bowtie^f J$ which is strongly $(*)$-ring. Thus $I \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J) = (P_1 \cap ... \cap P_n) \bowtie^f J$ and hence I has a primary decomposition.
The "moreover" statement is similar to the one of (3).

(vi) Suppose that R is a strongly $(*)$-ring, and let H be an invertible ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 4.2 and Lemma 4.1, we get $H = I \bowtie^f J$ where I is an invertible ideal of R. Since R is a strongly $(*)$-ring, it follows that $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = I \bowtie^f J = (P_1 \cap ... \cap P_n) \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J)$. Consequently, $R \bowtie^f J$ is a strongly $(*)$-ring.

(vii) If $f^{-1}(J) = 0$, then $R \bowtie^f J \cong f(R) + J$, as desired.

□

Corollary 4.4. *Let R and S be two rings, J an ideal of S and $f : R \to S$ be a ring homomorphism.*

(i) Suppose that $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$. If $R \bowtie^f J$ is an almost weak π-ring, then so is R.

(ii) Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is an almost weak π-ring, then so is $R \bowtie^f J$.

Proof. (i) Let $I =< r >$ be a regular ideal of R, so $I \bowtie^f f(I)J$ is a regular principal ideal of $R \bowtie^f J$ by hypothesis and by Lemma 4.2. Now let $M \in Max(R, I)$, two cases are then possible:
If $M \in Max(R, I) \backslash V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ where $M'^f = M \bowtie^f J$, as desired.
If $M \in Max(R, I) \cap V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ where $T_M = f(R-M)+J$ is a multiplicative set of S and f_M is the ring homomorphism induced by f for all $M \in Max(R)$. By applying the same reasoning of the proof of Theorem 4.3(1), we obtain the desired result.

(ii) Let $H =< (r, f(r)+j) >$ be a regular ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 4.2 we get $H = I \bowtie^f J$ where $I =< r >$ is a regular ideal of R. As R is almost weak π-ring, we get $I_M = P_{1M}...P_{nM}$ for all $M \in Max(R, I)$ where P_l's are prime ideals of R. Two cases are then possible:
If $M \in Max(R) \backslash V(f^{-1}(J))$, so $H_{M'^f} = (I \bowtie^f J)_{M'^f} \cong I_M$, as desired.
If $M \in Max(R) \cap V(f^{-1}(J))$, so $H_{M'^f} = (I \bowtie^f J)_{M'^f} \cong I_M \bowtie^{f_M} J_{T_M}$. Moreover we have $I_M \bowtie^{f_M} J_{T_M} = P_{1M}...P_{nM} \bowtie^{f_M} J_{T_M}$. By the same way of the proof of theorem 4.3(2), we get $P_{1M}...P_{nM} \bowtie^{f_M} J_{T_M} = P_{1M} \bowtie^{f_M} J_{T_M}...P_{nM} \bowtie^{f_M} J_{T_M}$.
Finally, $R \bowtie^f J$ is an almost weak π-ring.

□

We next present necessary and sufficient conditions for $A \bowtie I$ to be a weak π-ring, a $(*)$-ring and a strongly $(*)$-ring.

Theorem 4.5. *Let R be a ring and I be an ideal of R.*

(i) If $R \bowtie I$ is a weak π-ring, then so is R.

(ii) Suppose that $I = aI$ for all $a \in Reg(R)$. If R is a weak π-ring, then so is $R \bowtie I$.

(iii) If R is an arithmetical weak π-ring and I a finitely generated ideal of R satisfying $I_M = 0$ for all $M \in Max(R, I)$, then $R \bowtie I$ is a weak π-ring.

(iv) Suppose that $I = aI$ for all $a \in Reg(R)$. $R \bowtie I$ is a $(*)$-ring if and only if so is R.

(v) Suppose that $I = aI$ for all $a \in Reg(R)$. $R \bowtie I$ is a strongly $(*)$-ring if and only if so is R.

Proof. (i) Holds by Theorem 4.3(1).

(ii) Suppose that R is a weak π-ring.
If I is regular, then there exists $x \in I$ a regular element of R, since $I = aI$ for all $a \in Reg(R)$, we get $x = xk$ for some $k \in I$ which implies that $k = 1$ and hence $I = R$. Finally, we obtain $R \bowtie I = R \times R$ which is a weak π-ring if and only if so is R.
If I is not regular, let $H =< (r, r + i) >$ be a regular principal ideal of $R \bowtie I$. We claim that r and $r + i$ are regular elements.
Assume that there exists $x \in R - \{0\}$ such that $rx = 0$:

- If $x \in Ann(I)$, then $(r, r + i)(x, x) = (0, 0)$, contradiction since $(r, r + j)$ is regular.
- If is not, then there exists $k \in I$ such that $xk \neq 0$, so $(r, r + i)(xk, 0) = (0, 0)$, contradiction since $(r, r + j)$ is regular.

Assume that there exists $y \in R - \{0\}$ such that $(r + i)y = 0$:

- If $y \in Ann(I)$, then $(r + i)y = 0 \Rightarrow ry = 0$, contradiction since r is regular.
- If is not, then there exists $k \in I$ such that $yk \neq 0$, so $(r, r + i)(0, yk) = (0, 0)$, contradiction since $(r, r + j)$ is regular.

As $I = aI$ for all $a \in Reg(R)$, we get for all $k \in I$ there exists $k' \in I$ such that $k = (r+i)k'$, so $(0, k) = (0, k')(r, r + i) \in H$ and $0 \times I \subseteq H$ and hence $H = J \bowtie I$ where $J =< r >$ is a regular principal ideal of R. As R is weak π-ring, we get $J = P_1...P_n$ where P_l's are prime ideals, so $H = J \bowtie I = P_1...P_n \bowtie I \supseteq (P_1 \bowtie I)...(P_n \bowtie I)$. Now let $(a, a + j) \in P_1...P_n \bowtie I$, so $(a, a + j) = (\Sigma_{l=1}^m p_1^l...p_n^l, \Sigma_{l=1}^m p_1^l...p_n^l + j) = \Sigma_{l=1}^m (p_1^l, p_1^l)...(p_n^l, p_n^l) + (0, j) = \Sigma_{l=1}^m (p_1^l, p_1^l)...(p_n^l, p_n^l) + (a, a)...(a, a)(0, j')$ since a^{n-1} is a regular element and $j = a^{n-1}j'$ for some $j' \in I$, hence $P_1...P_n \bowtie I \subseteq (P_1 \bowtie I)...(P_n \bowtie I)$. Consequently, $R \bowtie I$ is a weak π-ring.

(iii) Let H be a regular prime ideal of $R \bowtie I$.
Case 1: $H = P' = P \bowtie I$, so P is a regular prime. Since R is a weak π-ring, it follows that $K \subseteq P$ for some invertible prime ideal K of R. Now we show that the prime ideal $K \bowtie I$ is invertible. Since I is finitely generated, we have $K \bowtie I$ is a finitely generated regular ideal. By [6, Corollary 3.8] $R \bowtie I$ is an arithmetical ring, thus $K \bowtie I$ is locally principal. Consequently, $K \bowtie I$ is an invertible prime ideal.
Case 2: $H = \overline{P}$, so P is a regular prime. Since R is a weak π-ring, it follows that $K \subseteq P$ for some invertible prime ideal K of R. It's clear that $\overline{K}$ is regular. Since K and I are finitely generated, it follows that $\overline{K}$, indeed :
Let $K =< (k_j)_{j=1}^n >$, $I =< (i_l)_{l=1}^m >$ and $(r, r + i) \in \overline{K}$, we have $r + i = \Sigma_{j=1}^n a_j k_j$ and $i = \Sigma_{l=1}^m b_l i_l$, so $r = \Sigma_{j=1}^n a_j k_j - \Sigma_{l=1}^m b_l i_l$, then:
$(r, r + i) = (\Sigma_{j=1}^n a_j k_j - \Sigma_{l=1}^m b_l i_l, \Sigma_{j=1}^n a_j k_j) = \Sigma_{j=1}^n (a_j, a_j)(k_j, k_j) - \Sigma_{l=1}^m (b_l, b_l)(i_l, 0)$.
Since R is an arithmetical ring, it follows that $\overline{K}$ is locally principal, and hence an invertible prime ideal. Finally, $R \bowtie I$ is a weak π-ring.

(iv) Suppose $R \bowtie I$ a $(*)$-ring and let $J =< r >$ be a regular ideal of R. By the same way in the proof of statement (2), $< (r, r) >= J \bowtie I$ is regular. Hence by hypothesis, $J \bowtie I = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J) = (P_1 \cap ... \cap P_n) \bowtie^f J$ where P_l's are primary ideals. Therefore, J has a primary decomposition.

Conversely, suppose R is a $(*)$-ring, and let $H =< (r, r + i) >$ be a regular ideal. Similarly to the proof of Theorem 4.3, $H = J \bowtie I$ where $J =< r >$ is a regular ideal of R. We get then $J = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so

$H = (P_1 \cap ... \cap P_n) \bowtie I = (P_1 \bowtie I) \cap ... \cap (P_n \bowtie I)$ where $P_l \bowtie I$ are primary ideals for all $l = 1, ..., n$. Consequently, $R \bowtie I$ is a $(*)$-ring.

(v) Assume that $R \bowtie I$ is a strongly $(*)$-ring and let J be an invertible ideal of R. By applying lemma 4.2, we get $J \bowtie I$ an invertible ideal, and so $J \bowtie I = (P_1 \bowtie I) \cap ... \cap (P_n \bowtie I) = (P_1 \cap ... \cap P_n) \bowtie I$, as desired.
Conversely, suppose R is a strongly $(*)$-ring, and let H be an invertible ideal of $R \bowtie I$. Similarly to the proof of Theorem 4.3, $H = J \bowtie I$ where J is an invertible ideal of R. We get then $J = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = (P_1 \cap ... \cap P_n) \bowtie I = (P_1 \bowtie I) \cap ... \cap (P_n \bowtie I)$ where $P_l \bowtie I$ are primary ideals for all $l = 1, ..., n$, which completes the proof.

□

The condition "$I = aI$ for all $a \in Reg(R)$" is necessary in Theorem 4.5(2).

Example 4.6. Let $A = \mathbb{Z}_2$, $E = A/2A$, $R = A \propto E$ and $I = 0 \propto E$.
We have $R \bowtie I = R \propto I$ is a total quotient ring, hence a weak π-ring. Clearly for all $r \in Reg(R)$, $rI \subsetneqq I$. Similarly to the proof of Example 2.3, we prove that $R \bowtie I$ is not a weak π-ring.

The result in the next corollary follows at once from Theorem 4.5.

Corollary 4.7. *Let R be a ring, I an ideal of R and $R \bowtie I$.*

(i) If $R \bowtie I$ is an almost weak π-ring, then so is R.

(ii) Suppose that $I = aI$ for all $a \in Reg(R)$. If R is an almost weak π-ring, then so is $R \bowtie I$.

Proof. (i) Follows from Corollary 4.4(1).

(ii) Let $H =< (r, r + i) >$ be a regular ideal. Similarly to the previous proof we have $H = J \bowtie I$ where $J =< r >$ is a regular ideal of R. Since R is almost weak π-ring, we get $J_M = P_{1M}...P_{nM}$ for all $M \in Max(R, J)$ where P_l's are prime ideals of R. Two cases are then possible:
If $M \in Max(R) \backslash V(I)$, so $H_{M'} = (J \bowtie I)_{M'} \cong J_M$ where $M' = M \bowtie I$, as desired.
If $M \in Max(R) \cap V(I)$, so $H_{M'} = (J \bowtie I)_{M'} \cong J_M \bowtie I_M$. As R is almost weak π-ring, we have $J_M \bowtie I_M = P_{1M}...P_{nM} \bowtie I_M = P_{1M} \bowtie I_M...P_{nM} \bowtie I_M$, that is to say $R \bowtie^f J$ is an almost weak π-ring.

□

We close this section with some examples of application of our results. We construct the following examples of a weak π-ring (a strongly $(*)$-ring) for the amalgamated duplication along an ideal and the amalgamation of rings along an ideal with respect to f.

Example 4.8. Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$ and $I = 0 \propto 2\mathbb{Z}/4\mathbb{Z}$. Thus $R \bowtie I$ is a weak π-ring and hence a strongly $(*)$-ring.

Proof. By Example 2.3, R is a weak π-ring. We claim that $I = aI$ for all $a \in Reg(R)$. Indeed, since $Reg(R) = \{(2k + 1, \overline{s}^*) \in \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z} | k \in \mathbb{Z}\}$ and $J =< (0, \overline{2}) >$, we get $(0, \overline{2}) = (0, \overline{4k} + \overline{2}) = (2k + 1, \overline{z})(0, \overline{2})$ for all $k, z \in \mathbb{Z}$. Hence $R \bowtie I$ is a weak π-ring and hence a strongly $(*)$-ring. □

Example 4.9. Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$, $S = \mathbb{Z}/6\mathbb{Z}$, f a ring homomorphism defined by $f((r, \overline{s}^*)) = \overline{r}$ and $J = 3\mathbb{Z}/6\mathbb{Z}$. Therefore $R \bowtie^f J$ is a weak π-ring and thus a strongly $(*)$-ring.

Proof. By Example 2.3, R is a weak π-ring. We claim that $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. Indeed, let $(a, f(a) + j) \in Reg(R \bowtie^f J)$, since $(3, \overline{0}^*) \in f^{-1}(J)$, it follows that $Ann(f^{-1}(J)) = 0$, by Lemma 4.2 a is a regular element of R. Now we prove that if $(a, f(a)+j)$ is regular, then $j = \overline{0}$. Indeed, as $Reg(R) = \{(2k + 1, \overline{s}^*) \in \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z} | k \in \mathbb{Z}\}$, it follows that $f(Reg(R)) = \{\overline{1}, \overline{3}, \overline{5}\}$. Clearly $J = \{\overline{0}, \overline{3}\}$, so three cases are possible:
$f(a) + j = \overline{1} + \overline{3} = \overline{4}$, so $(a, f(a) + j)(0, \overline{3}) = (0, \overline{0})$, contradiction.

$f(a)+j=\overline{3}+\overline{3}=\overline{0}$, so $(a, f(a)+j)(0,\overline{3})=(0,\overline{0})$, contradiction.
$f(a)+j=\overline{5}+\overline{3}=\overline{8}$, so $(a, f(a)+j)(0,\overline{3})=(0,\overline{0})$, contradiction.
We get then, $\overline{3}=\overline{1}\times\overline{3}$, $\overline{3}=\overline{3}\times\overline{3}$ and $\overline{3}=\overline{5}\times\overline{3}$. Hence $R\bowtie^f J$ is a weak π-ring and thus a strongly $(*)$-ring. □

Example 4.10. Let $R=\mathbb{Z}/8\mathbb{Z}$ and $I=2\mathbb{Z}/8\mathbb{Z}$. R is a weak π-ring since R is a total quotient ring. Clearly I which is not regular satisfies $I=\overline{a}I$ for all $\overline{a}\in Reg(R)$. Therefore $R\bowtie I$ is a weak π-ring and thus a strongly $(*)$-ring.

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