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More Properties of (Multiplicative) Hom-Lie Algebras

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Abstract The concept of (regular) (multiplicative) Hom-Lie algebras is studied further. Many new examples are given to illustrate this structure. It is shown, among other things, that if L is any Hom-Lie algebra over any field of characteristic $\neq 2$, then its bracket makes it a Lie algebra too. We define infinite direct products of Hom-Lie algebras and provide some of properties about it. Finally, it is proved that the isomorphism theorems for Lie algebras are also satisfied for Hom-Lie algebras.

1 Introduction

The notion of Hom-Lie algebras was originally introduced and studied by Hartwig, Larsson, and Silvestrov in [8] in 2006. It is one of generalizations for the notion of Lie algebras. In recent years, they have become an interesting topic of mathematics and physics.

A Hom-Lie algebra is a vector space equipped with a linear map and an alternating non associative bilinear map satisfying Hom-Jacobi identity. (Restricted) Hom-Lie (color-)(super-)algebras have been widely studied during the last years (see e.g. [2, 5, 6, 14, 15, 18, 20], and references therein).

In this paper, we discuss (Multiplicative) Hom-Lie algebras. The direct product of Hom-Lie algebras (not necessarily finite) is introduced and many of their properties are established. Many known theorems and results concerning Hom-Lie algebras are reviewed with possible improvements. Also, many examples are given for the purpose of illustration. Finally, many properties involving Hom-Lie algebra morphisms are studied. We show that there are isomorphism theorems for Hom-Lie algebras as there for vector spaces, for groups, for rings, and for Lie algebras.

2 Basic Definitions and Examples

Let F be a ground field. A Hom-Lie algebra over F is a triple $(L,\ [\ ,\ ],\ \alpha)$ where L is a vector space over F, $\alpha : L \to L$ is a linear map, and $[\ ,\ ] : L \times L \to L$ is a bilinear map (called a bracket), satisfying the following properties:

(i) $[x,\ y] = -[y,\ x]$ for all $x, y \in L$ (skew-symmetry property).

(ii) $[\alpha(x),\ [y,\ z]]+[\alpha(y),\ [z,\ x]]+[\alpha(z),\ [x,\ y]] = 0$, for all $x, y, z \in L$ (Hom-Jacobi identity).

It is clear that every Lie algebra (see e.g. [4, 16]) is a Hom-Lie algebra by setting $\alpha = id_L$ (The identity map). On the other hand the bracket corresponding to a Hom-Lie algebra need not to define a Lie algebra [5]. For a Hom-Lie algebra L over a field F of characteristic $\neq 2$, as in the setting of Lie algebras one can show that $[x,\ x] = 0$ for each $x \in L$. Also for an arbitrary Hom-Lie algebra L, we have $[x,\ 0] = [0, x] = 0$ for each $x \in L$.

Example 2.1. ([8]) Let L be a vector spaces L and α any linear map. Then $(L,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra, where $[x,\ y] = 0$ for all $x, y \in L$. Such Hom-Lie algebras are called abelian or commutative Hom-Lie algebras. ■

Example 2.2. Let L be a vector space and $[\ ,\ ] : L \times L \to L$ be any skew-symmetric bilinear map. If $\alpha : L \to L$ is the zero map, then $(L,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra. ■

Example 2.3. Let L be a vector space over F with basis $\{e_1,\ e_2,\ e_3\}$. Define the linear map $\alpha : L \to L$ by setting $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = e_1 + e_2 + e_3$. Let $[\ ,\ ] : L \times L \to L$ be the skew-symmetric bilinear map such that

$$[e_1,\ e_2] = [e_1,\ e_3] = [e_2,\ e_3] = e_1 + e_2 + e_3,$$

and $[e_i,\ e_i] = 0$ for all $i = 1, 2, 3$. Then $(L,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra. Indeed the Hom-Jacobi identity follows directly from the fact that for $x, y \in L$ we have $[x,\ y]$ and $\alpha(x)$ will be scalar multiples of $(e_1 + e_2 + e_3)$, and so for any $x, y, z \in L$ we obtain

$$[\alpha(x),\ [y,\ z]] + [\alpha(y),\ [z,\ x]] + [\alpha(z),\ [x,\ y]] = 0.$$

■

Example 2.4. Let L be a vector space over F with basis $\{e_1,\ e_2,\ e_3\}$. Define the linear map $\alpha : L \to L$ by setting $\alpha(e_1) = e_2$ and $\alpha(e_2) = \alpha(e_3) = 0$. Let $[\ ,\ ] : L \times L \to L$ be the skew-symmetric bilinear map such that

$$[e_1,\ e_2] = [e_2,\ e_3] = 0,\ [e_1,\ e_3] = e_1$$

and $[e_i,\ e_i] = 0$ for all $i = 1, 2, 3$. Then $(L,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra. We note that $[x,\ y]$ is a scalar multiple of e_1 for each $x, y \in L$, and also $\alpha(x)$ is a scalar multiple of e_2 for each $x \in L$. Thus $[\alpha(x),\ [y,\ z]] = 0$ for each $x, y, z \in L$. This implies that the Hom-Jacobi identity is satisfied. ■

3 Example of a Hom-Lie Algebra that is not a Lie Algebra

The following example, appeared in [5] without proof, shows that the bracket corresponding to a Hom-Lie algebra need not to define a Lie algebra.

Example 3.1. Let L be a vector space over F with basis $\{e_1,\ e_2,\ e_3\}$. Define the linear map $\alpha : L \to L$ by setting $\alpha(e_1) = e_1$, $\alpha(e_2) = 2e_2$, and $\alpha(e_3) = 2e_3$. Let $[\ ,\ ] : L \times L \to L$ be the skew-symmetric bilinear map such that

$$[e_1,\ e_2] = e_1,\ [e_1,\ e_3] = 2e_2,\ [e_2,\ e_3] = 2e_3,$$

and for all $i = 1, 2, 3$ we have $[e_i,\ e_i] = 0$. Then $(L,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra. Indeed for $y = b_1e_1 + b_2e_2 + b_3e_3, z = c_1e_1 + c_2e_2 + c_3e_3 \in L$ ($b_i, c_i \in F, i = 1, 2, 3$), we have

$$\begin{aligned}
[y,\ z] &= [b_1e_1 + b_2e_2 + b_3e_3,\ c_1e_1 + c_2e_2 + c_3e_3] \\
&= b_1c_1[e_1,\ e_1] + b_1c_2[e_1,\ e_2] + b_1c_3[e_1,\ e_3] \\
&\quad + b_2c_1[e_2,\ e_1] + b_2c_2[e_2,\ e_2] + b_2c_3[e_2,\ e_3] \\
&\quad + b_3c_1[e_3,\ e_1] + b_3c_2[e_3,\ e_2] + b_3c_3[e_3,\ e_3] \\
&= b_1c_2e_1 + 2b_1c_3e_2 - b_2c_1e_1 + 2b_2c_3e_3 - 2b_3c_1e_2 - 2b_3c_2e_3 \\
&= (b_1c_2 - b_2c_1)e_1 + 2(b_1c_3 - b_3c_1)e_2 + 2(b_2c_3 - b_3c_2)e_3.
\end{aligned}$$

Now, for an another arbitrary element $x = a_1e_1 + a_2e_2 + a_3e_3$ in L, we find

$$[\alpha(x),\ [y,\ z]] = [a_1e_1 + 2a_2e_2 + 2a_3e_3,\ (b_1c_2 - b_2c_1)e_1 + 2(b_1c_3 - b_3c_1)e_2 + 2(b_2c_3 - b_3c_2)e_3].$$

So that, after some calculations we can get

$$\begin{aligned}
[\alpha(x),\ [y,\ z]] &= 2(a_1b_1c_3 - a_1b_3c_1 - a_2b_1c_2 + a_2b_2c_1)e_1 \\
&\quad + 4(a_1b_2c_3 - a_1b_3c_2 - a_3b_1c_2 + a_3b_2c_1)e_2 \\
&\quad + 8(a_2b_2c_3 - a_2b_3c_2 - a_3b_1c_3 + a_3b_3c_1)e_3.
\end{aligned}$$

Now, according to the formula above and some calculations, we can get

$$[\alpha(x),\ [y,\ z]] + [\alpha(y),\ [z,\ x]] + [\alpha(z),\ [x,\ y]] = 0.$$

On the other hand $(L, [\ ,\ ])$ is not a Lie algebra. Indeed

$$\begin{aligned}[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] &= [e_1, 2e_3] + [e_2, -2e_2] + [e_3, e_1] \\ &= 4e_2 + 0 + (-2e_2) \\ &= 2e_2 \neq 0.\end{aligned}$$

■

4 2-Dimensional Hom-Lie Algebras

The following theorem was given in [11] without proof to the case where L is a 2-dimensional space over a field of characteristic 0. We are going to prove it if characteristic $(F) \neq 2$.

Theorem 4.1. *Let L be a 2-dimensional vector space over a field F with characteristic $\neq 2$. Then every skew-symmetric bilinear map $[\ ,\ ] : L \times L \to L$ defines a Hom-Lie algebra.*

Proof. Let L be a vector space over F with basis $\{e_1, e_2\}$. Suppose that $\alpha : L \to L$ is any arbitrary linear map. Then $\alpha(e_1) = n_1e_1 + m_1e_2$ and $\alpha(e_2) = n_2e_1 + m_2e_2$ for some $n_1, n_2, m_1, m_2 \in F$. Also we can assume $[e_1, e_2] = se_1 + te_2$ for some $s, t \in F$. So for $y = b_1e_1 + b_2e_2,\ z = c_1e_1 + c_2e_2 \in L$ where $b_1, b_2, c_1, c_2 \in F$, we have

$$\begin{aligned}[y, z] &= [b_1e_1 + b_2e_2, c_1e_1 + c_2e_2] \\ &= (b_1c_2 - b_2c_1)(se_1 + te_2).\end{aligned}$$

Now, for an another arbitrary element $x = a_1e_1 + a_2e_2 \in L$ where $a_1, a_2 \in F$, we find

$$[\alpha(x), [y, z]] = [(n_1a_1 + n_2a_2)e_1 + (m_1a_1 + m_2a_2)e_2, (b_1c_2 - b_2c_1)se_1 + (b_1c_2 - b_2c_1)te_2].$$

So that, after some calculations we can get

$$\begin{aligned}[\alpha(x), [y, z]] &= (t(n_1a_1b_1c_2 + n_2a_2b_1c_2 - n_1a_1b_2c_1 - n_2a_2b_2c_1) \\ &\quad -s(m_1a_1b_1c_2 + m_2a_2b_1c_2 - m_1a_1b_2c_1 - m_2a_2b_2c_1))[e_1, e_2].\end{aligned}$$

Now, according to the formula above and some calculations, we can get

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0.$$

□

Corollary 4.2. *The bracket defined in any 2-dimensional Hom-Lie algebra over a field of characteristic $\neq 2$ makes it a Lie algebra too.*

Proof. This follows from our calculations in the proof of the theorem above by setting $\alpha = \mathrm{id}_L$. □

5 Hom-Lie Subalgebras and Ideals

A subset H of a Hom-Lie algebra L is called a Hom-Lie subalgebra if H itself is a Hom-Lie algebra under the same operations corresponding to the Hom-Lie algebra L. One can easily prove the following result.

Theorem 5.1. *Let $(L, [\ ,\ ], \alpha)$ be a Hom-Lie algebra. A subspace H of L is a Hom-Lie subalgebra if $\alpha(H) \subseteq H$ and $[x, y] \in H$ for all $x, y \in H$.*

Definition 5.2. ([6]) A Hom-Lie subalgebra H is said to be a Hom-Lie ideal if $[x, y] \in H$ for all $x \in H$ and $y \in L$.

Example 5.3. Let $(L, [\ ,\ ], \alpha)$ be a Hom-Lie algebra. Then $\{0\}$ and L are Hom-Lie ideals of L. They are called the trivial ideals. ■

Example 5.4. In Example 3.1, the subspace $H = \text{Span}(\{e_1,\ e_2\})$ is a Hom-Lie subalgebra. Note that $\alpha(H) \subseteq H$ since $\alpha(e_1) = e_1 \in H$ and $\alpha(e_2) = 2e_2 \in H$. Also, $[e_1,\ e_2] = e_1 \in H$ implies $[x,\ y] \in H$ for any $x, y \in H$. But H is not ideal. Indeed $[e_2,\ e_3] = 2e_3 \notin H$. ■

Example 5.5. In Example 2.3, the subspace $H = \text{Span}(\{e_1+e_2+e_3\})$ is a Hom-Lie ideal. Indeed $\alpha(H) \subseteq H$ since $\alpha(e_1 + e_2 + e_3) = 3(e_1 + e_2 + e_3) \in H$. Also, for all $x = ae_1 + be_2 + ce_3 \in L$ and for any $y = d(e_1 + e_2 + e_3) \in H$ we find $[y,\ x] = d(2c - 2a)(e_1 + e_2 + e_3)$. This means that $[x,\ y] \in H$. ■

Suppose H and K are Hom-Lie ideals of a Hom-Lie algebra L. The sum and multiplication of H and K are defined as follows:

$$H + K = \{h + k \mid h \in H \text{ and } k \in K\}$$

and

$$[H,\ K] = \text{Span}(\{[h,\ k] \mid h \in H \text{ and } k \in K\}).$$

Recall that $H \cap K$, $H + K$, and $[H,\ K]$ are subspaces of the space L.

Theorem 5.6. *([6]) Let H and K be Hom-Lie ideals of a Hom-Lie algebra $(L,\ [\ ,\ ],\ \alpha)$. Then,*

(i) $H \cap K$ and $H + K$ are Hom-Lie ideals

(ii) $[H,\ K] \subseteq H \cap K$

Remark 5.7. Suppose that H and K are Hom-Lie ideals of a Hom Lie algebra L. Then in the contrast of the case of Lie algebras, $[H,\ K]$ need not be a Hom-Lie subalgebra. Consider the Hom-Lie algebra given in Example 2.4, then $[L,\ L] = \text{Span}(\{e_1\})$ is not a Hom-Lie subalgebra because $\alpha(e_1) = e_2 \notin [L,\ L]$.

The center of a Hom-Lie algebra $(L,\ [\ ,\ ],\ \alpha)$ is the vector subspace $Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$ ([5]). Recall that the center of any Lie algebra is an ideal. The following example shows that the center of a Hom-Lie algebra need not be Hom-Lie subalgebra.

Example 5.8. We consider Example 2.3. For arbitrary $ae_1 + be_2 + ce_3 \in L$ ($a, b, c \in F$), we get $[e_1 - e_2 + e_3,\ ae_1 + be_2 + ce_3] = (b + c + a - c - a - b)(e_1 + e_2 + e_3) = 0$. On the other hand, if $x = se_1 + te_2 + ke_3 \in Z(L)(s, t, k \in F)$, then $[x,\ e_1] = [x,\ e_2] = [x,\ e_3] = 0$ and so $(-t-k)(e_1+e_2+e_3) = (s-k)(e_1+e_2+e_3) = (s+t)(e_1+e_2+e_3) = 0$. Because $\{e_1,\ e_2,\ e_3\}$ are linearly independent, it follows $-t - k = s - k = s + t = 0$. Thus $x = s(e_1 - e_2 + e_3) \in \text{Span}\{e_1 - e_2 + e_3\}$. This shows that

$$Z(L) = \text{Span}(\{e_1 - e_2 + e_3\}).$$

On the other hand, $Z(L)$ is not a Hom-Lie subalgebra of L, since $\alpha(e_1-e_2+e_3) = (e_1+e_2+e_3) \notin H$. ■

6 Multiplicative Hom-Lie Algebras and Important Example

Let $(L,\ [\ ,\ ],\ \alpha)$ be a Hom-Lie algebra. It is called **multiplicative** if $\alpha([x,\ y]) = [\alpha(x),\ \alpha(y)]$ for all $x, y \in L$. A multiplicative Hom-Lie algebra is called **regular** if α is bijective. The following theorem is a generalization of a known theorem for Lie algebras in the case of regular Hom-Lie algebras.

Example 6.1. Let $F = \mathbb{C}$ be the field of complex numbers. Consider the vector space $\mathbb{C}^2$ and define the linear map

$$\alpha : \mathbb{C}^2 \to \mathbb{C}^2;\ (x,\ y) \mapsto (-y,\ -x).$$

We define a map $[\ ,\ ] : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$, where

$$[(x_1,\ x_2),\ (y_1,\ y_2)] = (i(x_1y_2 - x_2y_1),\ i(x_1y_2 - x_2y_1)).$$

Then $(\mathbb{C}^2,\ [\ ,\ ],\ \alpha)$ is a regular Hom-Lie algebra. The bracket is skew-symmetric, since

$$\begin{aligned}[(x_1,\ x_2),\ (y_1,\ y_2)] &= (i(x_1y_2 - x_2y_1),\ i(x_1y_2 - x_2y_1)) \\ &= -(i(x_2y_1 - x_1y_2),\ i(x_2y_1 - x_1y_2)) \\ &= -[(y_1,\ y_2),\ (x_1,\ x_2)]\end{aligned}$$

The bracket is bilinear map, since for any $c_i, x_i, y_i, z_i \in \mathbb{C}, i = 1, 2$, we have

$$[c_1(x_1,\ x_2) + c_2(y_1,\ y_2),\ (z_1,\ z_2)] = [(c_1x_1 + c_2y_1,\ c_1x_2 + c_2y_2),\ (z_1,\ z_2)]$$

$$\begin{aligned}&= (i(c_1x_1z_2 + c_2y_1z_2 - c_1x_2z_1 - c_2y_2z_1),\ i(c_1x_1z_2 + c_2y_1z_2 - c_1x_2z_1 - c_2y_2z_1)) \\ &= c_1(i(x_1z_2 - x_2z_1),\ i(x_1z_2 - x_2z_1)) + c_2(i(y_1z_2 - y_2z_1),\ i(y_1z_2 - y_2z_1)) \\ &= c_1[(x_1,\ x_2),\ (z_1,\ z_2)] + c_2[(y_1,\ y_2),\ (z_1,\ z_2)].\end{aligned}$$

Also, the Hom-Jacobi identity is satisfied. For any $x_i, y_i, z_i \in \mathbb{C}, i = 1, 2$, we get
$[\alpha(x_1,\ x_2),\ [(y_1,\ y_2),\ (z_1,\ z_2)]]$

$$\begin{aligned}&= [(-x_2,\ -x_1),\ (i(y_1z_2 - y_2z_1),\ i(y_1z_2 - y_2z_1))] \\ &= ((y_1z_2 - y_2z_1)(x_2 - x_1),\ (y_1z_2 - y_2z_1)(x_2 - x_1)) \\ &= (x_2y_1z_2 - x_2y_2z_1 - x_1y_1z_2 + x_1y_2z_1,\ x_2y_1z_2 - x_2y_2z_1 - x_1y_1z_2 + x_1y_2z_1).\end{aligned}$$

Therefore
$[\alpha(x_1,\ x_2),\ [(y_1,\ y_2),\ (z_1,\ z_2)]] + [\alpha(y_1,\ y_2),\ [(z_1,\ z_2),\ (x_1,\ x_2)]] + [\alpha(z_1,\ z_2),\ [(x_1,\ x_2),\ (y_1,\ y_2)]]$

$$\begin{aligned}&= (x_2y_1z_2 - x_2y_2z_1 - x_1y_1z_2 + x_1y_2z_1,\ x_2y_1z_2 - x_2y_2z_1 - x_1y_1z_2 + x_1y_2z_1) \\ &\quad + (y_2z_1x_2 - y_2z_2x_1 - y_1z_1x_2 + y_1z_2x_1,\ y_2z_1x_2 - y_2z_2x_1 - y_1z_1x_2 + y_1z_2x_1) \\ &\quad + (z_2x_1y_2 - z_2x_2y_1 - z_1x_1y_2 + z_1x_2y_1,\ z_2x_1y_2 - z_2x_2y_1 - z_1x_1y_2 + z_1x_2y_1) \\ &= (0,\ 0).\end{aligned}$$

Thus, $(\mathbb{C}^2,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra. Also it is regular. Indeed one can easily prove that α is bijective, and also

$$\begin{aligned}\alpha[(x_1,\ x_2),\ (y_1,\ y_2)] &= \alpha(i(x_1y_2 - x_2y_1),\ i(x_1y_2 - x_2y_1)) \\ &= (i(x_2y_1 - x_1y_2),\ i(x_2y_1 - x_1y_2)) \\ &= [(-x_2,\ -x_1),\ (-y_2,\ -y_1)] \\ &= [\alpha(x_1,\ x_2),\ \alpha(y_1,\ y_2)].\end{aligned}$$

■

Theorem 6.2. *([6]) If $(L,\ [\ ,\ ],\ \alpha)$ is a regular Hom-Lie algebra, then $Z(L)$ is a Hom-Lie ideal.*

7 New Hom-Lie Algebras From Old Lie Algebras

The following theorem gives a way to construct a new Hom-Lie algebras, beginning from a Lie algebra and a homomorphism of Lie algebras.

Theorem 7.1. *([5]) Let $(L,\ [\ ,\ ])$ be a Lie algebra and $\alpha : L \to L$ be a Lie algebra homomorphism. Then $(L,\ [\ ,\ ]_\alpha,\ \alpha)$ is a multiplicative Hom-Lie algebra, where $[\ ,\ ]_\alpha = \alpha \circ [\ ,\ ]$.*

Example 7.2. Consider the associative algebra $M_n(F)$ of $n \times n$ matrices over F under the ordinary multiplication of matrices. Define the following skew-symmetric bilinear bracket map

$$[\ ,\ ] : M_n(F) \times M_n(F) \to M_n(F);\ (A,\ B) \mapsto [A,\ B]$$

where $[A,\ B] = AB - BA$. It is well known that $(M_n(F),\ [\ ,\ ])$ is a Lie algebra (see for example [4]). Now, define the linear map

$$\alpha : M_n(F) \to M_n(F);\ A \mapsto -A^T.$$

Since

$$\begin{aligned} \alpha([A,\ B]) &= -(AB - BA)^T = (BA)^T - (AB)^T \\ &= A^T B^T - B^T A^T \\ &= [A^T,\ B^T] = [\alpha(A),\ \alpha(B)], \end{aligned} \tag{7.1}$$

it follows that α is a Lie algebra homomorphism. Thus, using the theorem above, we have $(M_n(F),\ [\ ,\ ]_\alpha,\ \alpha)$ is a multiplicative Hom-Lie algebra, where $[A,\ B]_\alpha = \alpha([A,\ B]) = A^T B^T - B^T A^T$ by the relation (7.1) for any $A, B \in M_n(F)$. ■

8 Direct Sum and Direct Product of Hom-Lie Algebras

Let $S = \{L_\lambda \mid \lambda \in \Delta\}$ be a family of vector spaces. The (external) direct product of S is the vector space $\prod_{\lambda\in\Delta} L_\lambda = \{f : \Delta \to \bigcup_{\lambda\in\Delta} L_\lambda \mid f(\lambda) \in L_\lambda\}$ with the ordinary addition and scalar multiplication of functions ([13]).

Theorem 8.1. *Let $(L_\lambda,\ [\ ,\ ]_\lambda,\ \alpha_\lambda)$ be a Hom-Lie algebra for each $\lambda \in \Delta$. Then $(\prod_{\lambda\in\Delta} L_\lambda,\ [\ ,\ ],\ \alpha)$ is a Hom-Lie algebra by setting*

$$\alpha : \prod_{\lambda\in\Delta} L_\lambda \to \prod_{\lambda\in\Delta} L_\lambda;\ f \mapsto f_\alpha,$$

where $f_\alpha(\lambda) = \alpha_\lambda(f(\lambda))$ $(\in L_\lambda)$ for all $\lambda \in \Delta$, and also the multiplication

$$[\ ,\ ] : \prod_{\lambda\in\Delta} L_\lambda \times \prod_{\lambda\in\Delta} L_\lambda \to \prod_{\lambda\in\Delta} L_\lambda;\ (f,\ g) \mapsto [f,\ g],$$

where $[f,\ g](\lambda) = [f(\lambda),\ g(\lambda)]_\lambda$ $(\in L_\lambda)$.

Proof. Let $f, g \in \prod_{\lambda\in\Delta} L_\lambda$ and $a, b \in F$. For any $\lambda \in \Delta$ we have,

$$\begin{aligned} (\alpha(af + bg))(\lambda) &= (af + bg)_\lambda(\lambda) = \alpha_\lambda((af + bg)(\lambda)) \\ &= a\alpha_\lambda(f(\lambda)) + b\alpha_\lambda(g(\lambda)) = (a\alpha(f) + b\alpha(g))(\lambda). \end{aligned}$$

Thus, $\alpha(af + bg) = a\alpha(f) + b\alpha(g)$, which implies that α is a linear map. Also, $[\ ,\ ]$ is a skew-symmetric bilinear map because $[\ ,\ ]_\lambda$ is a skew-symmetric bilinear map for all $\lambda \in \Delta$.
Now, for any $f, g, h \in \prod_{\lambda\in\Delta} L_\lambda$ and any $\lambda \in \Delta$, we obtain

$$([\alpha(f),\ [g,\ h]] + [\alpha(g),\ [h,\ f]] + [\alpha(h),\ [f,\ g]])(\lambda)$$

$$\begin{aligned} &= [f_\alpha(\lambda),\ [g(\lambda),\ h(\lambda)]_\lambda]_\lambda + [g_\alpha(\lambda),\ [h(\lambda),\ f(\lambda)]_\lambda]_\lambda + [h_\alpha(\lambda),\ [f(\lambda),\ g(\lambda)]_\lambda]_\lambda \\ &= [\alpha_\lambda(f(\lambda)),\ [g(\lambda),\ h(\lambda)]_\lambda]_\lambda + [\alpha_\lambda(g(\lambda)),\ [h(\lambda),\ f(\lambda)]_\lambda]_\lambda \\ &\quad + [\alpha_\lambda(h(\lambda)),\ [f(\lambda),\ g(\lambda)]_\lambda]_\lambda \\ &= 0 \end{aligned}$$

□

Let $S = \{L_\lambda \mid \lambda \in \Delta\}$ be a family of Hom-Lie algebras. The (external) direct sum of S is the vector space $\bigoplus_{\lambda\in\Delta} L_\lambda = \{f : \Delta \to \bigcup_{\lambda\in\Delta} L_\lambda \mid f(\lambda) \in L_\lambda$ and $f(\lambda) = 0$ for all but finitely many $\lambda\}$. In fact $\bigoplus_{\lambda\in\Delta} L_\lambda$ is a subspace of $\prod_{\lambda\in\Delta} L_\lambda$ ([13]).

Theorem 8.2. *Let* $(L_\lambda,\ [\ ,\ ]_\lambda,\ \alpha_\lambda)$ *be a Hom-Lie algebra for all* $\lambda \in \Delta$. *Then* $(\bigoplus_{\lambda\in\Delta} L_\lambda,\ [\ ,\ ],\ \alpha)$ *is a Hom-Lie algebra*

$$\alpha : \bigoplus_{\lambda\in\Delta} L_\lambda \to \bigoplus_{\lambda\in\Delta} L_\lambda;\ f \mapsto f_\alpha,$$

where $f_\alpha(\lambda) = \alpha_\lambda(f(\lambda))$ *for each* $\lambda \in \Delta$, *and also the skew-symmetric bilinear map*

$$[\ ,\ ] : \bigoplus_{\lambda\in\Delta} L_\lambda \times \bigoplus_{\lambda\in\Delta} L_\lambda \to \bigoplus_{\lambda\in\Delta} L_\lambda;\ (f,\ g) \mapsto [f,\ g],$$

where $[f,\ g](\lambda) = [f(\lambda),\ g(\lambda)]_\lambda$.

Proof. Note that, $\bigoplus_{\lambda\in\Delta} L_\lambda$ is a Hom-Lie ideal of $\prod_{\lambda\in\Delta} L_\lambda$. Indeed For any $f \in \bigoplus_{\lambda\in\Delta} L_\lambda$, we have $\alpha(f) = f_\alpha \in \bigoplus_{\lambda\in\Delta} L_\lambda$ because $f_\alpha(\lambda) = \alpha_\lambda(f(\lambda)) = 0$ for all but finitely many λ ($f(\lambda) = 0$ for all but finitely many λ and $\alpha_\lambda(0) = 0$). Also, For any $f \in \bigoplus_{\lambda\in\Delta} L_\lambda$ and $g \in \prod_{\lambda\in\Delta} L_\lambda$, $[f,\ g](\lambda) = [f(\lambda),\ g(\lambda)]_\lambda = 0$ for all but finitely many λ (because $f(\lambda) = 0$ for all but finitely many λ and $[0,\ g(\lambda)] = 0$ for all $\lambda \in \Delta$). Thus, $[f,\ g] \in \bigoplus_{\lambda\in\Delta} L_\lambda$. □

Remark 8.3. The direct product and the direct sum are the same for a finite family $\{L_i \mid i = 1, 2, \ldots, n\}$ of Hom-Lie algebras; that is

$$L_1 \times L_2 \times \cdots \times L_n = L_1 \oplus L_2 \oplus \cdots \oplus L_n.$$

Corollary 8.4. *Given* n *Hom-Lie algebras* $(L_i,\ [\ ,\ ]_i,\ \alpha_i)$,$i = 1, \ldots, n$, *there is a Hom-Lie algebra* $(L_1 \oplus L_2 \oplus \ldots \oplus L_n,\ [,],\ \alpha_1 + \alpha_2 + \ldots + \alpha_n)$, *where the skew-symmetric bilinear map*

$$[,]\ :\ (L_1 \oplus L_2 \oplus \ldots \oplus L_n) \times (L_1 \oplus L_2 \oplus \ldots \oplus L_n) \to (L_1 \oplus L_2 \oplus \ldots \oplus L_n)$$

is given by

$$[(x_1,\ x_2, \ldots,\ x_n),\ (y_1,\ y_2, \ldots,\ y_n)] = ([x_1,\ y_2]_1,\ [x_1,\ y_2]_2, \ldots,\ [x_n,\ y_n]_n),$$

for all $x_i, y_i \in L_i$, $i = 1, 2, \ldots, n$, *and the linear map*

$$(\alpha_1 + \alpha_2 + \ldots + \alpha_n)\ :\ (L_1 \oplus L_2 \oplus \ldots \oplus L_n) \to (L_1 \oplus L_2 \oplus \ldots \oplus L_n)$$

is given by

$$(\alpha_1 + \alpha_2 + \ldots + \alpha_n)(x_1,\ x_2, \ldots,\ x_n) = (\alpha_1(x_1),\ \alpha_2(x_2), \ldots,\ \alpha_n(x_n)),$$

for all $x_i \in L_i$, $i = 1, 2, \ldots, n$.

In the special case where $n = 2$, we obtain [17, Proposition 2.2].

Theorem 8.5. *Let* $(L_\lambda,\ [\ ,\ ]_\lambda,\ \alpha_\lambda)$ *be a Hom-Lie algebra for all* $\lambda \in \Delta$.

(i) If H_λ *is a Hom-Lie subalgebra of* L_λ *for each* $\lambda \in \Delta$, *then* $\prod_{\lambda\in\Delta} H_\lambda$ *is a Hom-Lie subalgebra of* $\prod_{\lambda\in\Delta} L_\lambda$ *and* $\bigoplus_{\lambda\in\Delta} H_\lambda$ *is a Hom-Lie subalgebra of* $\bigoplus_{\lambda\in\Delta} L_\lambda$.

(ii) If H_λ *is a Hom-Lie ideal of* L_λ *for each* $\lambda \in \Delta$, *then* $\prod_{\lambda\in\Delta} H_\lambda$ *is a Hom-Lie ideal of* $\prod_{\lambda\in\Delta} L_\lambda$ *and* $\bigoplus_{\lambda\in\Delta} H_\lambda$ *is a Hom-Lie ideal of* $\bigoplus_{\lambda\in\Delta} L_\lambda$.

Proof.

(i) Let $h \in \prod_{\lambda\in\Delta} H_\lambda$. Then for each $\lambda \in \Delta$, $h(\lambda) \in H_\lambda$ and so $h_\alpha(\lambda) = \alpha_\lambda(h(\lambda)) \in H_\lambda$, which implies $\alpha(h) = h_\alpha \in \prod_{\lambda\in\Delta} H_\lambda$. If $h, g \in \prod_{\lambda\in\Delta} H_\lambda$, then $[h,\ g](\lambda) = [h(\lambda),\ g(\lambda)]_\lambda \in H_\lambda$ for each $\lambda \in \Delta$. Which implies $[h,\ g] \in \prod_{\lambda\in\Delta} H_\lambda$. Thus, $\prod_{\lambda\in\Delta} H_\lambda$ is a Hom-Lie subalgebra of $\prod_{\lambda\in\Delta} L_\lambda$. Similarly, $\bigoplus_{\lambda\in\Delta} H_\lambda$ is a Hom-Lie subalgebra of $\bigoplus_{\lambda\in\Delta} L_\lambda$.

(ii) Similar to the proof of (i). □

Theorem 8.6. *Let* $(L_\lambda,\ [\ ,\ ]_\lambda,\ \alpha_\lambda)$ *be a multiplicative Hom-Lie algebra for all* $\lambda \in \Delta$. *Then the Hom-Lie algebras* $(\prod_{\lambda\in\Delta} L_\lambda,\ [\ ,\ ],\ \alpha)$ *and* $(\bigoplus_{\lambda\in\Delta} L_\lambda,\ [\ ,\ ],\ \alpha)$ *are multiplicative.*

Proof. Let $(L_\lambda,\ [\ ,\ ]_\lambda,\ \alpha_\lambda)$ be a multiplicative Hom-Lie algebra for all $\lambda \in \Delta$. If $f, g \in \prod_{\lambda\in\Delta} L_\lambda$, then for any $\lambda \in \Delta$,

$$\begin{aligned}(\alpha([f,\ g]))(\lambda) &= \alpha_\lambda([f,\ g](\lambda)) = \alpha_\lambda([f(\lambda),\ g(\lambda)]_\lambda)\\ &= [\alpha_\lambda(f(\lambda)),\ \alpha_\lambda(g(\lambda))]_\lambda = [(\alpha(f))(\lambda),\ (\alpha(g))(\lambda)]_\lambda\\ &= ([\alpha(f),\ \alpha(g)])(\lambda).\end{aligned}$$

Thus $\alpha([f,\ g]) = [\alpha(f),\ \alpha(g)]$. Consequently, $\prod_{\lambda\in\Delta} L_\lambda$ is a multiplicative Hom-Lie algebra. Also, $\bigoplus_{\lambda\in\Delta} L_\lambda$ is a multiplicative Hom-Lie algebra too, because it is a Hom-Lie subalgebra of $\prod_{\lambda\in\Delta} L_\lambda$. □

9 Hom-Lie Algebras Morphism

Definition 9.1. ([8]) Let $(L_1,\ [\ ,\ ]_1,\ \alpha_1)$ and $(L_2,\ [\ ,\ ]_2,\ \alpha_2)$ be Hom-Lie algebras. A linear map $\varphi : L_1 \to L_2$ is called a morphism of Hom-Lie algebras if the following two identities are satisfied:

(i) $\varphi([x,\ y]_1) = [\varphi(x),\ \varphi(y)]_2$ for all $x,\ y \in L_1$.

(ii) $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$.

Example 9.2. (i) Let L_1 and L_2 be Hom-Lie algebras over F. Then

$$\varphi : L_1 \to L_2;\ x \mapsto 0$$

is a morphism of Hom-Lie algebras.

(ii) Let L be any Hom-Lie algebra. Then the identity map

$$\mathrm{id}_L : L \to L;\ x \mapsto x$$

is a morphism of Hom-Lie algebras. ■

Example 9.3. If $(L,\ [\ ,\ ],\ \alpha)$ is a multiplicative Hom-Lie algebra, then α is a morphism of Hom-Lie algebras. ■

Example 9.4. Let us consider the Hom-Lie algebra

$$L_1 = L = \mathrm{Span}\{e_1,\ e_2,\ e_3\}$$

with the linear map $\alpha_1 : L \to L$ defined by setting $\alpha_1(e_1) = \alpha_1(e_2) = \alpha_1(e_3) = e_1 + e_2 + e_3$ and the skew-symmetric bracket $[\ ,\ ]_1$ defined as $[e_1,\ e_2]_1 = [e_1,\ e_3]_1 = [e_2,\ e_3]_1 = e_1 + e_2 + e_3$ and also $[e_i,\ e_i]_1 = 0$ for each $i = 1, 2, 3$ given in Example 2.3. Also let us take the Hom-Lie algebra $L_2 = L$ where the linear map α_2 on L defined as $\alpha_2(e_1) = e_2$ and $\alpha_2(e_2) = \alpha_2(e_3) = 0$, and also the skew-symmetric bracket $[\ ,\ ]_2$ on L defined as $[e_1,\ e_3]_2 = e_1$ and $[e_1,\ e_2]_2 = [e_2,\ e_3]_2 = [e_1,\ e_1]_2 = [e_2,\ e_2]_2 = [e_3,\ e_3]_2 = 0$ given in Example 2.4. Define the linear map

$$\varphi : L_1 \to L_2$$

by setting $\varphi(e_1) = e_2,\ \varphi(e_2) = -e_2 + e_3$ and $\varphi(e_3) = -e_3$. Then φ is a morphism of Hom-Lie algebras. For $x, y \in L\ (= L_1)$, we have $\alpha_1(x) = a(e_1 + e_2 + e_3)$, $\varphi(x) = be_2 + ce_3$, $\varphi(y) = de_2 + fe_3$, and $[x, y]_1 = k(e_1 + e_2 + e_3)$ for some $a, b, c, d, f, k \in F$. Consequently

$$\varphi(\alpha_1(x)) = a\varphi(e_1 + e_2 + e_3) = 0 = \alpha_2(be_2 + ce_3) = \alpha_2(\varphi(x)),$$

and also

$$\varphi([x,\ y]_1) = k\varphi(e_1 + e_2 + e_3) = 0 = [be_2 + ce_3,\ de_2 + fe_3]_2 = [\varphi(x),\ \varphi(y)]_2.$$

■

Example 9.5. Let $(L_1,\ [\ ,\ ]_1,\ \alpha_1)$ and $(L_2,\ [\ ,\ ]_2,\ \alpha_2)$ be Hom-Lie algebras. Then the projection map

$$\varphi : (L_1 \oplus L_2,\ [\ ,\ ],\ \alpha_1 + \alpha_2) \to (L_1,\ [\ ,\ ]_1,\ \alpha_1);\ (x,\ y) \mapsto x$$

is a morphism of Hom-Lie algebras. For any $(x_1,\ y_1), (x_2,\ y_2) \in L_1 \oplus L_2$ and $c_1, c_2 \in F$, we have, $\varphi(c_1(x_1,\ y_1) + c_2(x_2,\ y_2)) = \varphi(c_1x_1 + c_2x_2,\ c_1y_1 + c_2y_2) = c_1x_1 + c_2x_2 = c_1\varphi(x_1,\ y_1) + c_2\varphi(x_2,\ y_2)$. Hence φ is linear.
Also, $\varphi[(x_1,\ y_1),\ (x_2,\ y_2)] = \varphi([x_1,\ x_2]_1,\ [y_1,\ y_2]_2) = [x_1,\ x_2]_1 = [\varphi(x_1,\ y_1),\ \varphi(x_2,\ y_2)]_1$.
Next $\varphi \circ (\alpha_1 + \alpha_2) = \alpha_1 \circ \varphi$. Indeed for any $(x,\ y) \in L_1 \oplus L_2$, $\varphi \circ (\alpha_1 + \alpha_2)(x,\ y) = \varphi(\alpha_1(x),\ \alpha_2(y)) = \alpha_1(x) = \alpha_1 \circ \varphi(x,\ y)$. ■

We omit the proof for the following theorem because it is straightforward.

Theorem 9.6. *Let $(L_1,\ [\ ,\ ]_1,\ \alpha_1)$ and $(L_2,\ [\ ,\ ]_2,\ \alpha_2)$ be Hom-Lie algebras and let $\varphi : L_1 \to L_2$ be a morphism of Hom-Lie algebras. Then,*

(i) If A is a Hom-Lie subalgebra of L_1, then $\varphi(A)$ is a Hom-Lie subalgebra of L_2.

(ii) If φ is onto, and A is a Hom-Lie ideal of L_1, then $\varphi(A)$ is a Hom-Lie ideal of L_2.

(iii) If B is a Hom-Lie subalgebra of L_2, then $\varphi^{-1}(B) = \{x \in L_1 \mid \varphi(x) \in B\}$ is a Hom-Lie subalgebra of L_1.

(iv) If B is a Hom-Lie ideal of L_2, then $\varphi^{-1}(B)$ is a Hom-Lie ideal of L_1.

(v) Ker(φ) *is a Hom-Lie ideal of L_1.*

(vi) Im(φ) *is a Hom-Lie subalgebra of L_2.*

10 Isomorphisms of Hom-Lie Algebras

Let L_1 and L_2 be a Hom-Lie algebras. A linear map $\varphi : L_1 \to L_2$ is called an isomorphism of Hom-Lie algebras if φ is a bijection morphism of Hom-Lie algebras. We say L_1 and L_2 are isomorphic and write $L_1 \equiv L_2$.

Example 10.1. Let L be any Hom-Lie algebra. Then the identity map

$$\mathrm{id}_L : L \to L;\ x \mapsto x$$

is isomorphism. ■

Example 10.2. Let $F = \mathbb{C}$. Consider the set

$$L_1 = \left\{ \begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix} \mid x, y \in \mathbb{C} \right\}$$

with the linear map

$$\alpha_1 : L_1 \to L_1;\ A \mapsto -A^T,$$

and the skew-symmetric bilinear map

$$[\ ,\ ]_1\ :\ L_1 \times L_1 \to L_1;\ (A,\ B) \mapsto [A,\ B]_1,$$

where $[A,\ B]_1 = A^TB^T - B^TA^T$. For any $x, y, z, w \in \mathbb{C}$

$$\alpha_1\left(\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix}\right) = \begin{bmatrix} \frac{i((-x)+(-y))}{2} & (-y) \\ (-x) & \frac{-i((-x)+(-y))}{2} \end{bmatrix} \in L_1$$

and

$$\left[\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix},\ \begin{bmatrix} \frac{i(z+w)}{2} & z \\ w & \frac{-i(z+w)}{2} \end{bmatrix}\right]_1$$

$$= \begin{bmatrix} \frac{i(x+y)}{2} & y \\ x & \frac{-i(x+y)}{2} \end{bmatrix} \begin{bmatrix} \frac{i(z+w)}{2} & w \\ z & \frac{-i(z+w)}{2} \end{bmatrix} - \begin{bmatrix} \frac{i(z+w)}{2} & w \\ z & \frac{-i(z+w)}{2} \end{bmatrix} \begin{bmatrix} \frac{i(x+y)}{2} & y \\ x & \frac{-i(x+y)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-(x+y)(z+w)}{4} + yz & \frac{iw(x+y)}{2} - \frac{iy(z+w)}{2} \\ \frac{ix(z+w)}{2} - \frac{iz(x+y)}{2} & xw - \frac{(x+y)(z+w)}{4} \end{bmatrix} - \begin{bmatrix} \frac{-(x+y)(z+w)}{4} + xw & \frac{iy(z+w)}{2} - \frac{iw(x+y)}{2} \\ \frac{iz(x+y)}{2} - \frac{ix(z+w)}{2} & yz - \frac{(x+y)(z+w)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} yz - xw & i(xw - yz) \\ i(xw - yz) & xw - yz \end{bmatrix} \in L_1 \quad \left(\frac{i(i(xw-yz) + i(xw-yz))}{2} = yz - xw \right).$$

This shows that L_1 is a Hom-Lie subalgebra of the Hom-Lie algebra $M_2(\mathbb{C})$ given in Example 7.2.

Let us take the Hom-Lie algebra given in Example 6.1, $L_2 = \mathbb{C}^2$ with the linear map

$$\alpha_2 : L_2 \to L_2;\ (x,\ y) \mapsto (-y,\ -x),$$

and the skew-symmetric bilinear map

$$[\ ,\ ]\ :\ L_2 \times L_2 \to L_2;\ ((x,\ y), (z,\ w)) \mapsto [(x,\ y),\ (z,\ w)]_2,$$

where $[(x,\ y),\ (z,\ w)]_2 = (i(xw - yz),\ i(xw - yz))$. Define the linear map

$$\varphi : L_1 \to L_2;\ \begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix} \mapsto (x,\ y).$$

Then φ is an isomorphism of Hom-Lie algebras. It is easy to show that φ is a bijection linear map. For any $x, y \in \mathbb{C}$, we have

$$\begin{aligned} \varphi\left(\alpha_1\left(\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix}\right)\right) &= \varphi\left(\begin{bmatrix} \frac{i((-x)+(-y))}{2} & (-y) \\ (-x) & \frac{-i((-x)+(-y))}{2} \end{bmatrix}\right) \\ &= (-y,\ -x) \\ &= \alpha_2(x,\ y) \\ &= \alpha_2\left(\varphi\left(\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix}\right)\right). \end{aligned}$$

Also for any $x, y, z, w \in \mathbb{C}$,

$$\varphi\left(\left[\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix}, \begin{bmatrix} \frac{i(z+w)}{2} & z \\ w & \frac{-i(z+w)}{2} \end{bmatrix}\right]_1\right)$$

$$\begin{aligned} &= \varphi\left(\begin{bmatrix} yz - xw & i(xw-yz) \\ i(xw-yz) & xw - yz \end{bmatrix}\right) \\ &= (i(xw-yz),\ i(xw-yz)) = [(x,\ y),\ (z,\ w)]_2 \\ &= \left[\varphi\left(\begin{bmatrix} \frac{i(x+y)}{2} & x \\ y & \frac{-i(x+y)}{2} \end{bmatrix}\right), \varphi\left(\begin{bmatrix} \frac{i(z+w)}{2} & z \\ w & \frac{-i(z+w)}{2} \end{bmatrix}\right)\right]_2. \end{aligned}$$

■

Theorem 10.3. *Isomorphism of Hom-Lie algebras is an equivalence relation.*

Proof. A Hom-Lie algebra L is isomorphic to itself via $\mathrm{id}_L : L \to L;\ x \mapsto x$. So isomorphism is reflexive. If $\varphi : (L_1,\ [\ ,\ ]_1,\ \alpha_1) \to (L_2,\ [\ ,\ ]_2,\ \alpha_2)$ is an isomorphism of Hom-Lie algebras, then $\varphi^{-1} : L_2 \to L_1$ is an isomorphism of Hom-Lie algebras. It is enough to show that φ^{-1} is a morphism of Hom-Lie algebras. Since $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$ so $\varphi^{-1} \circ \alpha_2 = \alpha_1 \circ \varphi^{-1}$. For any $x, y \in L_2$, there exists $a, b \in L_1$ such that $\varphi(a) = x$ and $\varphi(b) = y$, and so $\varphi^{-1}[x,\ y]_2 = \varphi^{-1}[\varphi(a),\ \varphi(b)]_2 = \varphi^{-1}\varphi[a,\ b]_1 = [a,\ b]_1 = [\varphi^{-1}(x),\ \varphi^{-1}(y)]_1$. Thus, the isomorphism is symmetric. If $\varphi_1 : (L_1,\ [\ ,\ ]_1,\ \alpha_1) \to (L_2,\ [\ ,\ ]_2,\ \alpha_2)$ and $\varphi_2 : (L_2,\ [\ ,\ ]_2,\ \alpha_2) \to (L_3,\ [\ ,\ ]_3,\ \alpha_3)$ are isomorphism of Hom-Lie algebras, then $\varphi_2 \circ \varphi_1 : L_1 \to Ł_3$ is an isomorphism of Hom-Lie algebras. Indeed $(\varphi_2 \circ \varphi_1) \circ \alpha_1 = \varphi_2 \circ \alpha_2 \circ \varphi_1 = \alpha_3 \circ (\varphi_2 \circ \varphi_1)$ and $\varphi_2 \circ \varphi_1[x,\ y]_1 = \varphi_2[\varphi_1(x),\ \varphi_1(y)]_2 = [\varphi_2 \circ \varphi_1(x),\ \varphi_2 \circ \varphi_1(y)]_3$ for each $x, y \in L_1$. Thus, the isomorphism is transitive. Therefore isomorphism is an equivalence relation on the collection of all Hom-Lie algebras. □

We have the following result.

Theorem 10.4. *Let $(L_1,\ [\ ,\ ]_1,\ \alpha_1)$ and $(L_2,\ [\ ,\ ]_2,\ \alpha_2)$ be Hom-Lie algebras and $\varphi : L_1 \to L_2$ be an isomorphism of Hom-Lie algebras. If L_1 is a multiplicative(regular) Hom-Lie algebra, then L_2 is a multiplicative(regular) Hom-Lie algebra.*

Proof. For any $x, y \in L_2$,

$$\begin{aligned}
\alpha_2\left([x,\ y]_2\right) &= \alpha_2(\varphi(\varphi^{-1}[x,\ y]_2)) \quad (\varphi \text{ is bijective map})\\
&= \varphi(\alpha_1(\varphi^{-1}[x,\ y]_2) \quad (\varphi \circ \alpha_1 = \alpha_2 \circ \varphi)\\
&= \varphi(\alpha_1[\varphi^{-1}(x),\ \varphi^{-1}(y)]_1)) \quad (\text{Lemma } 10.3)\\
&= \varphi[\alpha_1(\varphi^{-1}(x)),\ \alpha_1(\varphi^{-1}(y))]_1 \quad (L_1 \text{ is multiplicative})\\
&= [\varphi(\alpha_1(\varphi^{-1}(x))),\ \varphi(\alpha_1(\varphi^{-1}(y)))]_2 \quad (\varphi \text{ is morphism})\\
&= [\alpha_2(\varphi(\varphi^{-1}(x))),\ \alpha_2(\varphi(\varphi^{-1}(y)))]_2 \quad (\varphi \circ \alpha_1 = \alpha_2 \circ \varphi)\\
&= [\alpha_2(x),\ \alpha_2(y)]_2.
\end{aligned}$$

Since φ and φ^{-1} are bijection and $\alpha_2 = \alpha_2 \circ \varphi \circ \varphi^{-1} = \varphi \circ \alpha_1 \circ \varphi^{-1}$, so α_2 is a bijection map if and only if α_1 is a bijection map. Thus, If L_1 is a regular Hom-Lie algebra, then L_2 is too. □

Theorem 10.5. *([6]) Let $(L,\ [\ ,\ ],\ \alpha)$ be a Hom-Lie algebra and let H be a Hom Lie ideal. Then the quotient space $(L/H,\ \overline{[\ ,\ ]},\ \overline{\alpha})$ is a Hom-Lie algebra (and it is called the quotient Hom-Lie algebra of L by H) where*

$$\overline{[\ ,\ ]} : L/H \times L/H \to L/H;\ (x + H,\ y + H) \mapsto [x,\ y] + H,$$

and

$$\overline{\alpha} : L/H \to L/H;\ x + H \mapsto \alpha(x) + H.$$

Theorem 10.6. ***(First isomorphism theorem of Hom-Lie algebras)** Let*

$$\varphi : (L_1,\ [\ ,\ ]_1,\ \alpha_1) \to (L_2,\ [\ ,\ ]_2,\ \alpha_2)$$

be a morphism of Hom-Lie algebras. Then $L_1/\mathrm{Ker}(\varphi) \equiv \mathrm{Im}(\varphi)$.

Proof. Using Theorems 9.6 and 10.5, we have $(L_1/\mathrm{Ker}(\varphi),\ \overline{[\ ,\ ]_1},\ \overline{\alpha_1})$ is a Hom-Lie algebra where $\overline{\alpha}_1(x+\mathrm{Ker}(\varphi)) = \alpha_1(x)+\mathrm{Ker}(\varphi)$ and also $\overline{[x + \mathrm{Ker}(\varphi),\ y + \mathrm{Ker}(\varphi)]_1} = [x,\ y]_1+\mathrm{Ker}(\varphi)$ for all $x, y \in L_1$. We already know that the map

$$\Psi : L_1/\mathrm{Ker}(\varphi) \to \mathrm{Im}(\varphi);\ x + \mathrm{Ker}(\varphi) \mapsto \varphi(x).$$

is an isomorphism of vector spaces. Also, Ψ is a morphisms of Hom-Lie algebras. Indeed for any $x, y \in L_1$ we find

$$\begin{aligned}
\Psi(\overline{[x + \mathrm{Ker}(\varphi),\ y + \mathrm{Ker}(\varphi)]_1}) &= \Psi\left([x,\ y]_1 + \mathrm{Ker}(\varphi)\right)\\
&= \varphi([x,\ y]_1) = [\varphi(x),\ \varphi(y)]_2\\
&= [\Psi(x + \mathrm{Ker}(\varphi)),\ \Psi(y + \mathrm{Ker}(\varphi))]_2.
\end{aligned}$$

For any $x \in L_1$, we have

$$\begin{aligned}\Psi(\overline{\alpha}_1(x+\text{Ker}(\varphi))) &= \Psi(\alpha_1(x)+\text{Ker}(\varphi)) = \varphi(\alpha_1(x)) \\ &= \alpha_2(\varphi(x)) = \alpha_2(\Psi(x+\text{Ker}(\varphi))).\end{aligned}$$

□

Theorem 10.7. ***(Second isomorphism theorem of Hom-Lie algebras)*** *Let I and J be ideals of a Hom-Lie algebra $(L,\ [\ ,\ ],\ \alpha)$. Then $(I+J)/J \equiv I/(I\cap J)$.*

Proof. Using theorems 10.5 and 5.6, we have $((I+J)/J,\ \overline{[\ ,\ ]}_1,\ \overline{\alpha}_1)$ and $(I/(I\cap J),\ \overline{[\ ,\ ]}_2,\ \overline{\alpha}_2)$ are Hom-Lie algebras ($\overline{\alpha_1},\ \overline{[x+J,\ y+J]}_1,\ \overline{\alpha_2}$, and $\overline{[x+(I\cap J),\ y+(I\cap J)]}_2$ are defined as in Theorem 10.5). The map

$$\Psi : (I+J)/J \to I/(I\cap J);\ (i+j)+J \mapsto i+(I\cap J)$$

is a morphism of Hom-Lie algebras. Since for each $i_1, i_2 \in I$ and each $j_1, j_2 \in J$ we find

$$\begin{aligned}\Psi\overline{[(i_1+j_1)+J,\ (i_2+j_2)+J]}_1 &= \Psi\overline{[i_1+J,\ i_2+J]}_1 \\ &= \Psi([i_1,\ i_2]+J) = [i_1,\ i_2]+(I\cap J) \\ &= \overline{[i_1+(I\cap J),\ i_2+(I\cap J)]}_2 \\ &= \overline{[\Psi((i_1+j_1)+J),\ \Psi((i_2+j_2)+J)]}_2.\end{aligned}$$

Also for all $i \in I$ and $j \in J$,

$$\begin{aligned}\Psi(\overline{\alpha}_1((i+j)+J)) &= \Psi(\alpha(i)+J) = \alpha(i)+(I\cap J) \\ &= \overline{\alpha}_2(i+(I\cap J)) = \overline{\alpha}_2(\Psi((i+j)+J)).\end{aligned}$$

□

Lemma 10.8. *Let I and J be Hom-Lie ideals of a Hom-Lie algebra $(L,\ [\ ,\ ],\ \alpha)$ such that $I \subseteq J$. Then J/I is a Hom-Lie ideal of L/I.*

Proof. For each $j \in J$, $\overline{\alpha}(j+I) = \alpha(j)+I \in J/I$ $(\alpha(J) \subseteq J)$. Also, for any $j \in J$ and $l \in L$, we have $\overline{[j+I,\ l+I]} = [j,\ l]+I \in J/I$ (J is ideal). □

Theorem 10.9. ***(Third isomorphism theorem of Hom-Lie algebras)*** *Let I and J be a Hom-Lie ideals of a Hom-Lie algebra $(L,\ [\ ,\ ],\ \alpha)$ such that $I \subseteq J$. Then $(L/I)/(J/I) \equiv L/J$.*

Proof. According to Lemma 10.8, we have J/I is a Hom-Lie ideal of L/I. Also we have the following isomorphism of vector spaces

$$\Psi : ((L/I)/(J/I),\ \overline{\overline{[\ ,\ ]}},\ \overline{\overline{\alpha}}) \to ((L/J),\ \overline{[\ ,\ ]},\ \overline{\alpha});\ (x+I)+J/I \mapsto x+J,$$

where $\overline{\overline{\alpha}}((x+I)+J/I) = (\alpha(x)+I)+J/I$, $\overline{\overline{[(x+I)+J/I,\ (y+I)+J/I]}} = ([x,\ y]+I)+J/I$, $\overline{\alpha}(x+J) = \alpha(x)+J$, and $\overline{[x+J,\ y+J]} = [x,\ y]+J$ for all $x, y \in L$. In addition Ψ is a morphisms of Hom-Lie algebras. Indeed for all $x, y \in L$,

$$\begin{aligned}\Psi\overline{\overline{[(x+I)+J/I,\ (y+I)+J/I]}} &= \Psi(([x,\ y]+I)+J/I) \\ &= [x,\ y]+J \\ &= \overline{[x+J,\ y+J]} \\ &= \overline{[\Psi((x+I)+J/I),\ \Psi((y+I)+J/I)]},\end{aligned}$$

and

$$\begin{aligned}\Psi(\overline{\overline{\alpha}}((x+I)+J/I)) &= \Psi((\alpha(x)+I)+J/I) = \alpha(x)+J \\ &= \overline{\alpha}(x+J) = \overline{\alpha}(\Psi((x+I)+J/I)).\end{aligned}$$

□

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