Palestine Journal of Mathematics

Vol. 12(2)(2023) , 597–608 © Palestine Polytechnic University-PPU 2023

Statistics on Multisets

Shashikant Mulay and Carl Wagner

Communicated by Ayman Badawi

MSC 2010 Classifications: 05A05, 05A15.

Keywords and phrases: Cell decompositions, Inversions.

Abstract: This paper was inspired by Donald Knuth's celebrated explanation of the remarkable connection between q-binomial coefficients and integer partitions. In the spirit of Knuth's proof, we offer a new proof of the well-known result that a certain q-analogue of multinomial coefficients furnishes a q-counting of the set of permutations of an associated multiset of positive integers, according to the number of inversions in such arrangements. Our proof uses the fact that such q-multinomial coefficients enumerate certain classes of chains of subspaces of a finite dimensional vector space over a finite field of cardinality q. Additionally, we investigate the function that counts the number of permutations of a multiset having a fixed number of inversions.

1. Introduction.

The notational conventions of this paper are as follows: $\mathbb{N}$ and $\mathbb{P}$ denote the set of non-negative integers and the set of positive integers respectively. If $n \in \mathbb{P}$, then $[n] := \{1, \ldots, n\}$,

$$n_q := \sum_{i=0}^{n-1} q^i \quad \text{and} \quad n_q^! := \prod_{m=1}^{n} m_q.$$

We follow the convention that $[0] = \emptyset$, $0_q = 0$ and $0_q^! = 1$. If $n \in \mathbb{N}$, $r \in \mathbb{P}$ and $(e_1, \ldots, e_r) \in \mathbb{N}^r$ is such that $e_1 + \cdots + e_r \leq n$ then define

$$\binom{n}{e_1, \ldots, e_r}_q := \frac{n_q^!}{\prod_{i=1}^{r+1} (e_i)_q^!} \quad \text{where } e_{r+1} := n - \textstyle\sum_{i=1}^{r} e_i. \tag{1.1}$$

Note that if $r = n - 1$ and $e_1 = \cdots = e_r = 1$, then

$$\binom{n}{e_1, \ldots, e_{n-1}}_q = n_q^!.$$

When $r = 1$, we write $e_1 = e$ and so,

$$\binom{n}{e}_q = \frac{n_q^!}{e_q^! (n-e)_q^!}. \tag{1.2}$$

Since $\binom{n}{0}_q = 1 = \binom{n}{n}_q$ for all $n \in \mathbb{N}$, and (1.2) implies that

$$\binom{n}{e}_q = \binom{n-1}{e-1}_q + q^e \binom{n-1}{e}_q \quad \text{for } 0 < e < n, \tag{1.3}$$

it follows that $\binom{n}{e}_q$ is a polynomial in q of degree $e(n-e)$ with coefficients in $\mathbb{N}$. Furthermore, letting $e_0 := 0$, observe that

$$\binom{n}{e_1, \ldots, e_r}_q = \prod_{i=1}^{r} \binom{n - e_1 - \cdots - e_{i-1}}{e_i}_q.$$

Hence $\binom{n}{e_1,\dots,e_r}_q$ is also a polynomial in q having coefficients in $\mathbb{N}$ and of degree

$$\deg\binom{n}{e_1,\dots,e_r}_q = \binom{n}{2} - \sum_{i=1}^{r+1}\binom{e_i}{2} = \sum_{1\le i<j\le r+1} e_ie_j.$$

When $q = 1$, the polynomials n_q, $n_q^!$, $\binom{n}{e}_q$ and $\binom{n}{e_1,\dots,e_r}_q$ evaluate, respectively, to n, $n!$, the binomial coefficient $\binom{n}{e}$, and the $(r+1)$-nomial coefficient (in abbreviated notation) $\binom{n}{e_1,\dots,e_r}$. We will have more to say about this apparently trivial observation in what follows.

2. Combinatorial Statistics.

Suppose that Δ is a finite set of discrete structures and $s : \Delta \to \mathbb{N}$ is a statistic on Δ that records some nonnegative integral property of each structure $\delta \in \Delta$. The *distribution polynomial* $p(\Delta, s, q)$ of s is defined by

$$p(\Delta, s, q) := \sum_{\delta\in\Delta} q^{s(\delta)} = \sum_{j\in\mathbb{N}} |\{\delta \in \Delta \mid s(\delta) = j\}|q^j. \tag{2.1}$$

Of course, $p(\Delta, s, 1) = |\Delta|$ and, if Δ is equipped with the uniform probability distribution, then the expected value μ_s of s is equal to $|\Delta|^{-1} \cdot \frac{d}{dq}|_{q=1}p(\Delta, s, q)$.

As an example, suppose that n and e are positive integers, with $e < n$, and let $M(e, n-e)$ denote the family of all multisets containing at most $n-e$ positive integers, each of which is no larger than e. Note that $|M(e, n-e)| = \binom{n}{e}$. For each $M \in M(e, n-e)$, let $\Sigma(M)$ denote the sum of all members of M.

The formula

$$\sum_{M\in M(e,n-e)} q^{\Sigma(M)} = \binom{n}{e}_q \tag{2.2}$$

is a well-known result in combinatorial number theory. Indeed, the equivalent formula

$$\sum_{m\in\mathbb{N}} p(e, n-e, m)q^m = \binom{n}{e}_q, \tag{2.3}$$

(where $p(e, n-e, m)$ denotes the number of partitions of the integer m with at most $n-e$ parts, each part being no larger than e) appears as early as 1882 in a paper of Sylvester and Franklin [6, p.269].

More recently Knuth [3] has proved the polynomial identity (2.2) by demonstrating that it holds whenever q is a power of a prime number. Knuth first notes that the q-binomial coefficient enumerates the family $\mathcal{E}$ of all e-dimensional subspaces of an n-dimensional vector space over a finite field of cardinality q. He then makes use of the unique $e \times n$ row-echelon matrix associated with each such subspace to define a natural mapping $\kappa : \mathcal{E} \to M(e, n-e)$ for which each $M \in M(e, n-e)$ has exactly $q^{\Sigma(M)}$ preimages with respect to κ.

Suppose that $\theta := (\theta(1), \dots, \theta(n))$ is any sequence of positive integers. Let $inv(\theta)$ denote the number of inversions of θ, *i.e.*, the number of ordered pairs (i, j) with $1 \le i < j \le n$ such that $\theta(i) > \theta(j)$. Let

$$M := \{1^{e_1}, 2^{e_2}, \dots, r^{e_r}, (r+1)^{e_{r+1}}\} \tag{2.4}$$

be the multiset consisting of e_i copies of i, with $e_i \in P$, for $1 \le i \le r+1$, and $e_1 + e_2 + \cdots + e_r + e_{r+1} = n$, and let $S(M)$ denote the set of all permutations of M.

Our aim in this paper is to present, in the spirit of Knuth's proof, what Stanley [5.p.27] calls a "direct" combinatorial proof of the well-known identity (see Wagner [7, Theorem 11.9.5])

$$\sum_{\theta\in S(M)} q^{inv(\theta)} = \binom{n}{e_1,\dots,e_r}_q, \tag{2.5}$$

and, as a corollary, the well-known identity (see Wagner [7, Theorem 11.9.2])

$$\sum_{\theta\in S_n} q^{inv(\theta)} = n_q^! = (1)(1+q)\cdots(1+q+\cdots+q^{n-1}). \tag{2.6}$$

A brief outline of our approach follows. For the above multiset M, define d to be the sequence $d_1 < \cdots < d_r$, where $d_i := e_1 + \cdots + e_i$ for $1 \le i \le r$. Our proofs use the fact that the q-multinomial coefficient enumerates chains $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r$ of subspaces (so-called *flags* in the language of algebraic geometry) of an n-dimensional vector space over a finite field of cardinality q such that V_i has dimension d_i for $1 \le i \le r$. Underlying our proof is a cell-decomposition of flag spaces. Of course, such cell-decompositions have been known in algebraic geometry since the 19th century. To the best of our knowledge, however, no proof such as the one we provide here has appeared in the literature.

In the last section we investigate the heretofore neglected function $I_n(d; j)$ which counts the number of permutations of M having exactly j inversions; this affords a generalization of the more familiar function $I_n(j)$ which counts the number of permutations of $[n]$ having exactly j inversions. The equations, the estimates and the other properties we establish in Theorem 5.3 merely scratch the surface of a potentially deeper combinatorial analysis of $I_n(d; j)$.

3. Cell decomposition of flag spaces.

Let n be a positive integer. Henceforth we tacitly assume that $n \ge 2$. Let V be an n-dimensional vector space over a (not necessarily finite) field k. Given a strictly increasing sequence $d : d_1 < d_2 < \cdots < d_r$, a sequence $V_1 \subset V_2 \subset \cdots \subset V_r$ of k-subspaces of V is called a *d-flag* in V if $\dim V_i = d_i$ for $1 \le i \le r$. The set of all d-flags in V, called the *space of d-flags in V,* is denoted by $\mathrm{FL}(d, V)$. If $r = 1$, *i.e.*, when the sequence d consists of a single integer (also denoted by d), the corresponding set $\mathrm{FL}(d, V)$ is usually denoted by $\mathrm{Gr}(d, V)$ and it is called the *Grassmannian* of d-dimensional subspaces of V. If $r = n$, *i.e.*, when d is the sequence $1 < 2 < \cdots < n$, the corresponding d-flag is called a *full flag* in V. The space of full flags in V is usually denoted simply by $\mathrm{FL}(V)$.

Remark 3.1: If $r = 1$ and $d_1 = n$, then $\mathrm{FL}(d, V) = \{V\}$. Note that if $r \ge 2$ and $d_r = n$, then $\mathrm{FL}(d, V)$ is in bijective correspondence with the flag-space $\mathrm{FL}(d_1 < \cdots < d_{r-1}, V)$. So, it suffices to restrict consideration to the spaces $\mathrm{FL}(d, V)$, where d satisfies the additional requirement that $d_r < n$. By choosing a k-isomorphism of vector spaces, V can be identified with k^n and then this induces an identification of $\mathrm{FL}(d, V)$ with $\mathrm{FL}(d, k^n)$. In what follows, we tacitly assume $V = k^n$ and $d_r < n$ (whence $r < n$).

For positive integers r, s let $\mathbb{M}(r, s, k)$ be the vector space over k of all $r \times s$ matrices with entries in k. Let $\mathbb{M}(r, k) := \mathbb{M}(r, r, k)$, and as usual, let $\mathrm{GL}(r, k)$ be the multiplicative group of $r \times r$ invertible matrices with entries in k.

Let $d : d_1 < d_2 < \cdots < d_r$ be as above. Define $d_0 := 0$, $d_{r+1} := n$ and $e_i := d_i - d_{i-1}$ for $1 \le i \le r + 1$. Then, $e_1 + \cdots + e_{r+1} = n$ and since $d_r < n$, $e_{r+1} \ge 1$. Define $P(n, d, k)$ to be the set of all $g \in \mathrm{GL}(n, k)$ such that g is an $(r + 1) \times (r + 1)$ block-matrix $[M_{ij}]$, where $M_{ij} = 0$ for $1 \le j < i \le r + 1$ and $M_{ii} \in \mathrm{GL}(e_i, k)$ for $1 \le i \le r + 1$. Observe that $P(n, d, k)$ is a subgroup of $\mathrm{GL}(n, k)$ and $P(n, 1 < 2 < \cdots < n - 1, k)$ is the subgroup of upper-triangular matrices. Also, $P(n, 1 < 2 < \cdots < n - 1, k) \le P(n, d, k)$ for all sequences d.

For a nonnegative integer e, let $\mathcal{R}(n, e, k) \subset \mathbb{M}(n, e, k)$ be the subset of matrices of rank e. If $A \in \mathcal{R}(n, e, k)$, then the column-space of A, denoted by $C(A)$, is an e-dimensional k-subspace of k^n. Conversely, any e-dimensional k-subspace of k^n is the column-space of some $A \in \mathcal{R}(n, e, k)$. Furthermore, given $B \in \mathcal{R}(n, e, k)$, we have $C(B) = C(A)$ if and only if $A = Bg$ for some $g \in \mathrm{GL}(e, k)$.

Fix a sequence d as above and let $e_1, \ldots, e_{r+1}$ be the sequence associated to d (as defined above). An element of k^n is thought of as an n-rowed column-matrix. Given $A \in \mathrm{GL}(n, k)$, write $A := [A_1, \ldots, A_{r+1}]$ with the understanding that A_j is the $n \times e_j$ matrix made up of columns $d_{j-1} + 1, \ldots, d_j$ of A. By $\Phi(A)$ denote the d-flag $V_1 \subset \cdots \subset V_r$, where

$$V_m := \sum_{i=1}^{m} C(A_i) \quad \text{for } 1 \le i \le m.$$

It is important to note that the sum appearing on the right in the above equation is an internal direct sum of subspaces of k^n.

Theorem 3.2: The following holds.

(i) The map $\Phi : \mathrm{GL}(n,k) \to \mathrm{FL}(d,V)$ given by $A \to \Phi(A)$ is surjective.

(ii) $\Phi(A) = \Phi(B)$ if and only if $A = Bg$ for some $g \in P(n,d,k)$.

Proof: Fix a d-flag $\mathcal{F} : V_1 \subset \cdots \subset V_r$. Then, find an ordered k-basis $b(1), \ldots, b(n)$ of k^n such that $V_m = \oplus_{1 \le i \le d_m} k \cdot b(i)$ for $1 \le m \le r$. Define A_{j+1} to be the $n \times e_{j+1}$ matrix $[b(d_j + 1), \ldots, b(d_{j+1})]$ for $0 \le j \le r$ and let $A := [A_1, \ldots, A_{r+1}]$. Clearly, $A \in \mathrm{GL}(n,k)$ and $\Phi(A) = \mathcal{F}$. Thus Φ is surjective. Suppose $g \in P(n,d,k)$ is the $(r+1) \times (r+1)$ block-upper-triangular matrix $[M_{ij}]$ (as in the definition of $P(n,d,k)$) and let $[B_1, \ldots, B_{r+1}] =: B \in \mathrm{GL}(n,k)$. Then, $Bg = [B_1^*, \ldots, B_{r+1}^*]$, where

$$B_j^* \;:=\; B_1 M_{1j} + B_2 M_{2j} + \cdots + B_j M_{jj} \quad \text{for } 1 \le j \le r+1.$$

Since M_{jj} is invertible, $C(B_j M_{jj}) = C(B_j)$ for $1 \le j \le r+1$. By a straightforward induction,

$$C(B_1^*) + \cdots + C(B_m^*) \;=\; C(B_1) + \cdots + C(B_m) \quad \text{for } 1 \le m \le r+1.$$

Hence $\Phi(B) = \Phi(Bg)$. Conversely, suppose

$$A \;:=\; [A_1, \ldots, A_{r+1}] \quad \text{and} \quad [B_1, \ldots, B_{r+1}] \;=:\; B \in \mathrm{GL}(n,k)$$

are such that $\Phi(A) = \Phi(B) := V_1 \subset \cdots \subset V_r$. Let $V_{r+1} := k^n$. Now since $C(A_1) = V_1 = C(B_1)$, there is a $M_{11} \in \mathrm{GL}(e_1, k)$ such that $A_1 = B_1 M_{11}$. Inductively, assume that $A_j = B_1 M_{1j} + \cdots + B_m M_{jj}$, where $M_{ij} \in \mathbb{M}(e_i, e_j, k)$ and $M_{jj} \in \mathrm{GL}(e_j, k)$ for $1 \le i \le j \le m \le r$. Since $\Phi(A) = \Phi(B)$, we have

$$V_m \oplus C(A_{m+1}) \;=\; \oplus_{i=1}^{m+1} C(A_i) \;=\; V_{m+1} \;=\; \oplus_{i=1}^{m+1} C(B_i) \;=\; V_m \oplus C(B_{m+1})$$

and hence there are natural k-linear isomorphisms

$$C(A_{m+1}) \;\cong\; \frac{V_{m+1}}{V_m} \;\cong\; C(B_{m+1}).$$

Consequently, there exists $M_{(m+1)(m+1)} \in \mathrm{GL}(e_{m+1}, k)$ such that

$$C(A_{m+1} - B_{m+1} M_{(m+1)(m+1)}) \;\subseteq\; V_m.$$

In other words, for $1 \le i \le m$, there exist matrices $M_{ij} \in \mathbb{M}(e_i, e_j, k)$ with $A_{m+1} = B_1 M_{1(m+1)} + \cdots + B_{m+1} M_{(m+1)(m+1)}$. This proves 2. □

Definitions 3.3: Fix a positive integer e not exceeding n. Let $s : s_1 < \cdots < s_e$ be a sequence of integers in $[n]$.

(i) Define $A[s] := [a_{ij}] \in \mathcal{R}(n,e,k)$ by setting

$$a_{ij} \;=\; \begin{cases} 1 & \text{if } i = s_j, \\ 0 & \text{if } i \neq s_j \end{cases} \qquad \text{for all } (i,j) \in [n] \times [e].$$

(ii) $M := [m_{ij}] \in \mathbb{M}(n,e,k)$ is said to be in *s-reduced form* (resp. *anti s-reduced* form) provided

$$m_{ij} \;=\; \begin{cases} 1 & \text{if } i = s_j, \\ 0 & \text{if } i = s_p \text{ for } p \neq j \text{ and} \\ 0 & \text{if } i < s_j \text{ (resp. } i > s_j) \end{cases}$$

for all $(i,j) \in [n] \times [e]$. M is said to be an *s-reduced* form of $N \in \mathcal{R}(n,e,k)$ if M is in s-reduced form and there exists a matrix $g \in \mathrm{GL}(e,k)$ such that $M = Ng$.

Lemma 3.4: The following holds.

(i) For $s : s_1 < \cdots < s_e$ in $[n]$, the matrix $A[s]$ is in s-reduced form (resp. in anti s-reduced form).

(ii) If $s : s_1 < \cdots < s_e$ and $\sigma : \sigma_1 < \cdots < \sigma_e$ are in $[n]$ and $M \in \mathcal{R}(n, e, k)$, $g \in \mathrm{GL}(e, k)$ are such that M is in s-reduced (resp. anti s-reduced) form and Mg is in σ-reduced (resp. anti-reduced) form, then $s = \sigma$ and $g = I$ (the identity matrix).

(iii) Given $N \in \mathcal{R}(n, e, k)$, there exists a unique sequence $s : s_1 < \cdots < s_e$ in $[n]$ such that N has an s-reduced (resp. anti s-reduced) form.

Proof: Assertion (i) clearly holds. Assertions (ii),(iii) are verified by a straightforward use of column-reduction to obtain the 'reduced column-echelon form'; the 'anti' versions of (ii), (iii) can be established similarly. Below we present an essential outline of the process (inviting the reader to formulate its 'anti' version).

For $1 \leq j \leq e$, we describe a three-step process. Step 1j: by a suitable permutation of the columns $j, \ldots, e$, ensure that the j-th column has a nonzero entry in some (say) s_j-th row whereas the rows above the s_j-th row have only 0 as their entry in columns $j, \ldots, e$. Step 2j: multiply the j-th column by the reciprocal of the entry in the s_j-th row. Step 3j: subtracting suitable multiples of the j-th column from each of the remaining columns make sure that the entries appearing in the s_j-th row and columns other than the j-th column, are all 0. To obtain the reduced form of a given $N \in \mathcal{R}(n, e, k)$, perform the above three-step process starting from column $j = 1$ of N and then perform the process for column $j = 2$ on the updated matrix, and then perform the process for column $j = 3$ on the updated matrix and so on. $\square$.

Definitions 3.5: Let $d : d_1 < d_2 < \cdots < d_r$ and $e_1, \ldots, e_{r+1}$ be as above.

(i) For a positive integer e, let $S[e, n]$ denote the set of all sequences $s : s_1 < \cdots < s_e$ in $[n]$. An e-element subset of $[n]$ is viewed (via the natural ordering of its elements) as a member of $S[e, n]$.

(ii) Given a subset (possibly empty) $H \subseteq [n]$, a sequence $s \in S[e, n]$ and a matrix $M \in \mathcal{R}(n, e, k)$, we say M is in (s, H)-*reduced* (resp. *anti* (s, H)-*reduced*) form if M is in s-reduced (resp. anti s-reduced) form and for each $i \in H$, the i-th row of M is 0.

(iii) Given subsets $\sigma_1, \ldots, \sigma_m$ of $[n]$, we write $\sigma_1 + \cdots + \sigma_m = [n]$ to mean that $\sigma_1, \ldots, \sigma_m$ form a partition of $[n]$.

(iv) Let $\pi(d)$ be the subset of $S[e_1, n] \times S[e_2, n] \times \cdots \times S[e_{r+1}, n]$ consisting of $(\sigma_1, \ldots, \sigma_{r+1})$ such that $\sigma_1 + \cdots + \sigma_{r+1} = [n]$.

(v) A matrix $A \in G := \mathrm{GL}(n, k)$ is said to be in $(\sigma_1, \ldots, \sigma_{r+1})$-*form* (resp. *anti* $(\sigma_1, \ldots, \sigma_{r+1})$-*form*) provided $(\sigma_1, \ldots, \sigma_{r+1}) \in \pi(d)$ and $A = [A_1, \ldots, A_{r+1}]$, where A_j is in $(\sigma_j,\ \sigma_1 \cup \cdots \cup \sigma_{j-1})$-reduced form (resp. anti $(\sigma_1, \ldots, \sigma_{r+1})$-form) for $1 \leq j \leq r + 1$ (convention: if $j = 1$, then $\sigma_1 \cup \cdots \cup \sigma_{j-1} := \emptyset$).

Remark 3.6: Note that for there to be an M in (s, H)-reduced (resp. anti (s, H)-reduced) form, it is necessary that $s \cap H = \emptyset$.

Lemma 3.7: Let $A \in \mathrm{GL}(n, k)$. Then, there exists a unique $(\sigma_1, \ldots, \sigma_{r+1}) \in \pi(d)$ and a unique $g \in P(n, d, k)$ such that Ag is in $(\sigma_1, \ldots, \sigma_{r+1})$-form (resp. anti $(\sigma_1, \ldots, \sigma_{r+1})$-form).

Proof: Again, the 'anti' version of the proof is left to the reader; it is easily obtained by suitable modification of the following arguments. By induction on $m \leq r + 1$ we find matrices $M_{ij} \in \mathbb{M}(e_i, e_j, k)$, with $M_{ii} \in \mathrm{GL}(e_i, k)$ such that $B_j := A_1 M_{1j} + \cdots + A_j M_{jj}$ is in $(\sigma_j,\ \sigma_1 \cup \cdots \cup \sigma_{j-1})$-reduced form for $1 \leq i \leq j \leq m$. Case $m = 1$: thanks to Lemma 3.4, there is a unique $M_{11} \in \mathrm{GL}(e_1, k)$ and a unique $\sigma_1 \in S[e_1, n]$ such that $B_1 := A_1 M_{11}$ is in $(\sigma_1,\ \emptyset)$-reduced form. Case $m \geq 2$: By induction we assume that we have found the desired M_{ij} for $1 \leq i \leq j \leq m-1$. For $1 \leq t \leq m - 1$, let $N_t \in \mathbb{M}(e_t, e_m, k)$ be such that for each $i \in \sigma_1 \cup \cdots \cup \sigma_{m-1}$, the i-th row of $C_m := A_m + (B_1N_1 + \cdots + B_{m-1}N_{m-1})$ is a zero row. Let $M_{mm} \in \mathrm{GL}(e_m, k)$ be the unique matrix and let $\sigma_m \in S[e_m, n]$ be the unique sequence such that C_mM_{mm} is in σ_m-reduced form. Then, C_mM_{mm} is automatically in $(\sigma_m,\ \sigma_1 \cup \cdots \cup \sigma_{m-1})$-reduced form. Letting $B_m := C_mM_{mm}$

$$M_{im} \ := \left(\sum_{q=i}^{m-1} M_{iq}N_q \right) M_{mm} \quad \text{for } 1 \le i \le m-1$$

we infer that $B_m = A_1M_{1m} + \cdots + A_mM_{mm}$. Also, letting g be the $(r+1) \times (r+1)$ block-upper triangular matrix $[M_{ij}]$, we have $g \in P(n, d, k)$ and $[B_1, \ldots, B_{r+1}] = [A_1, \ldots, A_{r+1}]g$. This establishes the existence part of our assertion.

We proceed to prove the asserted uniqueness. Suppose $A := [A_1, \ldots, A_{r+1}]$ is in $(\sigma_1, \ldots, \sigma_{r+1})$-form and there is $g \in P(n, d, k)$ such that $[B_1, \ldots, B_{r+1}] := Ag$ is in $(\tau_1, \ldots, \tau_{r+1})$-form. Say g is the $(r+1) \times (r+1)$ block-upper triangular matrix $[M_{ij}]$. By (ii) of Lemma 3.4, we at once infer that $\sigma_1 = \tau_1$ and $M_{11} = I$. Hence $B_1 = A_1$. By induction, assume that $\sigma_j = \tau_j$, $M_{jj} = I$ and $M_{ij} = 0$ for $1 \le i < j \le m-1$. Then, we must also have $B_j = A_j$ for $1 \le j \le m-1$ and hence

$$B_m - A_mM_{mm} \ = \ B_{m-1}M_{(m-1)m} + \cdots + B_1M_{1m}.$$

Consider a $1 \le j \le m-1$. For each $i \in \sigma_1 \cup \cdots \cup \sigma_{j-1} = \tau_1 \cup \cdots \cup \tau_{j-1}$, the i-th row of the matrix on the left (in the above equation) as well as the matrix $B_{m-1}M_{(m-1)m} + \cdots + B_jM_{jm}$, is 0. When $j = 2$, using the fact that B_1 is $\tau_1 = \sigma_1$-reduced, we get $M_{1m} = 0$. Repeating this argument for each of $j = 3, \ldots, m-1$, we infer that $M_{im} = 0$ for $1 \le i \le m-1$. Consequently, $B_m = A_mM_{mm}$. Now by (ii) of Lemma 3.4, $\sigma_m = \tau_m$ and $M_{mm} = I$. It follows that $g = I$. □

Definition 3.8: Let $d : d_1 < d_2 < \cdots < d_r$ and $\pi(d)$ be as above. To each partition

$$\sigma \ := \ (\sigma_1, \ldots, \sigma_{r+1}) \ \in \ \pi(d)$$

we associate subsets, or *cells*, W_σ and $\widehat{W}_\sigma$ defined by

$$\begin{aligned} W_\sigma \quad &:= \quad \{A \in \mathrm{GL}(n,k) \mid A \text{ is in } (\sigma_1, \ldots, \sigma_{r+1})\text{-form}\}, \\ \widehat{W}_\sigma \quad &:= \quad \{A \in \mathrm{GL}(n,k) \mid A \text{ is in anti } (\sigma_1, \ldots, \sigma_{r+1})\text{-form}\}. \end{aligned}$$

Theorem 3.9: Given $d : d_1 < d_2 < \cdots < d_r$ in $S[r, n]$, the following holds.

(i) We have the cell-decompositions

$$\frac{\mathrm{GL}(n,k)}{P(n,d,k)} \ \cong \ \bigsqcup_{\sigma \in \pi(d)} W_\sigma \ = \ \bigsqcup_{\sigma \in \pi(d)} \widehat{W}_\sigma.$$

(ii) Let $\Phi : \mathrm{GL}(n,k) \to \mathrm{FL}(d, V)$ be the map as in Theorem 3.2. Then,

$$\mathrm{FL}(d, V) \ = \ \bigsqcup_{\sigma \in \pi(d)} \{\Phi(A) \mid A \in W_\sigma\} \ = \ \bigsqcup_{\sigma \in \pi(d)} \left\{\Phi(A) \mid A \in \widehat{W}_\sigma\right\}.$$

In (i) and (ii) above, the symbol $\bigsqcup$ denotes disjoint union.

Proof: By Lemma 3.7, for a left-coset L of $P(n, d, k)$ in $\mathrm{GL}(n, k)$, there is a unique $\sigma \in \pi(d)$ and a unique $A \in W_\sigma$ (resp. $A \in \widehat{W}_\sigma$) such that $L = AP(n, d, k)$. Mapping a left-coset L to its representative yields the bijective correspondence asserted in (i). In view of Theorem 3.2, (ii) follows from (i). □

4. Dimension Counting.

Fix $d : d_1 < d_2 < \cdots < d_r$ in $S[r, n]$ with $d_r < n$. As before, let $d_0 := 0$ and $d_{r+1} := n$. For notational simplicity, a sequence $s \in S[e, n]$ is henceforth written as $s(1) < s(2) < \cdots < s(e)$.

Definitions 4.1: Let $\sigma := (\sigma_1, \ldots, \sigma_{r+1}) \in \pi(d)$. As before, let $S(M)$ be the set of permutations of the multiset $M := \{1^{e_1}, 2^{e_2}, \ldots, r^{e_r}, (r+1)^{e_{r+1}}\}$.

(i) For $j \in [n]$, define

$$\sigma(j) \;:=\; \sigma_m(j - d_m) \quad \text{if } d_m < j \leq d_{m+1} \text{ for some } 0 \leq m \leq r.$$

(ii) For $j \in [n]$ with $d_m < j \leq d_{m+1}$ (where $0 \leq m \leq r$), define

$$\Delta(\sigma, j) \;:=\; \{t \in [n] \mid t > \sigma(j)\} \setminus \{\sigma(1), \ldots, \sigma(d_{m+1})\}$$

and let $\delta(\sigma, j) \;:=\; |\Delta(\sigma, j)|$.

(iii) Let $\lambda(\sigma) := \sum_{j=1}^{j=n} \delta(\sigma, j)$.

(iv) Define $\Theta_\sigma \in S(M)$ by $\Theta_\sigma(i) := n - \sigma(i) + 1$ for all $i \in [n]$.

Remarks 4.2: Let $\nu(d) := \sum_{1 \leq i < j \leq r+1} e_i e_j$.

(i) Observe that σ, as defined in the first of the above definitions, is a permutation of $[n]$. The corresponding permutation Θ_σ is also called the 'opposite' or the 'dual' of σ.

(ii) Clearly, if $d_{m-1} < j \leq d_m$, then $\delta(\sigma, j) \leq n - d_m = e_{m+1} + \cdots + e_{r+1}$. In particular, $\delta(\sigma, j) = 0$ for $j > d_r$ and $\lambda(\sigma) \leq \nu(d)$. Moreover, $\delta(\sigma, j) = n - d_{\mu(j)}$ for $1 \leq j \leq d_r$ if and only if $\lambda(\sigma) = \nu(d)$ if and only if σ is the identity permutation. At the opposite extreme, $\delta(\sigma, i) = 0$ for $1 \leq i \leq j$ if and only if $\lambda(\sigma) = 0$ if and only if $\sigma(i) : n - d_i + 1 < \cdots < n - d_{i-1}$ for $1 \leq i \leq r+1$.

(iii) Consider the case where $r = 1$, *i.e.*, $e_1 = d_1$ and $e_2 = n - d_1$. Suppose k is an integer with $0 \leq k \leq \nu(d) = e_1 e_2$. Let $a, b \in \mathbb{N}$ be such that $k = a e_2 + b$ and $b < e_2$. Let $\tau := (\tau_1, \tau_2) \in \pi(d)$ be such that

$$\tau_1 \;=\; \{j \;\mid\; 1 \leq j \leq a\} \cup \{n - j \;\mid\; 0 \leq j \leq e_1 - a - 1\} \cup \{n - e_1 + a - b\}.$$

Then, it is easy to verify that $\lambda(\tau) = k$.

Theorem 4.3: The following holds.

(i) Let $A := [a_{ij}] \in \mathrm{GL}(n, k)$ and $\sigma \in \pi(d)$. Then, $A \in W_\sigma$ if and only if

$$a_{ij} \;=\; \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{if } i \neq \sigma(j) \text{ and } i \notin \Delta(\sigma, j). \end{cases}$$

(ii) $\Theta : \pi(d) \to S(M)$ defined by $\sigma \to \Theta_\sigma$ is a bijective map. Moreover, letting $\theta := \Theta_\sigma$, we have $\lambda(\sigma) = inv(\theta)$.

(iii) Letting $\theta := \Theta_\sigma$, we have $\lambda(\sigma) = inv(\theta)$. So,

$$W_\sigma \;\cong\; k^{\lambda(\sigma)} \;=\; k^{inv(\theta)}.$$

Proof: If $A \in W_\sigma$, then it is straightforward to verify that the entries of A satisfy (i). Conversely, suppose entries of A satisfy (i). As before, let $A := [A_1, \ldots, A_{r+1}]$ and let $\sigma := (\sigma_1, \ldots, \sigma_{r+1})$. Let m be an integer such that $0 \leq m \leq r$. If $d_m < j \leq d_{m+1}$, then the j-th column of A is the $(j - d_m)$-th column of A_{m+1}. Consider the q-th column of A_{m+1}. Letting $j := q + d_m$, we have $a_{ij} = 1$ if $i = \sigma(j) = \sigma_{m+1}(q)$, $a_{ij} = 0$ if $i < \sigma(j) = \sigma_{m+1}(q)$ and $a_{ij} = 0$ if $\sigma(j) = \sigma_{m+1}(q) < i \in \sigma_1 \cup \cdots \cup \sigma_{m+1}$. Thus A_{m+1} is in $(\sigma_{m+1},\ \sigma_1 \cup \cdots \cup \sigma_m)$-reduced form. This proves (i).

It is straightforward to verify that Θ is a bijective map. Fix a $\sigma \in \pi(d)$ and let $\theta := \Theta_\sigma$. For $1 \leq i < j \leq n$, we have $\mu(i) \leq \mu(j)$ and $\theta(i) > \theta(j)$ if and only if $\sigma(i) < \sigma(j)$ if and only if $\sigma(j) \in \Delta(\sigma, i)$. In other words, for $i \in [n]$, the set $\{j \in [n] \;\mid\; i < j, \quad \theta(i) > \theta(j)\}$ is in one-to-one correspondence with the set $\Delta(\sigma, i)$. Now it readily follows that $\lambda(\sigma) = inv(\theta)$. In view of Theorem 3.9, (iii) follows from (i) and (ii). □

Corollary 4.4: Let $d : d_1 < \cdots < d_r$ be in $S[r, n]$ with $d_r < n$, $d_0 := 0$, $d_{r+1} := n$ and let $e_i := d_i - d_{i-1}$ for $1 \leq i \leq r+1$. Assume k is a finite field with $|k| = q$. Then, the following holds.

(i)

$$|\mathrm{FL}(d,V)| \;=\; \frac{\prod_{i=0}^{n-1}(q^n-q^i)}{\prod_{i=1}^{r+1}\prod_{j=0}^{e_i-1}(q^{e_i}-q^j)\prod_{1\le i<j\le r+1} q^{e_ie_j}} \;=\; \binom{n}{e_1,\ldots,e_r}_q .$$

(ii)

$$|\mathrm{FL}(d,V)| \;=\; \sum_{\theta\in S(M)} q^{inv(\theta)},$$

(iii) and so

$$\sum_{\theta\in S(M)} q^{inv(\theta)} \;=\; \binom{n}{e_1,\ldots,e_r}_q .$$

Proof: By Theorem 3.9, $|\mathrm{FL}(d,V)| = |\mathrm{GL}(n,k)/P(n,d,k)|$. Since we have $|\mathrm{GL}(n,k)| = \prod_{i=0}^{n-1}(q^n-q^i)$ and

$$|P(n,d,k)| \;=\; \left(\prod_{i=1}^{r+1}\prod_{j=0}^{e_i-1}(q^{e_i}-q^j)\right)\left(\prod_{1\le i<j\le r+1} q^{e_ie_j}\right),$$

the first equality in (i) follows. The second equality asserted in (i) is essentially the equality (1.1). In view of Theorem 3.9 and (iii) of Theorem 4.3, (ii) holds. Since (i), (ii) and hence (iii), hold for infinitely many q, it follows in particular that (iii) is a polynomial identity. □

Remark 4.5: It can be easily verified that $\widehat{W}_\sigma \;\cong\; k^{inv(\sigma)}$. So, decomposing $\mathrm{FL}(d,V)$ into cells $\widehat{W}_\sigma$, we can identify $\pi(d)$ with $S(M)$ in a straightforward manner (*i.e.*, without Θ). Our preference for the cells $\widehat{W}_\sigma$ is rooted in the belief that σ-forms (of matrices) are more familiar than anti σ-forms.

5. Multiset-permutations with fixed number of inversions.

Let t be an indeterminate and let $w := (w_1,\ldots,w_n)$ be an n-tuple of positive integers (where n is also a positive integer). For $m\in\mathbb{Z}$, let $D_w(m)$ be defined by the equation

$$\prod_{i=1}^{n}\frac{1}{(1-t^{w_i})} \;=\; \sum_{m\in\mathbb{Z}} D_w(m)t^m.$$

Since the rational function on the left is a power-series in t with coefficients in $\mathbb{N}$, we have $D_w(m)\in\mathbb{N}$ for all $m\in\mathbb{Z}$ and $D_w(m)=0$ for $m<0$. Also, observe that $D_w(0)=1$. For $m\in\mathbb{N}$, the integer $D_w(m)$ is known as the *Sylvester's denumerant*; clearly,

$$D_w(m) \;=\; |\{(i_1,\ldots,i_n)\in\mathbb{N}^n \;\mid\; i_1w_1+\cdots+i_nw_n=m\}|.$$

It is well known that if $\lambda := lcm(w_1,\ldots,w_n)$, then for each j with $0\le j\le\lambda-1$, there is a polynomial $Q_j(t)\in\mathbb{Q}[t]$ of degree $n-1$ such that $D_w(m)=Q_j(m)$ provided $m\equiv j \ mod\ \lambda$. So,

$$D_w(m) = P_0(w;m) + P_1(w;m)m + \cdots + P_{n-1}(w;m)m^{n-1},$$

where each $P_i(w;m)$ is a $\mathbb{Q}$-valued periodic function of m. For more on this topic the reader is referred to [1], [2] and their list of references.

Definitions 5.1: Let n be a positive integer and let $d: d_1<\cdots<d_r$ be a sequence of positive integers with $d_r<n$; as before, $d_0:=0$, $d_{r+1}:=n$

(i) Given $a,b\in\mathbb{Z}$, let

$$\binom{a}{b} := \begin{cases} 0 & \text{if } \min\{a,b\}<0,\\ 0 & \text{if } a<b \text{ and}\\ \prod_{0\le i\le b-1}\frac{(a-i)}{i+1} & \text{if } 0\le b\le a.\end{cases}$$

(ii) For a subset $E \subseteq [n]$, let $\omega(E) := \sum_{i \in E} i$. For $r \in \mathbb{N}$, let

$$\psi_n(r) := \sum_{E \in \Omega(n,r)} (-1)^{|E|}, \quad \text{where} \quad \Omega(n,r) := \{E \subseteq [n] \mid \omega(E) = r\}.$$

(iii) Define

$$\varepsilon(d,n) := (w_1, \ldots, w_n), \quad \text{where } w_i = i - d_j \text{ provided } d_j < i \leq d_{j+1}.$$

(iv) For $k \in \mathbb{N}$, let $I_n(d;k)$ denote the number of permutations $\theta \in S(M)$ with $inv(\theta) = k$. In the special case where $r = n-1$ (and hence $d : 1 < 2 < \cdots < n-1$), we write $I_n(k)$ instead of $I_k(d;n)$. It is convenient to allow d to be the empty sequence (in which case, $\pi(d)$ is (by convention) the trivial subgroup of S_n).

(v) A sequence $d^* : d_1^* < \cdots < d_s^* < n$ is called a *refinement of* d if each d_i is a member of d^*.

(vi) For $n \in \mathbb{N}$, let $f_n(t) := (1-t)(1-t^2)\cdots(1-t^n) \in \mathbb{Z}[t]$ and let $G_n(t) \in \mathbb{Z}[t]$ be defined by $G_n(t) := f_n(t)/(1-t)^n$.

Remarks 5.2:

(i) It is easily seen that $\psi_n(r)$ is the coefficient of t^r in $f_n(t)$. From the identity $f_n(t) = (-1)^n \cdot t^{n(n+1)/2} \cdot f_n(1/t)$, we deduce that

$$\psi_n(r) = (-1)^n \cdot \psi_n\left(\frac{n(n+1)}{2} - r\right).$$

Applying Euler's pentagonal number theorem, it can be easily verified that

$$\text{for } 1 \leq r \leq n, \quad \psi_n(r) = \begin{cases} (-1)^s & \text{if } 2r = s(3s \pm 1) \text{ with } s \in \mathbb{N}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer k, define the *restricted divisor-sum*

$$\sigma_n(k) := \sum_{1 \leq d \leq n,\ d|k} d$$

and let $\alpha_n(k) := \sigma_n(k)/k$. Then, the (formal) identity $f_n(t) = exp(log(f_n(t))$ provides the formula

$$\psi_n(r) = \sum_{i_1 + 2i_2 + \cdots + mi_m = r} \frac{(-1)^{i_1+i_2+\cdots+i_m}}{i_1! i_2! \cdots i_m!} \alpha_n(1)^{i_1} \alpha_n(2)^{i_2} \cdots \alpha_n(m)^{i_m}.$$

There is yet another such formula that can be derived from the pentagonal number expansion; but it is equally complicated and perhaps of little use. We also have the obvious inequality

$$|\psi_n(r)| \leq \binom{n-1+r}{n-1} \quad \text{for all } r \in \mathbb{N}.$$

For $n < r < n(n-1)/2$, very little seems to be known regarding the size or sign of $\psi_n(r)$.

(ii) Note that $I_n(k)$ is the number of permutations of $\{1, \ldots, n\}$ having exactly k inversions and so,

$$\sum_{k \in \mathbb{N}} I_n(k) t^k = \binom{n}{1, \ldots, 1}_t = G_n(t).$$

Since $G_n(t) = t^{n(n-1)/2} \cdot G_n(1/t) = G_{n-1}(t)(1 + t + \cdots + t^{n-1})$, we have

$$I_n(k) = I_n\left(\frac{n(n-1)}{2} - k\right) \quad \text{and} \quad I_n(k) = \sum_{j=\max\{0,k-n+1\}}^{k} I_{n-1}(j).$$

In particular, $I_n(k) \geq 1$ for $0 \leq k \leq n(n-1)/2$.

Theorem 5.3: Let the notation be as above; in particular, let $w := (w_1, \ldots, w_n)$ be an n-tuple of positive integers and let $d : d_1 < \cdots < d_r$ be a sequence of positive integers with $d_r < n$. As before, let $e_i := d_i - d_{i-1}$ for $1 \le i \le r+1$, where $d_0 = 0$ and $d_{r+1} = n$.

(i) For $r \in \mathbb{N}$, we have

$$\frac{\prod_{i=1}^{r}(1-t^i)}{\prod_{i=1}^{n}(1-t^{w_i})} = \sum_{m\in\mathbb{N}} \left(\sum_{E\subseteq[r]} (-1)^{|E|} D_w(m-\omega(E)) \right) t^m.$$

(ii) For $k \in \mathbb{N}$, we have

$$I_n(d;k) = \sum_{E\subseteq[n]} (-1)^{|E|} D_{\varepsilon(d,n)}(k-\omega(E)) = \sum_{i=0}^{k} \psi_n(i) D_{\varepsilon(d,n)}(k-i).$$

(iii) For $k \in \mathbb{N}$, we have

$$I_n(k) = \sum_{E\subseteq[n]} (-1)^{|E|} \binom{n-1+k-\omega(E)}{n-1} = \sum_{i=0}^{k} \psi_n(i) \binom{n-1+k-i}{n-1}.$$

(iv) Let d^* be a refinement of d and let

$$\mathcal{A}_m := \{ j := (j_1, \ldots, j_{r+1}) \in \mathbb{N}^{r+1} \mid j_1 + \cdots + j_{r+1} = m \}.$$

Then, for $1 \le i \le r+1$, there is a (possibly empty) sequence $d[i]^*$ of positive integers $< e_i$ determined by d^* (in a canonical manner). Furthermore,

$$I_n(d;k) = I_n(d^*;k) - \sum_{m=1}^{k} \left(\sum_{j\in\mathcal{A}_m} \prod_{m=1}^{r+1} I_{e_m}(d[m]^*; j_m) \right) I_n(d;k-m)$$

for all $k \in \mathbb{N}$. In particular, $I_n(d;k) \le I_n(d^*;k)$.

(v) Let $\nu(d) := \sum_{1\le i<j\le r+1} e_i e_j$. Then, $\nu(d) \le n(n-1)/2$,

$$I_n(d;k) = I_n(d;\nu(d)-k) \quad \text{for all } k \in \mathbb{N}$$

and $I_n(d;k) \ge 1$ for $0 \le k \le \nu(d)$.

Proof: We prove (i) by induction on r. If $r = 0$, then since $[r] = \emptyset$ and $\prod_{i=1}^{r}(1-t^i) = 1$, assertion (i) trivially holds. Fix a positive integer r such that (i) holds for $r-1$. If $f := \sum c(m)t^m \in \mathbb{Q}[[t]]$, then observe that the coefficient of t^m in the product $(1-t^r)f$ is $c(m) - c(m-r)$. In particular, the coefficient of t^m in the product

$$(1-t^k) \cdot \sum_{m=0} \left(\sum_{E\subseteq[r-1]} (-1)^{|E|} D_w(m-\omega(E)) \right) t^m$$

is the difference

$$\sum_{S\subseteq[r-1]} (-1)^{|S|} D_w(m-\omega(S)) - \sum_{S\subseteq[r-1]} (-1)^{|S|} D_w(m-\omega(S)-r).$$

Given a subset $E \subseteq [r]$, letting $S := E \cap [r-1]$, we have either $E = S$ or $E = S \cup \{r\}$. In the first case $\omega(E) = \omega(S)$ and in the second case, $\omega(E) = \omega(S) + r$. So, in view of our induction hypothesis, (i) holds for r.

Let d be as in (ii) and let $d_0 := 0$, $d_{r+1} := n$ and let $e_i := d_i - d_{i-1}$ for $1 \le i \le r+1$. Combining our earlier observations, we get

$$\sum_{k\ge 0} I_n(d;k)t^k = \binom{n}{e_1, \ldots, e_r}_t = \frac{\prod_{i=1}^{n}(1-t^i)}{\prod_{i=1}^{r+1}\prod_{j=1}^{e_i}(1-t^j)}.$$

Consequently, (ii) is a special case of (i) in which $w = \varepsilon(d, n)$. As is well known,

$$D_{(1,\ldots,1)}(m) = \binom{n-1+m}{n-1} \qquad \text{for all } m \in \mathbb{N}.$$

Hence (iii) is a special case of (i) in which w is the n-tuple $(1, \ldots, 1)$.

As in (iv), let d^* be a refinement of d. Given $1 \leq i \leq r+1$, suppose $d_{i-1} < d_j^* < \cdots < d_{j+s}^* < d_i$ and then, let

$$d[i]^* :\ (d_j^* - d_{i-1}) < (d_{j+1}^* - d_{i-1}) < \cdots < (d_{j+s}^* - d_{i-1}).$$

Note that $d[i]^*$ may be empty. Let $e_1^*, \ldots, e_{p+1}^*$ be the sequence of the consecutive differences of the members of d^* (with $d_0^* = 0$ and $d_{p+1}^* = n$). Likewise, for $1 \leq i \leq r+1$, let $e[i]_1^*, \ldots, e[i]_{p[i]+1}^*$ be the sequence of the consecutive differences of the members of $d[i]^*$ (with $d[i]_0^* = 0$ and $d[i]_{p[i]+1}^* = e_i$). Observe that

$$\binom{n}{e_1, \ldots, e_{r+1}}_t = \binom{n}{e_1, \ldots, e_{p+1}}_t \prod_{i=1}^{r+1} \binom{n}{e[i]_1^*, \ldots, e[i]_{p[i]+1}^*}_t.$$

Now (iv) follows by equating the coefficients of like powers on the both sides of this equation.

Clearly, we have

$$\nu(d) = \frac{n(n+1)}{2} - \sum_{i=1}^{r+1} \frac{e_i(e_i+1)}{2} = \frac{n(n+1)}{2} - n + \sum_{i=1}^{r+1} \frac{e_i(e_i-1)}{2} \leq \frac{n(n-1)}{2}.$$

Also, it is straightforward to verify that

$$\binom{n}{e_1, \ldots, e_{r+1}}_t = t^{\nu(d)} \cdot \binom{n}{e_1, \ldots, e_{r+1}}_{1/t} = \sum_{k \in \mathbb{N}} I_n(d; \nu(d) - k) t^k$$

and hence the second part of assertion (v) readily follows. From the last remark preceding Theorem 4.3 it follows that the coefficient of t^i in $\binom{n}{e}_t$ is a positive integer for $0 \leq i \leq e(n-e)$. Since $\binom{n}{e_1,\ldots,e_{r+1}}_t$ is a product of polynomials of the type $\binom{n}{e}_t$, we infer that the coefficient of t^k in it is also positive, *i.e.*, $I_n(d; k) \geq 1$, for $0 \leq k \leq \nu(d)$. □

We continue to use the above notation. Define

$$\eta(d) := \frac{e_1(e_1-1) + \cdots + e_{r+1}(e_{r+1}-1)}{2}$$

Also, for a positive integer k, define

$$[k]+ := \{0 \leq i \leq k \mid \psi_n(i) > 0\} \quad \text{and} \quad [k]- := \{i \in [k] \mid \psi_n(i) < 0\}.$$

From [1; Corollary 3.5], we get

$$\frac{1}{e_1! \cdots e_{r+1}!} \binom{n-1+m}{n-1} \leq D_{\varepsilon(d,n)}(m) \leq \frac{1}{e_1! \cdots e_{r+1}!} \binom{n-1+\eta(d)+m}{n-1}$$

for all $m \in \mathbb{N}$. Hence $I_n(d; k)$ is bounded below by

$$\frac{1}{\prod_{i=1}^{r+1} e_i!} \left\{ \sum_{i \in [k]+} \psi_n(i) \binom{n-1+k-i}{n-1} + \sum_{i \in [k]-} \psi_n(i) \binom{n-1+\eta(d)+k-i}{n-1} \right\}$$

and bounded above by

$$\frac{1}{\prod_{i=1}^{r+1} e_i!} \left\{ \sum_{i \in [k]-} \psi_n(i) \binom{n-1+k-i}{n-1} + \sum_{i \in [k]+} \psi_n(i) \binom{n-1+\eta(d)+k-i}{n-1} \right\}.$$

In particular, if $k \leq n$ (or by symmetry, $k \geq n(n-1)/2$), then these bounds are explicit. Observe that if $d : 1 < \cdots < n-1$, or equivalently if $\eta(d) = 0$, then each of the two bounds for $I_n(d;k)$ coincides with $I_n(k)$. Suppose $n \geq 3$, $k \geq 2$ and d is such that $\eta(d) \geq 1$. Then, since $\psi_n(1) = -1$ and

$$I_n(k) \leq \frac{n(n-1)}{2} \leq \frac{n-1}{k} \cdot \binom{n-1+k-1}{n-1},$$

it is easy to deduce that the above lower bound for $I_n(d;k)$ is ≤ 0. The above upper bound for $I_n(d;k)$ can be $< I_n(k)$, *e.g.*, a MAPLE computation shows that in the case of $d_0 = 0 < d_1 = 1 < d_2 = 10 = n$, we have $I_{10}(12) = 47043$ whereas the above upper bound for $I_n(d;k)$ evaluates to a number less than 44871 and $I_{10}(20) = 230131$ whereas the same upper bound evaluates to a number less than 182032. Under refinements of d, the above upper bound for $I_n(d;k)$ may increase or decrease, *e.g.*, letting $n = 5$, $k = 6$ if $d : 2$, then this upper bound is < 84, if $d : 1 < 2$, then it is 104 and if $d : 1 < 2 < 3$, then it is 77.

Remarks 5.4:

(i) In view of the first remark preceding Theorem 5.3, the Netto-Knuth formula for $I_n(k)$ with $k \leq n$ (see [4]) is easily obtained from assertion (iii) of Theorem 5.3.

(ii) For $n \geq 2$, the sequence $\psi_n(i)$ need not be log-concave, *e.g.*, $\psi_6(5) = 1$, $\psi_6(7) = 2$ whereas $\psi_6(6) = 0$.

(iii) For fixed n, the sequence $I_n(k)$ is known to be log-concave (since its generating function is a product of easily verified log-concave polynomials). In contrast, for fixed n and d with $\eta(d) \geq 1$, the sequence $I_n(d;k)$ need not be log-concave, *e.g.*, for $k = 0, 1, 2, \ldots\ldots$, we have $I_7(2 < 4; k) :\ 1, 2, 5, 8, 13, \ldots\ldots.$

REFERENCES:

(i) Agnarsson, G., *On the Sylvester denumerants for general restricted partitions*, Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2002). Congr. Numer. 154 (2002), 49-60.

(ii) Baldoni, V.; Berline, N.; De Loera, J. A.; Dutra, B. E.; Koppe, M.; Vergne, M., *Coefficients of Sylvester's denumerant*, Integers 15 (2015), Paper No. A11, 32 pp.

(iii) Knuth, D., *Subspaces, subsets, and partitions*, J. Combinatorial Theory 10 (1971), 178-180.

(iv) Knuth, D., *The Art of Computer Programming*, Vol. 3, Addison-Wesley, 1998 Reading, MA.

(v) Stanley, R., *Enumerative Combinatorics, Volume 1*, Second edition. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, 2012, Cambridge.

(vi) Sylvester, J. and Franklin, F., *A constructive theory of partitions, arranged in three acts, an interact and an exodian*, Amer. J. Math. 5 (1882), 251-330.

(vii) Wagner, C., *A First Course in Enumerative Combinatorics*, American Mathematical Society, 2020, Providence, RI.

Author information

Shashikant Mulay and Carl Wagner, The University of Tennessee, Knoxville, TN 37996, USA.
E-mail: smulay@tennessee.edu; cwagner@tennessee.edu

Received : 2022-10-29.
Accepted: 2023-04-04.