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A NOTE ON THE CONSTANT $1/e$

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Abstract In this short Note we discuss the main context in which the constant $1/e$ arises (that is, in certain types of probability problems), and offer derivations of it from a pedagogic point of view.

1 Introduction

The constant $1/e = 0.367879$ (6 d.p.) is listed on the celebrated O.E.I.S. (On-Line Encyclopedia of Integer Sequences) [14] in terms of its decimal expansion as Seq. No. A068985, namely, $\{3, 6, 7, 8, 7, 9, 4, 4, 1, 1, 7, \ldots\}$ (its Engel expansion is Seq. No. A059193[1]). It would appear to have its main (that is to say, popular) interpretation in the so called Sultan's Dowry Problem described thus, and—as we hope to show in this paper—it possesses context and attributes of interest to the mathematical community.

Sultan's Dowry Problem. A sultan has granted a commoner the chance to marry one of his n daughters. The commoner will be presented with the daughters one at a time and, when each daughter is viewed, will be told the daughter's dowry (which is fixed in advance) and must immediately decide whether to accept or reject her (he is not allowed to return to a previously rejected daughter for reconsideration). However, the sultan will allow the marriage to take place only if the commoner picks the daughter with the overall highest dowry. Assuming he knows nothing about the distribution of dowries, the commoner's best strategy is to wait until he has seen 37 of the daughters, then pick the first daughter with a dowry that is bigger than any preceding one—in doing so his odds of choosing the daughter with the highest dowry are surprisingly high, at about 37%; as the number of daughters increases, the probability tends towards the value $1/e$.

Remark 1. The problem has also been called the Secretary Problem, whereby an administrator wants to hire for a position the best secretary out of n rankable applicants who are interviewed one by one, in random order—a decision about each particular applicant is to be made immediately after the interview and, once rejected, an applicant cannot be recalled. The probability of selecting the best applicant in the classic problem converges towards $1/e$, and the optimal win probability is always at least this value. The optimal stopping rule prescribes always rejecting the first $n(1/e)$ applicants that are interviewed, and then stopping at the first applicant who is better than every applicant interviewed so far (or continuing to the last applicant if this never occurs), so that the best candidate is selected about 37% of the time. This and the Dowry Problem offer examples of 'optimal stopping strategies'. The constant $1/e$ is also the probability that, upon n people mixing up their hats in a box, nobody (for large n) randomly selects their own hat, a fact which falls under the broad umbrella of derangement problems in discrete mathematics.

Remark 2. From the O.E.I.S. we find other descriptions of the constant, including the trivial observations that it is the (right-half plane) x location of the minimum value (of $e^{-1/e}$) of the function x^x ($x > 0$), the outcome of the integral $\int_0^1 x\sinh(x)\,dx$ (and also $1 - \int_0^1 x\cosh(x)\,dx$), and the sum $\cos(i) + i\sin(i)$, while a more sophisticated interpretation is that it gives the asymptotic density of numbers with an odd number of trailing zeros in their factorial base representation; listed, too, are versions of the constant in summation, functional and limit form, some more

[1] The Engel expansion—also referred to as the Egyptian product—of a positive real number x is that formed by the unique increasing sequence of positive integers $\{a_1, a_2, a_3, \ldots\}$ for which $x = 1/(a_1) + 1/(a_1a_2) + 1/(a_1a_2a_3) + \cdots$.

exotic than others.

We choose at the outset to state a probability problem in which the constant $1/e$ appears as part of the solution, and in doing so work through one of its limit forms; other forms, and matters of interest, then follow. Note that the '$1/e$ strategy' alluded to above has been analysed as one which is not optimal for 'best choice' problems of a related, but different, nature to those set out above (see the recent work of Bruss and Rogers in [2], and references therein).

Problem. Suppose n numbers are selected randomly (with replacement) from a list of the first n integers, each time with probability $1/n$. Then, for large n, any particular integer appears within the random n-long selection with probability $1 - 1/e$.

Solution. Let $\mathcal{S}$ be the set of the first n integers $\mathcal{S} = \{1, 2, 3, \ldots, n\} = \{s_1, s_2, s_3, \ldots, s_n\}$, say, and choose randomly an element s_j of $\mathcal{S}$ to be put into a set $\mathcal{C}$ so that, repeating the exercise n times, the set $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_n\}$ has at most n distinct elements with (for $i = 1, \ldots, n$), $c_i = s_j$ for some $j \in \{1, 2, 3, \ldots, n\}$. Then, for any $p \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned}\Pr\{s_p \notin \mathcal{C}\} &= \Pr\{s_p \neq c_1\} \cdot \Pr\{s_p \neq c_2\} \cdot \;\cdots\; \cdot \Pr\{s_p \neq c_n\} \\ &= \left(\frac{n-1}{n}\right) \cdot \left(\frac{n-1}{n}\right) \cdot \;\cdots\; \cdot \left(\frac{n-1}{n}\right) \\ &= \left(\frac{n-1}{n}\right)^n, \qquad\qquad \text{(S.1)}\end{aligned}$$

whence

$$\Pr\{s_p \in \mathcal{C}\} = 1 - \Pr\{s_p \notin \mathcal{C}\} = 1 - \left(\frac{n-1}{n}\right)^n = 1 - 1/e \qquad \text{(S.2)}$$

for large n, since $\lim_{n\to\infty}\{[(n-1)/n]^n\} = 1/e$.

For completeness here, we prove this last result formally.

Result I.

$$1/e = \lim_{n\to\infty}\{[(n-1)/n]^n\}.$$

Proof. Consider the limit $L_1 = \lim_{x\to\infty}\{[(x-1)/x]^x\}$, which we write as

$$L_1 = \lim_{x\to\infty}\{f(x)^{g(x)}\}, \qquad \text{(I.1)}$$

where $f(x) = (x-1)/x$ (with $\lim_{x\to\infty}\{f(x)\} = 1$) and $g(x) = x$ (with $\lim_{x\to\infty}\{g(x)\} = \infty$); we seek to show that $L_1 = 1/e$. Given an evaluated indeterminate form of 1^∞, a standard transformation allows us to write L_1 as $L_1 = \exp(\gamma)$, for which

$$\begin{aligned}\gamma &= \lim_{x\to\infty}\left\{\frac{\ln[f(x)]}{1/g(x)}\right\} \\ &= \lim_{x\to\infty}\left\{\frac{\ln(1-\frac{1}{x})}{1/x}\right\} \\ &= \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}\{\ln(1-\frac{1}{x})\}}{\frac{d}{dx}\{1/x\}}\right\} \qquad \text{(by L'Hôpital's Rule)} \\ &= \lim_{x\to\infty}\left\{\frac{1}{\frac{1}{x}-1}\right\} \\ &= -1, \qquad\qquad \text{(I.2)}\end{aligned}$$

as required. □

2 A Poisson Consideration of the Optimal Stopping Problem

The Secretary/Sultan problem is a famous one that has been long understood using optimal problem theory [7]. It has captured the interest of many statisticians who have since developed and generalised the problem to cover a range of optimalisation problems and applications [11, 3, 9, 4]. The optimal stopping problem has, for example, been considered as a Poisson process [10, 5, 12, 6, 1], with variations of the model including both non-informative [6] and informative [1] priors. The Poisson distribution, with parameter $\lambda > 0$, has a probability density function (P.D.F.) given by $f(y;\lambda) = \lambda^y \text{exp}(-\lambda)/y! = \Pr\{Y = y\}$, where $y \geq 1$ is the number of occurrences of a discrete random variable Y and λ is the rate of success. Commonly the Poisson P.D.F. is denoted $\text{Poi}(\lambda)$, and it is known that $\lambda = \text{E}(Y) = \text{Var}(Y)$.

As a point of interest, we present a generalised variation of an uninformative Poisson model within a Bayesian framework, and investigate the approximation of the rate of success λ for an increasing sample size n. Consider Y to be a random sample, taking the form of counts—the number of n applicants for a job position, for instance. The optimal stopping rule then prescribes that the optimal sample size, k, is n/e applicants. For example, a random sample Y with sample sizes $n = 3, 4, 5$ would correspond to optimal sample sizes $k = 3/e, 4/e, 5/e$. In the context of the Secretary Problem this would amount to rejecting n/e applicants and hiring the relatively 'better' candidate, as outlined in Section 1, where it has been shown that the probability of selecting the best applicant converges towards $1/e$ as $n \to \infty$ [12]. Continuing with the accepted optimal sample size k as being n/e, we show that the rate of success λ that corresponds to the Poisson distribution converges to $1/e$ for large n, describing another context for this constant.

2.1 Bayesian Inference

Prior Distribution

In a Bayesian framework we need to specify prior distributions. In explorations of priors for this problem it has been shown that a gamma prior is plausible for the rate of success parameter. Thus, we assign a gamma distribution to λ, so that $\lambda \sim \Gamma(a, b)$ where shape parameter a and rate parameter b are assigned values $a = b = 1$ to express prior ignorance. An advantage of a Bayesian approach is that priors can be adjusted to the problem at hand with external information using elicitation techniques, or set up sequentially so that the previous posterior distribution is current prior. These are our future considerations.

Posterior Distribution

We obtain our inference based on the posterior distribution of the parameter vector space $\psi = (y_i; \lambda)$ $(y_1, \ldots, y_n$ are instances of $Y)$. The posterior distribution function of ψ can be obtained as

$$f_{post}(\psi|y) \propto f_{like}(y|\psi) f_{prior}(\psi), \tag{2.1}$$

with likelihood function

$$\begin{aligned} f_{like}(y|\psi) &= \prod_{i=1}^{n} \text{Poi}(y_i|\lambda) \\ &= \prod_{i=1}^{n} [\text{exp}(-\lambda)\lambda^{y_i}]/y_i! \\ &= \prod_{i=1}^{n} \frac{\text{exp}(-n\lambda)\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!} \\ &\propto \text{exp}(-n\lambda)\lambda^t, \end{aligned} \tag{2.2}$$

with $t = \sum_{i=1}^{n} y_i$ and the prior $f_{prior}(\psi)$ given by $f_{prior}(\psi) = f(\lambda)$. Inference on the parameter λ is based on the posterior summaries of the marginal posterior distribution such as the mean, median, standard deviation and quantiles.

A program has been built, using the statistical software R (Version 4.1.2) [13], to apply the Poisson model within a Bayesian framework and explore the inference on λ under varying values of n. To do this we obtain Y, comprising count data, with increasing sample sizes $n = 5, 10, 100, 1,000, \ldots, 1,000,000$. The Poisson distribution is fitted to the set of optimal sample sizes, k, and rounded to the nearest whole number (in this case it represents the number of rejected applicants). We do not show our Bayesian Poisson posterior distribution to be analytically tractable, so we perform Markov Chain Monte Carlo methods [8] to generate samples from the posterior distribution and indirectly obtain elements such as the location and dispersion to provide central values and variability. The location measure is computed using the mean and median along with the measure of dispersion using standard deviation and 95% creditable intervals. The posterior mean, for example, is the expected value of λ under ψ. The below table provides summary statistics of the posterior distribution of λ, indicating that the rate of success converges to $1/e$ for large n (values listed are the posterior mean, median, 95% credible intervals, and standard deviations (all displayed to 3 d.p.)).

Samp. Size n	**Mean** λ	**Median**	**2.5% Quart.**	**97.5% Quart.**	**Stand. Dev.**
5	0.397	0.341	0.064	1.031	0.252
10	0.396	0.367	0.115	0.840	0.186
50	0.374	0.367	0.230	0.557	0.084
100	0.370	0.367	0.261	0.493	0.059
1,000	0.369	0.368	0.330	0.411	0.020
10,000	0.368	0.368	0.357	0.380	0.006
100,000	0.368	0.368	0.364	0.372	0.002
1,000,000	0.368	0.368	0.367	0.369	0.001

3 Further Results

We next look at a more interesting limiting form of $1/e$.

Result II.

$$1/e = \lim_{n\to\infty}\{\sqrt[n]{n!}/n\}.$$

Proof IIA. The proof proceeds along similar lines to those of Result I.

Proof. Consider the limit $L_2 = \lim_{x\to\infty}\{\sqrt[x]{x!}/x\} = \lim_{x\to\infty}\{[(x!)^{1/x}]/x\}$. Stirling's approximation states that, for x large, $x! \sim \sqrt{2\pi x}(x/e)^x$, so that

$$[(x!)^{1/x}]/x \sim (\alpha\sqrt{x})^{1/x}(1/e), \tag{II.1}$$

with $\alpha = \sqrt{2\pi} > 0$; we thus need to establish that

$$\lim_{x\to\infty}\{(\alpha\sqrt{x})^{1/x}]\} = 1. \tag{II.2}$$

The limit is written as

$$M = \lim_{x\to\infty}\{f(x)^{g(x)}\}, \tag{II.3}$$

where $f(x) = \alpha\sqrt{x}$ (with $\lim_{x\to\infty}\{f(x)\} = \infty$) and $g(x) = 1/x$ (with $\lim_{x\to\infty}\{g(x)\} = 0$), having an evaluated indeterminate form of ∞^0. To show $M = 1$, this time we note that $M =$

$\exp(\delta)$, with δ given as

$$\begin{aligned}
\delta &= \lim_{x\to\infty}\left\{\frac{g(x)}{1/\ln[f(x)]}\right\} \\
&= \lim_{x\to\infty}\left\{\frac{\frac{1}{x}}{1/\ln(\alpha\sqrt{x})}\right\} \\
&= \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}\{\frac{1}{x}\}}{\frac{d}{dx}\{1/\ln(\alpha\sqrt{x})\}}\right\} \qquad \text{(by L'Hôpital's Rule)} \\
&= 2\lim_{x\to\infty}\left\{\frac{\ln^2(\alpha\sqrt{x})}{x}\right\}, \qquad \text{(II.4)}
\end{aligned}$$

after a little algebra. Since $\lim_{x\to\infty}\{\ln^2(\alpha\sqrt{x})/x\} = 0$ (left as a reader exercise (L'Hôpital's Rule may be applied again)), then $\delta = 0$ and so $M = 1$. □

Proof IIB.

Proof. The Cauchy-d'Alembert criterion states that the nth root of any (positive) sequence $\{a_n\}_{n\geq 1}$ has its limit as $\lim_{x\to\infty}\{a_{n+1}/a_n\}$—provided the latter exists—and so, setting $a_n = n!/n^n$, a routine proof follows:

$$\begin{aligned}
\lim_{n\to\infty}\{a_{n+1}/a_n\} &= \lim_{n\to\infty}\left\{\frac{n^n}{(n+1)^n}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{(1+\frac{1}{n})^n}\right\} \\
&= 1/e \qquad \text{(II.5)}
\end{aligned}$$

(since (assumed known) $\lim_{n\to\infty}\{(1+\frac{1}{n})^n\} = e) = \lim_{n\to\infty}\{\sqrt[n]{a_n}\} = \lim_{n\to\infty}\{\sqrt[n]{n!/n^n}\} = L_2$.[2] □

Proof IIC. This proof is somewhat different.

Proof. Consider $\ln(\sqrt[n]{n!}/n) = \ln[(n!)^{1/n}/n] = \frac{1}{n}\ln(n!) - \ln(n) = \frac{1}{n}[\ln(1) + \ln(2) + \cdots + \ln(n) - n\ln(n)]$. We further write

$$\begin{aligned}
\ln(\sqrt[n]{n!}/n) &= (1/n)\{[\ln(1) - \ln(n)] + [\ln(2) - \ln(n)] + \cdots + [\ln(n) - \ln(n)]\} \\
&= (1/n)[\ln(1/n) + \ln(2/n) + \cdots + \ln(n/n)] \\
&= (1/n)\sum_{k=1}^{n}\ln(k/n), \qquad \text{(II.6)}
\end{aligned}$$

which (in the limit $n\to\infty$) is a Riemann sum for the definite integral $\int_0^1 \ln(x)\, dx = -1$.[3] Thus, $\ln(\sqrt[n]{n!}/n) = -1$ as $n\to\infty$, whence $-1 = \lim_{n\to\infty}\{\ln(\sqrt[n]{n!}/n)\} = \ln(\lim_{n\to\infty}\{\sqrt[n]{n!}/n\}) = \ln(L_2)$, or $L_2 = e^{-1}$. □

Proof IID. The proof is another constructive one, utilising the following lemma (whose derivation is straightforward and omitted).

Lemma 1. *Suppose* $\{a_n\}_{n\geq 1} = \{a_1, a_2, a_3, \ldots\}$ *is a positive (real) sequence, with* $\lim_{n\to\infty}\{a_n\} = A$. *Then* $\lim_{n\to\infty}\{\sqrt[n]{a_1a_2a_3\cdots a_n}\} = A$.

[2] We are essentially using the inequalities $\liminf_{n\to\infty}\{a_{n+1}/a_n\} \leq \liminf_{n\to\infty}\{\sqrt[n]{a_n}\} \leq \limsup_{n\to\infty}\{\sqrt[n]{a_n}\} \leq \limsup_{n\to\infty}\{a_{n+1}/a_n\}$ for a real positive sequence $\{a_n\}_{n\geq 1}$. For $a_n = n!/n^n$, then $\liminf_{n\to\infty}\{a_{n+1}/a_n\} = \limsup_{n\to\infty}\{a_{n+1}/a_n\} = 1/e$, and so $\lim_{n\to\infty}\{\sqrt[n]{a_n}\} = L_2 = 1/e$.

[3] Note that $\int_\varepsilon^1 \ln(x)\, dx = \varepsilon - \varepsilon\ln(\varepsilon) - 1 =$ (as $\varepsilon \to 0$) $0 - 0 - 1 = -1$.

Proof. We choose $a_n = (1+\frac{1}{n})^n$, with $\lim_{n\to\infty}\{a_n\} = e$, and write (using Lemma 1)

$$\begin{aligned}
e &= \lim_{n\to\infty}\{\sqrt[n]{a_1 a_2 a_3 \cdots a_n}\} \\
&= \lim_{n\to\infty}\left\{\sqrt[n]{\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{n+1}{n}\right)^n}\right\} \\
&= \lim_{n\to\infty}\left\{\sqrt[n]{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\cdots\left(\frac{1}{n-1}\right)\left(\frac{(n+1)^n}{n}\right)}\right\} \\
&= \lim_{n\to\infty}\left\{\sqrt[n]{\frac{(n+1)^n}{n!}}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{n+1}{\sqrt[n]{n!}}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{n}{\sqrt[n]{n!}}\right\} + \lim_{n\to\infty}\left\{\frac{1}{\sqrt[n]{n!}}\right\}
\end{aligned} \tag{II.7}$$

$= 1/L_2 + 0 = 1/L_2$, so $L_2 = 1/e$. □

As the reader will have gathered, a familiarity with basic properties of the limit is assumed by the authors, though effort has been made to include explanatory steps in places so as to avoid confusion and aid clarity.

Proof IIE. This final proof is perhaps the most appealing.

Proof. It is well known that a (real valued) monotonically increasing function $f(x)$ satisfies the inequality chain

$$\int_{a-1}^{b} f(x)\,dx \le \sum_{k=a}^{b} f(k) \le \int_{a}^{b+1} f(x)\,dx, \tag{II.8}$$

where a, b are natural numbers. For $a = 2$, $b = n$, $f(x) = \ln(x)$, it gives

$$\int_1^n \ln(x)\,dx \le \sum_{k=2}^{n} \ln(k) = \sum_{k=1}^{n} \ln(k) \le \int_2^{n+1} \ln(x)\,dx < \int_1^{n+1} \ln(x)\,dx, \tag{II.9}$$

and in turn, writing $F(n) = \frac{1}{n}\sum_{k=1}^{n}\ln(k) - \ln(n)$,

$$\frac{1}{n}\int_1^n \ln(x)\,dx - \ln(n) \le F(n) < \frac{1}{n}\int_1^{n+1} \ln(x)\,dx - \ln(n), \tag{II.10}$$

that is,

$$\mathcal{B}_l(n) \le F(n) < \mathcal{B}_u(n), \tag{II.11}$$

with the lower and upper bounds evaluated as

$$\begin{aligned}
\mathcal{B}_l(n) &= \frac{1}{n}\int_1^n \ln(x)\,dx - \ln(n) &= \frac{1}{n} - 1, \\
\mathcal{B}_u(n) &= \frac{1}{n}\int_1^{n+1} \ln(x)\,dx - \ln(n) &= \ln\left(1+\frac{1}{n}\right) + \ln(n+1)/n - 1,
\end{aligned} \tag{II.12}$$

and forming, to order $1/n$, a bounding interval for $F(n)$ of length $\ln(n)/n$ (a Puiseux series gives $\mathcal{B}_u(n) = -1 + \frac{1}{n}[\ln(n)+1] + O(n^{-2})$ for n large, so that $\mathcal{B}_u(n) - \mathcal{B}_l(n) = \ln(n)/n + O(n^{-2})$). Since each of $\mathcal{B}_l(n), \mathcal{B}_u(n) \to -1^+$ as $n \to \infty$, so does $F(n)$—which has closed form $F(n) = \ln(\sqrt[n]{n!}/n)$ (see Proof IIC)—by the Sandwich Theorem applied to (II.11), completing the proof ($\ln(L_2) = \ln(\lim_{n\to\infty}\{\sqrt[n]{n!}/n\}) = \lim_{n\to\infty}\{\ln(\sqrt[n]{n!}/n)\} = \lim_{n\to\infty}\{F(n)\} = -1$, so $L_2 = e^{-1}$).[4] □

[4]The squeeze is relatively gentle, due largely to the slow convergence of $\mathcal{B}_u(n)$ compared with $\mathcal{B}_l(n)$; the difference $\mathcal{B}_u(n) - \mathcal{B}_l(n)$ first becomes $O(10^{-2})$ at $n = 36$, and decreases only gradually thereafter ($\mathcal{B}_u(100) - \mathcal{B}_l(100) = 4.6102 \times 10^{-2}$, for example).

Result III.

$$1/e = \lim_{n\to\infty}\left\{ \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right\}.$$

This result is a named one, the sequence $\{ \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\}_{n\geq 0}$ known as the Lalescu sequence after the Romanian mathematician Traian Lalescu (1882–1929). It would appear that he posed the convergent sequence value as Problem No. 579 in Vol. 6 of the Romanian journal *Gazeta Matematică* (Bukarest) in 1900 (variant sequences appeared in subsequent issues, beginning in 1913 with $\{(n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\}_{n\geq 0}$ of R.T. Ianculescu which converges to 1 (Problem No. 2042 of Vol. 19)).

Remark 3. As a point of interest, we note that the convergence rate of the expression in each of Results I,III is better than that of Result II. After 100 terms (that is, when $n = 100$) the sequence of values generated by Results I-III are within (resp.) 1.847×10^{-3}, 1.205×10^{-2} and 1.694×10^{-3} of the convergent value $1/e$.

Proof IIIA. The result is delivered from a powerful theorem.

Proof. A fairly high level pedagogic article by D.M. Bătineţu-Giurgiu and N. Stanciu ('New Methods for Calculations of Some Limits', *The Teach. Math.* **XVI**, 82–88 (2013)) contains Result III as a special case of a more general result, and also as an instance of the following which we have restated slightly to suit our needs (see Applications A3 therein, p. 85).

Theorem 1. *Given positive real sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ for which $a = \lim_{n\to\infty}\{x_{n+1}/(n^2x_n)\}$, and $b = \lim_{n\to\infty}\{y_{n+1}/(ny_n)\}$, then*

$$\lim_{n\to\infty}\left\{ \sqrt[n+1]{x_{n+1}/y_{n+1}} - \sqrt[n]{x_n/y_n} \right\} = \frac{a}{be}.$$

A derivation of this may be found in the Appendix. We simply choose $x_n = n!^2$, $y_n = n!$, with

$$\begin{aligned} a &= \lim_{n\to\infty}\{x_{n+1}/(n^2x_n)\} = \lim_{n\to\infty}\{[(n+1)/n]^2\} = 1^2 = 1, \\ b &= \lim_{n\to\infty}\{y_{n+1}/(ny_n)\} = \lim_{n\to\infty}\{(n+1)/n\} = 1, \end{aligned} \tag{III.1}$$

and, with $x_n/y_n = n!$, Lalescu's result is immediate by Theorem 1.[5] □

Proof IIIB. This proof is another technical one, but instructive in its own way.

Proof. Write $\ell_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$—so that we wish to prove $L_3 = \lim_{n\to\infty}\{\ell_n\} = 1/e$—and define

$$\begin{aligned} \alpha_n &= \sqrt[n]{n!}/n, \\ \beta_n &= \frac{\exp\{\ln[(n+1)!]/(n+1) - \ln(n!)/n\} - 1}{\ln[(n+1)!]/(n+1) - \ln(n!)/n}, \\ \gamma_n &= \{n\ln[(n+1)!] - (n+1)\ln(n!)\}/(n+1), \end{aligned} \tag{III.2}$$

from which it is seen (with some algebraic manipulation of β_n) that

$$\ell_n = \alpha_n\beta_n\gamma_n. \tag{III.3}$$

We initially re-write γ_n as

$$\gamma_n = [n\ln(n+1) - \ln(n!)]/(n+1) = u_n/v_n, \tag{III.4}$$

say (with $u_n = n\ln(n+1) - \ln(n!)$, $v_n = n+1$ each unbounded for n large), so that $(u_{n+1} - u_n)/(v_{n+1} - v_n) = \ln(p_n)$, where $p_n = [(n+2)/(n+1)]^{n+1} \to e$ as $n\to\infty$ (readily shown

[5] A 2020 article by Bătineţu-Giurgiu, Stanciu and Díaz-Barrero in the *Arhimede Mathematical Journal*—titled 'The Last Three Decades of Lalescu Limit' (**7**, 16–26)—contains essentially this route (though in a less general form), and presents other Lalescu-type limits drawn from "... several magazines around the world." It is a useful point of reference for those interested in the topic.

via L'Hôpital's Rule). Furthermore, $\lim_{n\to\infty}\{\gamma_n\} = \lim_{n\to\infty}\{u_n/v_n\}$ = (by the Stolz-Cesàro Theorem[6]) $\lim_{n\to\infty}\{(u_{n+1}-u_n)/(v_{n+1}-v_n)\} = \lim_{n\to\infty}\{\ln(p_n)\} = \ln(e) = 1$. Accordingly, with $\lim_{n\to\infty}\{\alpha_n\} = 1/e$ (Result II) and $\lim_{n\to\infty}\{\beta_n\} = 1$ (tedious reader exercise), (III.3) gives

$$L_3 = \lim_{n\to\infty}\{\ell_n\} = \lim_{n\to\infty}\{\alpha_n\beta_n\gamma_n\} = (1/e)\cdot 1\cdot 1 = 1/e, \tag{III.5}$$

and we have the result. □

Proof IIIC. We use the same type of approach as deployed in Proof IIE in so far as bounds are identified for the expression ℓ_n.

Proof. The result is immediate (from the Sandwich Theorem) upon establishing that $\mathcal{L}(n) \le \ell_n \le \mathcal{R}(n)$ for (resp.) left-hand and right-hand bounds $\mathcal{L}(n) = [n/(n+1)]^{n+1}$ and $\mathcal{R}(n) = [n/(n+1)]^n$, each of which $\to 1/e$ as $n\to\infty$ ($\mathcal{L}(n)$ does so from below, $\mathcal{R}(n)$ from above), with to $O(1/n)$ a bounding interval of length $1/(ne)$.[7]

The right-hand bound arrives from the two-variable weighted Arithmetic-Geometric Mean Inequality which states that, for x_1, x_2 positive reals, and positive weights w_1, w_2,

$$(w_1x_1 + w_2x_2)/w \ge \sqrt[w]{(x_1)^{w_1}(x_2)^{w_2}}, \tag{III.6}$$

where $w = w_1 + w_2 > 0$. Setting $x_1 = \sqrt[n]{n!}/n$, $x_2 = [n/(n+1)]^n$, with weights $w_1 = n$, $w_2 = 1$, it reads

$$\begin{aligned} n\cdot\frac{\sqrt[n]{n!}}{n} + 1\cdot\left(\frac{n}{n+1}\right)^n &\ge (n+1)\sqrt[n+1]{\left(\frac{\sqrt[n]{n!}}{n}\right)^n\left[\left(\frac{n}{n+1}\right)^n\right]^1} \\ &= (n+1)\sqrt[n+1]{n!/(n+1)^n} \\ &= \sqrt[n+1]{(n+1)!}, \end{aligned} \tag{III.7}$$

whence

$$\ell_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \le \left(\frac{n}{n+1}\right)^n = \mathcal{R}_n. \tag{III.8}$$

While the lower bound $\mathcal{L}_n$ is verifiable numerically, we have been unable to formulate its analytic form and leave this as an open problem for any interested reader to take on. □

We finish with a series representation of the constant $1/e$—this is the only one of its type that we have been able to locate.

Result IV.

$$1/e = 2(1/3! + 2/5! + 3/7! + 4/9! + \cdots).$$

Proof. Define a series

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty}[(n+1)/(2n+3)!]x^{2n+3} \\ &= (1/3!)x^3 + (2/5!)x^5 + (3/7!)x^7 + (4/9!)x^9 + \cdots. \end{aligned} \tag{IV.1}$$

[6] We apply the case for which the sequence $\{v_n\}_{n\ge1}$ is unbounded for large n, with $0 < v_1 < v_2 < v_3 < \cdots$; it is a discrete version of L'Hôpital's Rule. Lemma 1 (of Proof IID) has in fact been attributed to Cesàro (as the so called "Multiplicative" Cesàro Theorem).

[7] Large n Laurent series for the bounds are found to be $\mathcal{L}(n) = (1/e)[1 - 1/(2n) + 7/(24n^2) - \cdots]$ and $\mathcal{R}(n) = (1/e)[1 + 1/(2n) - 5/(24n^2) + \cdots]$.

Using the (known) series forms of $\cosh(x), \sinh(x)$, we consider

$$\begin{aligned} x\cosh(x) - \sinh(x) &= x\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right)x^{2n+1} \\ &= 2\sum_{n=0}^{\infty}\frac{n}{(2n+1)!}x^{2n+1} \\ &= 2\sum_{n=1}^{\infty}\frac{n}{(2n+1)!}x^{2n+1} \\ &= 2\sum_{n=0}^{\infty}\frac{n+1}{(2n+3)!}x^{2n+3} \\ &= 2S(x). \end{aligned} \tag{IV.2}$$

From (IV.1) $2S(1) = 2(1/3! + 2/5! + 3/7! + 4/9! + \cdots) =$ (by (IV.2)) $\cosh(1) - \sinh(1) =$ (by definition) $\frac{1}{2}(e + e^{-1}) - \frac{1}{2}(e - e^{-1}) = 1/e$, which is Result IV. □

Remark 4. The aforementioned O.E.I.S. Seq. No. A068985 has the result $\sum_{n=0}^{\infty}[(-1)^n/n!](1+n^2) = 1/e$ listed. Since $1/e = \sum_{n=0}^{\infty}(-1)^n/n!$ from the standard Maclaurin series form for e^x, this means that $\sum_{n=0}^{\infty}[(-1)^n/n!]n^2 = 0$. In fact this latter sum is but one of a class of infinite series whose convergent values are multiples of $1/e$ (in this case zero), and we observe the following: writing

$$S_p = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}n^p = \frac{\alpha_p}{e} \qquad p \geq 1, \tag{3.1}$$

the sequence of constants $\{\alpha_0 = 1, \alpha_1, \alpha_2 = 0, \alpha_3, \ldots\} = \{1, -1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, \ldots\}$ have been generated computationally (for $p \geq 1$) and can be found registered as Seq. No. A000587 (with (3.1) duly noted). It comprises the so called Rao Uppuluri-Carpenter numbers (or complementary Bell numbers), named after Venkata Ramamohana Rao Uppuluri and John A. Carpenter (they seem to have also been referred to as Rényi numbers). Among a variety of properties of the sequence $\{\alpha_p\}_{p\geq 0}$, we choose to highlight[8] that, for $p \geq 0$, $\alpha_p = [x^p/p!]\{E(x)\}$, with $E(x) = \exp[1 - \exp(x)]$ acting as its exponential generating function. The nearest to an explicit closed form for the sum is the ${}_{p-1}F_{p-1}(-1)$ hypergeometric series $S_p = -{}_{p-1}F_{p-1}(2, 2, \ldots, 2; 1, 1, \ldots, 1| - 1)$ ($p \geq 1$).

4 Summary

This paper has presented a discussion of the constant $1/e$, giving various mathematical formulations of it and discussing its statistical context. The authors hope that the work will be of interest to a wide readership.

Appendix: Proof of Theorem 1 (Bătineţu-Giurgiu and Stanciu)

Here we prove Theorem 1.

Proof. With sequences $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ as defined (that is, as (resp.) $\{n!^2\}_{n\geq 1}, \{n!\}_{n\geq 1}$), let

$$a_n = \sqrt[n]{x_n/y_n}. \tag{A.1}$$

[8]Thanks are due to Professor Dr. Wolfram Koepf for making these observations (using his bespoke routines within Maple) and communicating them to the author P.J.L. We are also grateful to Dr. James Clapperton for checking some of the computations related to this paper.

We establish first that

$$\lim_{n\to\infty}\{a_n/n\} = \lim_{n\to\infty}\{\sqrt[n]{x_n/y_n}/n\} = \frac{a}{be}. \tag{A.2}$$

This is readily achieved by defining $b_n = x_n/(n^n y_n)$. Then if $\{b_{n+1}/b_n\}_{n\geq 1}$ converges to a limit, so does (to the same limit) the sequence $\{\sqrt[n]{b_n}\}_{n\geq 1} = \{a_n/n\}_{n\geq 1}$ (Cauchy-d'Alembert). Writing the ratio $b_{n+1}/b_n = [x_{n+1}/((n+1)^{n+1}y_{n+1})]\cdot[(n^n y_n)/x_n]$ as $b_{n+1}/b_n = [x_{n+1}/(n^2 x_n)]\cdot[(ny_n)/y_{n+1}]\cdot[n/(n+1)]^{n+1}$, it is immediate that (using (III.1))

$$\lim_{n\to\infty}\{a_n/n\} = \lim_{n\to\infty}\{b_{n+1}/b_n\} = a\cdot(1/b)\cdot(1/e) = \frac{a}{be}. \tag{A.3}$$

Next, consider the ratio

$$\begin{aligned}
\left(\frac{a_{n+1}}{a_n}\right)^n &= \left(\frac{x_{n+1}}{y_{n+1}}\right)^{n/(n+1)}\left(\frac{y_n}{x_n}\right) \\
&= \left(\frac{x_{n+1}}{y_{n+1}}\right)^{n/(n+1)}\left(\frac{y_n}{x_n}\right)^{n/(n+1)}\left(\frac{y_n}{x_n}\right)^{1/(n+1)} \\
&= \left(\frac{x_{n+1}}{y_{n+1}}\right)^{n/(n+1)}\left(\frac{y_n}{x_n}\right)^{n/(n+1)}\left[\left(\frac{y_n}{x_n}\right)^{1/n}\right]^{n/(n+1)},
\end{aligned} \tag{A.4}$$

whence

$$\begin{aligned}
\lim_{n\to\infty}\left\{\left(\frac{a_{n+1}}{a_n}\right)^n\right\} &= \lim_{n\to\infty}\left\{\left[\left(\frac{x_{n+1}}{y_{n+1}}\right)\left(\frac{y_n}{x_n}\right)\left(\frac{y_n}{x_n}\right)^{1/n}\right]^{n/(n+1)}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{x_{n+1}}{n^2 x_n}\right\}\cdot\lim_{n\to\infty}\left\{\left(\frac{y_{n+1}}{ny_n}\right)^{-1}\right\}\cdot\lim_{n\to\infty}\left\{\left[\frac{1}{n}\left(\frac{x_n}{y_n}\right)^{1/n}\right]^{-1}\right\} \\
&= a\cdot b^{-1}\cdot\left(\frac{a}{be}\right)^{-1} \qquad \text{(invoking (A.2))} \\
&= e.
\end{aligned} \tag{A.5}$$

Finally, we write $\sqrt[n+1]{x_{n+1}/y_{n+1}} - \sqrt[n]{x_n/y_n} = a_{n+1} - a_n$ (by (A.1)) $= a_n[(a_{n+1}/a_n) - 1] = a_n[\exp\{\ln(a_{n+1}/a_n)\} - 1]$, and in turn

$$\begin{aligned}
\lim_{n\to\infty}\left\{\sqrt[n+1]{\frac{x_{n+1}}{y_{n+1}}} - \sqrt[n]{\frac{x_n}{y_n}}\right\} &= \lim_{n\to\infty}\left\{a_n[e^{\ln(a_{n+1}/a_n)} - 1]\right\} \\
&= \lim_{n\to\infty}\left\{\frac{a_n}{n}\cdot\frac{e^{\ln(a_{n+1}/a_n)} - 1}{\ln(a_{n+1}/a_n)}\cdot\ln[(a_{n+1}/a_n)^n]\right\},
\end{aligned} \tag{A.6}$$

whereupon, appealing to (A.2) and (A.5) (given that it is easy to show that $[\exp\{\ln(a_{n+1}/a_n)\} - 1]/\ln(a_{n+1}/a_n) \to 1$ as $n\to\infty$ (via L'Hôpital's Rule, for instance)),

$$\lim_{n\to\infty}\left\{\sqrt[n+1]{\frac{x_{n+1}}{y_{n+1}}} - \sqrt[n]{\frac{x_n}{y_n}}\right\} = a/(be)\cdot 1\cdot\ln(e) = a/(be), \tag{A.7}$$

as required. □

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