Palestine Journal of Mathematics

Vol. 12(2)(2023) , 620–634 © Palestine Polytechnic University-PPU 2023

Coefficient bounds for the family of bounded turning functions associated with tan hyperbolic function

T. Panigrahi, B. B. Mishra and A. Naik

Communicated by Deshna Loonker

MSC 2010 Classifications: Primary30C45; Secondary 30C50, 30C80

Keywords and phrases: Analytic function, k-fold symmetric, Starlike function, Tan hyperbolic function, Hankel determinant, Zalcman functional, Krushkal inequality.

Abstract. In the present paper, we introduce the class of bounded turning functions associated with tan hyperbolic functions. Sharp upper bounds for some of the initial coefficients, Fekete-Szegö functional, third Hankel determinant, Zalcman conjecture and Krushkal inequality are investigated for the class defined. Further, Fekete-Szeg$\ddot{o}$ inequality for the function $\frac{\xi}{h(\xi)}$ and inverse function h^{-1} for the above mentioned class are obtained.

1 Introduction and Motivation

Let $\mathcal{B}$ represent the family of holomorphic function $h(\xi)$ defined in the region of open unit disk $\Delta := \{\xi \in \mathbb{C} : |\xi| < 1\}$ having normalized by $h(0) = h'(0) - 1 = 0$. The function $h(\xi)$ can be expressed in Taylor-Maclaurin series as:

$$h(\xi) = \xi + \sum_{n=2}^{\infty} h_n \xi^n \quad (\xi \in \Delta). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{B}$ consisting of univalent functions. An analytic function h is subordinate to another analytic function g written as $h \prec g$ if there is an analytic function w with $w(0) = 0$ and $|w(\xi)| < 1$ such that $h(\xi) = g(w(\xi))$. If g is univalent, then $h \prec g$ if and only if $h(0) = g(0)$ and $h(\Delta) \subseteq g(\Delta)$.

In Geometric Function Theory, the subclasses of the class $\mathcal{S}$ which plays a significant role are the family $\mathcal{S}^*$ of starlike functions, $\mathcal{C}$ of convex functions and $\mathcal{R}$ of bounded turning functions defined in terms of subordination as follows:

$$\mathcal{S}^*(\phi) = \left\{ h \in \mathcal{S} : \frac{\xi h'(\xi)}{h(\xi)} \prec \phi(\xi) \right\},$$
$$\mathcal{C}(\phi) = \left\{ h \in \mathcal{S} : 1 + \frac{\xi h''(\xi)}{h'(\xi)} \prec \phi(\xi) \right\},$$
$$\mathcal{R}(\phi) = \{ h \in \mathcal{S} : h'(\xi) \prec \phi(\xi) \} \tag{1.2}$$

where the function ϕ is an analytic univalent function such that $\Re(\phi(\xi)) > 0$ in Δ with $\phi(0) = 1$ and $\phi'(0) > 0$ and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to real axis. The class $\mathcal{S}^*(\phi)$ was introduced and studied by Ma and Minda [16].

If we consider $\phi(\xi) = \frac{1+\xi}{1-\xi} = 1 + 2\sum_{n=2}^{\infty} \xi^n$ $(\xi \in \Delta)$ then the class $\mathcal{S}^*(\phi)$ is the well-known class of starlike functions.

By varying the function ϕ on the right hand side of (1.2) we get some subclasses of the set $\mathcal{S}$ which have several geometric properties. We here mention some of the families that are associated with different functions as follows:

- If we take $\phi(\xi) = \frac{1+A\xi}{1+B\xi}$ $(-1 \leq B < A \leq 1)$, we get the class $\mathcal{S}^*(A, B)$ studied by Janowski [7].
- If we take $\phi(\xi) = 1 + sin\xi$, we obtain the class $\mathcal{S}^*_{sin} = \mathcal{S}^*(1 + \sin \xi)$ introduced and studied by [5]
- Choosing $\phi(\xi) = e^{\xi}$, the class $\mathcal{S}^*_e = \mathcal{S}^*(e^{\xi})$ was introduced and studied by Arif et.al. [2] (also see [17]).
- The class $\mathcal{S}_N = \mathcal{S}^*(\phi(\xi))$ where $\phi(\xi) = 1+\xi-\frac{\xi^3}{3}$, a nephroid shaped domain was introduced and investigated by Wani and Swaminathan [29].
- Taking $\phi(\xi) = \sqrt{1+\xi}$, the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+\xi})$ was developed by Sokòl and Stankiewicz [27]. The function $\phi(\xi) = \sqrt{1+\xi}$ maps the region Δ onto the image domain which is bounded by $|w^2 - 1| < 1$ (Lemniscate of Bernoulli in right half plane).
- The family $\mathcal{S}_c = \mathcal{S}^*(\phi(\xi))$ with $\phi(\xi) = \xi+\sqrt{1+\xi^2}$ which maps Δ to crescent shaped region was introduced by Raina and Sokòl [23].
- For $\phi(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2$, the class $\mathcal{S}^*(\phi(\xi)) = \mathcal{S}^*_c$ was introduced by Sharma et. al. [25](also see [26]).
- If we take $\phi(\xi) = 1 + sinh^{-1}\xi$ the the class $\mathcal{S}^*_p = \mathcal{S}^*(1 + sinh^{-1}\xi)$ was studied by Kumar and Arora [14].
- Taking $\phi(\xi) = \frac{2}{1+e^{-\xi}}$, we get the class $\mathcal{S}^*(\phi(\xi)) = \mathcal{S}^*_{SG}$ was introduced by Goel and Kumar [6].
- For $\phi(\xi) = cosh\xi$, the class $\mathcal{S}^*_{cosh} = \mathcal{S}^*(\phi(\xi))$ was introduced by Alotaibi et. al. [1].

One of the interesting area of research in the Geometric Function Theory is the study of coefficient problem and Hankel determinant is one of the medium for the estimating it. It is very useful in the investigations of the singularities and power series with integral coefficients.
For the function $h \in \mathcal{B}$ of the form (1.1), Pommerenke [21, 22] introduced the Hankel determinant $H_{q,n}(h)$ as:

$$H_{q,n}(h) = \begin{vmatrix} h_n & h_{n+1} & \cdots & h_{n+q-1} \\ h_{n+1} & h_{n+2} & \cdots & h_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ h_{n+q-1} & h_{n+q} & \cdots & h_{n+2q-2} \end{vmatrix} (n, q \in \mathbb{N} := \{1, 2, 3, \cdots\},\ h_1 = 1).$$

In particular, for different values of q and n, we obtain Hankel determinant of various orders:
For $n = 1$ and $q = 2$,

$$H_{2,1}(h) = \begin{vmatrix} h_1 & h_2 \\ h_2 & h_3 \end{vmatrix} = h_3 - h_2^2$$

is popularly known as Feteke-Szegö functional. For different subclasses of $\mathcal{B}$, the upper bounds for $|H_{2,1}(h)|$ was investigated by different authors [10, 11, 20].
For $n = q = 2$, we have

$$H_{2,2}(h) = \begin{vmatrix} h_2 & h_3 \\ h_3 & h_4 \end{vmatrix} = h_2h_4 - h_3^2$$

is well-known as the second Hankel determinant. In recent time, many authors have contributed their results in form of research papers for finding the upper bounds of $|H_{2,2}(h)|$ for various subclasses of analytic function and their results are listed in literature For details, see [4, 8, 9, 15, 18].
For $n = 1$ and $q = 3$,

$$H_{3,1}(h) = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_2 & h_3 & h_4 \\ h_3 & h_4 & h_5 \end{vmatrix} = (h_2h_4 - h_3^2)h_3 - h_4(h_4 - h_2h_3) + h_5(h_3 - h_2^2) \quad (1.3)$$

is known as the third Hankel determinant. Babalola [3] obtained the upper bound of $|H_{3,1}(h)|$ for the family of $\mathcal{S}^*$, $\mathcal{C}$ and the class $\mathcal{R}$. Later, many researchers extended their ideas for finding the upper bounds of $|H_{3,1}(h)|$ for various subclasses of holomorphic functions which stood as a base for research in the field of Geometric Function Theory. For recent exposition works on third Hankel determinant, see [19, 24, 26, 28, 30].
Recently, a family of bounded turning function associated with modified sigmoid function was introduced by Khan et. al. [12] as

$$R_{SG} = \{h \in \mathcal{S} : h'(\xi) \prec \frac{2}{1+e^{-\xi}} \quad (\xi \in \Delta)\}.$$

Motivated by the aforementioned works, in this article we introduce the class R_{tanh} defined as follows:

Definition 1.1. A function $h \in \mathcal{B}$ is said to be in the class R_{tanh} if it satisfy the following subordination condition:

$$R_{tanh} = \{h \in \mathcal{B} : h'(\xi) \prec 1 + tanh\xi, \quad (\xi \in \Delta)\}. \quad (1.4)$$

We demonstrate by means of example to show that the class R_{tanh} defined above is non-empty. For that let $a > 0$. Let us consider the function

$$h_a(\xi) = \xi + a\xi^2 \in \mathcal{B}.$$

Then, the function $h_a \in R_{\text{tanh}}$ if and only if

$$\varphi(\xi) := 2a\xi \prec \tanh\xi =: \Phi(\xi).$$

We can see in the below Figure 1 made by using the MAPLE™ computer software, we have $\varphi(\Delta) \subset \Phi(\Delta)$ for $0 \le a \le 0.35$. Therefore, $h_a(\xi) = \xi + a\xi^2 \in R_{\text{tanh}}$ whenever $0 \le a \le 0.35$, hence $R_{\text{tanh}} \neq \emptyset$.

In this paper, we investigate some of the sharp results related to coefficient bounds, Fekete-Szegö inequality, second and third Hankel determinant, Zalcman conjecture and Krushkal inequality for the functional class R_{tanh}. Furthermore, Fekete-Szegö inequality for the function $\frac{\xi}{h(\xi)}$ and h^{-1} for the class R_{tanh} are obtained.

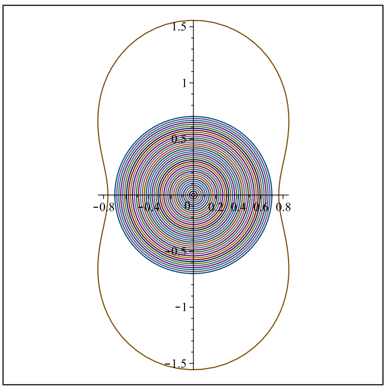

Figure 1. The image of $\varphi(\Delta)$ and $\Phi\left(e^{it}\right)$, $t \in [0, 2\pi]$

2 Preliminaries

Let $\mathcal{P}$ denote the set of all functions q that are analytic in Δ with $Re(q(\xi)) > 0$ and has the following form:

$$q(\xi) = 1 + \sum_{n=1}^{\infty} q_n \xi^n \quad (\xi \in \Delta). \tag{2.1}$$

We need the following lemmas in order to prove our main results.

Lemma 2.1. *[11] If $q(\xi) \in \mathcal{P}$ and has of the form (2.1), then*

$$|q_n| \leq 2 \quad for \quad n \geq 1, \tag{2.2}$$

$$|q_{n+k} - \delta q_n q_k| \leq \begin{cases} 2 & for \quad 0 \leq \delta \leq 1, \\ 2|2\delta - 1| & elsewhere, \end{cases} \tag{2.3}$$

$$|q_m q_n - q_k q_l| < 4 \quad for \quad m + n = l + k, \tag{2.4}$$

$$|q_{n+2k} - \delta q_n q_k^2| \leq 2(1 + 2\delta) \quad for \quad \delta \in \mathbb{R}, \tag{2.5}$$

and

$$\left| q_2 - \frac{q_1^2}{2} \right| \leq 2 - \frac{|q_1|^2}{2}. \tag{2.6}$$

Lemma 2.2. *(see[22]) Let the function $q \in \mathcal{P}$ be of the form (2.1). Then for complex number ν, we have*

$$|q_2 - \nu q_1^2| \leq 2max\{1, |2\nu - 1|\}. \tag{2.7}$$

Lemma 2.3. *[2] If $q \in \mathcal{P}$ and has the series of the form (2.1), then*

$$|Jq_1^3 - Kq_1q_2 + Lq_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|, \tag{2.8}$$

where J, K and L are real numbers.

Lemma 2.4. *[16] If $q \in \mathcal{P}$ and is of the form (2.1), then*

$$|q_2 - \mu q_1^2| \leq \begin{cases} -4\mu + 2 & (\mu \leq 0) \\ 2 & (0 \leq \mu \leq 1) \\ 4\mu - 2 & (\mu \geq 1). \end{cases} \tag{2.9}$$

3 Coefficient bounds and Fekete-Szegö inequality

In this section we will discuss some results related to coefficient bounds and Fekete-Szegö inequality for the function class R_{tanh}.

Theorem 3.1. *If the function $h \in \mathcal{B}$ of the form (1.1) belongs to the class R_{tanh} then*

$$|h_2| \leq \frac{1}{2}, \tag{3.1}$$

$$|h_3| \leq \frac{1}{3}, \tag{3.2}$$

$$|h_4| \leq \frac{1}{4}, \tag{3.3}$$

$$|h_5| \leq \frac{3}{5} \tag{3.4}$$

and

$$|h_6| \leq \frac{163}{180}. \tag{3.5}$$

The first three estimates are sharp.

Proof. Let the function $h(\xi) \in \mathcal{B}$ of the form (1.1) be in the class R_{tanh}. Then by Definition 1.1 there exists an analytic function $\omega(\xi)$ satisfying the conditions of Schwarz lemma (i.e. $\omega(0) = 0,\ |\omega(\xi)| < 1$) such that

$$h'(\xi) = 1 + \tanh \omega(\xi) \quad (\xi \in \Delta). \tag{3.6}$$

It follows from (1.1) that

$$h'(\xi) = 1 + 2h_2\xi + 3h_3\xi^2 + 4h_4\xi^3 + \cdots . \tag{3.7}$$

Define a function q such that

$$q(\xi) = \frac{1 + \omega(\xi)}{1 - \omega(\xi)} = 1 + q_1\xi + q_2\xi^2 + q_3\xi^3 + \cdots . \tag{3.8}$$

Clearly, $q(0) = 1$ and $\Re\{q(\xi)\} > 0$ which implies that $q \in \mathcal{P}$. From (3.8) we have

$$\begin{aligned}
\omega(\xi) &= \frac{q(\xi) - 1}{q(\xi) + 1} \\
&= \frac{q_1\xi + q_2\xi^2 + q_3\xi^3 + \cdots}{2 + q_1\xi + q_2\xi^2 + q_3\xi^3 + \cdots} \\
&= \frac{q_1}{2}\xi + \left(\frac{q_2}{2} - \frac{q_1^2}{4}\right)\xi^2 + \left(\frac{q_3}{2} - \frac{q_1q_2}{2} + \frac{q_1^3}{8}\right)\xi^3 \\
&+ \left(\frac{q_4}{2} - \frac{q_1q_3}{2} + \frac{3}{8}q_1^2q_2 - \frac{q_1^4}{16} - \frac{q_2^2}{4}\right)\xi^4 \\
&+ \left(\frac{q_5}{2} - \frac{q_1q_4}{2} + \frac{3}{8}q_1q_2^2 + \frac{3}{8}q_1^2q_3 - \frac{1}{4}q_1^3q_2 - \frac{q_2q_3}{2} + \frac{q_1^5}{32}\right)\xi^5 + \cdots .
\end{aligned} \tag{3.9}$$

Using relation (3.9) in the series expansion of $tanh\omega(\xi)$, we get

$$1+tanh\omega(\xi)=1+\frac{q_1}{2}\xi+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)\xi^2+\left(\frac{q_1^3}{12}-\frac{q_1q_2}{2}+\frac{q_3}{2}\right)\xi^3+\left(\frac{q_4}{2}-\frac{q_1q_3}{2}+\frac{1}{4}q_1^2q_2-\frac{q_2^2}{4}\right)\xi^4$$
$$+\left(\frac{q_5}{2}-\frac{q_1q_4}{2}+\frac{1}{4}q_1q_2^2+\frac{1}{4}q_1^2q_3-\frac{q_2q_3}{2}-\frac{13}{480}q_1^5\right)\xi^5+\cdots. \tag{3.10}$$

Using (3.7) and (3.10) in (3.6) and then comparing the coefficients of various powers of ξ, we obtain

$$h_2=\frac{q_1}{4}, \tag{3.11}$$

$$h_3=\frac{q_2}{6}-\frac{q_1^2}{12}, \tag{3.12}$$

$$h_4=\frac{q_1^3}{48}-\frac{q_1q_2}{8}+\frac{q_3}{8}, \tag{3.13}$$

$$h_5=\frac{q_4}{10}-\frac{q_1q_3}{10}+\frac{1}{20}q_1^2q_2-\frac{q_2^2}{20}, \tag{3.14}$$

and

$$h_6=\frac{q_5}{12}-\frac{q_1q_4}{12}+\frac{1}{24}q_1q_2^2+\frac{1}{24}q_1^2q_3-\frac{q_2q_3}{12}-\frac{13}{2880}q_1^5. \tag{3.15}$$

Using the relation (2.2) of Lemma 2.1 in (3.11) we get

$$|h_2|\le\frac{1}{2}.$$

From (3.12) we have

$$|h_3|=\frac{1}{6}\left|q_2-\frac{q_1^2}{2}\right|. \tag{3.16}$$

Applying relation (2.3) of Lemma 2.1 in (3.16) we get

$$|h_3|\le\frac{1}{3}.$$

An application of Lemma 2.3 in (3.13) gives the bounds for $|h_4|$. Rearranging the terms in (3.14) and application of triangle inequality followed by Lemma 2.1 yield

$$\begin{aligned}|h_5|&=\left|\frac{1}{10}\left(q_4-\frac{q_2^2}{2}\right)-\frac{q_1}{10}\left(q_3-\frac{q_1q_2}{2}\right)\right|\\&\le\frac{1}{10}\left|q_4-\frac{q_2^2}{2}\right|+\frac{|q_1|}{10}\left|q_3-\frac{q_1q_2}{2}\right|\\&\le\frac{3}{5}.\end{aligned}$$

By reordering the terms in (3.15), it yields

$$\begin{aligned}|h_6|&=\left|\left(\frac{1}{24}q_1^2q_3-\frac{13}{2880}q_1^5\right)+\left(\frac{q_1q_2^2}{24}-\frac{q_2q_3}{12}\right)+\left(\frac{q_5}{12}-\frac{q_1q_4}{12}\right)\right|\\&=\left|\frac{1}{24}q_1^2\left(q_3-\frac{13}{120}q_1^3\right)-\frac{q_2}{12}\left(q_3-\frac{q_1q_2}{2}\right)+\frac{1}{12}\left(q_5-q_1q_4\right)\right|.\end{aligned}$$

Application of triangle inequality and followed by Lemma 2.1 give

$$|h_6| \leq \frac{|q_1|^2}{24}\left|q_3 - \frac{13}{20}q_1^3\right| + \frac{|q_2|}{12}\left|q_3 - \frac{q_1q_2}{2}\right| + \frac{1}{12}|q_5 - q_1q_4|$$
$$\leq \frac{163}{180}.$$

The coefficient bounds of h_2, h_3 and h_4 are sharp. For that consider a function

$$h_n'(\xi) = 1 + tanh\xi^n \quad (n = 1,\ 2,\ 3). \tag{3.17}$$

Thus we have

$$h_1(\xi) = \int_0^\xi (1 + tanht)dt = \xi + \frac{\xi^2}{2} - \frac{\xi^4}{12} + \cdots, \tag{3.18}$$

$$h_2(\xi) = \int_0^\xi (1 + tanht^2)dt = \xi + \frac{\xi^3}{3} - \frac{\xi^7}{21} + \cdots, \tag{3.19}$$

$$h_3(\xi) = \int_0^\xi (1 + tanht^3)dt = \xi + \frac{\xi^4}{4} - \frac{\xi^{10}}{30} + \cdots. \tag{3.20}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *If h is of the form (1.1) belongs to the class R_{tanh}, then for any complex number μ, we have*

$$|h_3 - \mu h_2^2| \leq \frac{1}{3}max\{1, \frac{3}{4}|\mu|\}. \tag{3.21}$$

The estimate is sharp for the function h_2 defined in (3.19) for $|\mu| \leq \frac{4}{3}$ and the function h_1 defined in (3.18) for $|\mu| \geq \frac{4}{3}$.

Proof. From (3.11) and (3.12) we get

$$|h_3 - \mu h_2^2| = \left|\frac{q_2}{6} - \frac{q_1^2}{12} - \mu\frac{q_1^2}{16}\right| = \frac{1}{6}\left|q_2 - \frac{3\mu + 4}{8}q_1^2\right|.$$

Application of Lemma 2.2 we get

$$|h_3 - \mu h_2^2| \leq \frac{1}{3}max\left\{1, \left|2\left(\frac{3\mu - 4}{8}\right) - 1\right|\right\}.$$

Simple calculation shows

$$|h_3 - \mu h_2^2| \leq \frac{1}{3}max\left\{1, \frac{3|\mu|}{4}\right\}.$$

For the sharpness we consider the function

$$h_2(\xi) = \xi + \frac{\xi^3}{3} - \frac{\xi^7}{21} + \cdots,$$

which gives the equality in (3.21) when $|\mu| \leq \frac{4}{3}$, namely

$$|h_3 - \mu h_2^2| = |h_3| = \frac{1}{3} = \frac{1}{3}max\left\{1, \frac{3}{4}|\mu|\right\}.$$

For $|\mu| \geq \frac{4}{3}$, we take the function

$$h_1(\xi) = \xi + \frac{\xi^2}{2} - \frac{\xi^4}{12} + \cdots$$

which yields

$$|h_3 - \mu h_2^2| = |\mu h_2^2| = \frac{|\mu|}{4}.$$

The proof of Theorem 3.2 is thus completed. □

Letting $\mu = 1$ in Theorem 3.2 we get the following corollary:

Corollary 3.3. *If $h \in \mathcal{B}$ of the form (1.1) belongs to the class R_{tanh}, then*

$$|h_3 - h_2^2| \leq \frac{1}{3}.$$

The following theorem gives the Fekete-Szegö functional for the class R_{tanh} when μ is real.

Theorem 3.4. *Let $h(\xi) \in R_{tanh}$ be of the form (1.1). Then for $\mu \in \mathbb{R}$, we have*

$$|h_3 - \mu h_2^2| \leq \begin{cases} -\frac{\mu}{4} & \mu \leq -\frac{4}{3}, \\ \frac{1}{3} & -\frac{4}{3} \leq \mu \leq \frac{4}{3} \\ \frac{\mu}{4} & \mu \geq \frac{4}{3}. \end{cases}$$

Proof. From (3.11) and (3.12) we obtain

$$|h_3 - \mu h_2^2| = \frac{1}{6}|q_2 - \nu q_1^2|$$

where

$$\nu = \frac{3\mu + 4}{8}.$$

The assertion of Theorem 3.4 follows by virtue of Lemma 2.4. □

4 Coefficient Inequalities for the function h^{-1}

Theorem 4.1. *If the function $h \in R_{tanh}$ given by (1.1) and $h^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n$ is the analytic continuation to Δ of the inverse function of h with $|w| < r_0$ where $r_0 \geq \frac{1}{4}$ is the radius of the Koebe domain, then for any complex number μ, we have*

$$|l_2| \leq \frac{1}{2} \tag{4.1}$$

$$|l_3| \leq \frac{1}{2} \tag{4.2}$$

and

$$|l_3 - \mu l_2^2| \leq \frac{1}{3} max \left\{1, \frac{3}{4}|2 - \mu|\right\}. \tag{4.3}$$

Proof. Since

$$h^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n \tag{4.4}$$

is the inverse function of h, we have

$$h^{-1}(h(\xi)) = h(h^{-1}(\xi)) = \xi. \tag{4.5}$$

From (4.5) we have

$$h^{-1}(\xi + \sum_{n=2}^{\infty} h_n \xi^n) = \xi \tag{4.6}$$

From (4.4) and (4.6) we get

$$\xi + (h_2 + l_2)\xi^2 + (h_3 + 2h_2 l_2 + l_3)\xi^3 + \cdots = \xi \tag{4.7}$$

Equating the coefficients of ξ^2 and ξ^3 on both sides of (4.7) we get

$$l_2 = -h_2, \tag{4.8}$$

and

$$l_3 = -h_3 - 2h_2 l_2 = 2h_2^2 - h_3. \tag{4.9}$$

From relations (3.11), (3.12) and (4.8) and (4.9) we obtain

$$l_2 = -\frac{q_1}{4}, \tag{4.10}$$

and

$$\begin{aligned} l_3 &= 2h_2^2 - h_3 \\ &= \frac{q_1^2}{8} - \left(\frac{q_2}{6} - \frac{q_1^2}{12}\right) \\ &= -\frac{1}{6}\left(q_2 - \frac{5}{4}q_1^2\right). \end{aligned} \tag{4.11}$$

Taking modulus on both sides of (4.10) and (4.11) and applying inequality (2.2) and (2.3) of Lemma 2.1 respectively give the estimate (4.1) and (4.2).
Further, for any complex number μ, we have

$$\begin{aligned} |l_3 - \mu l_2^2| &= \left|\frac{5}{24}q_1^2 - \frac{q_2}{6} - \mu\frac{q_1^2}{16}\right| \\ &= -\frac{1}{6}\left|q_2 - \frac{10 - 3\mu}{8}q_1^2\right|. \end{aligned} \tag{4.12}$$

Taking modulus on the both sides of (4.12) and applying relation (2.7) of Lemma 2.2 we get

$$\begin{aligned} |l_3 - \mu l_2^2| &= \frac{1}{3}max\left\{1, \left|\frac{10 - 3\mu}{4} - 1\right|\right\} \\ &= \frac{1}{3}max\left\{1, \frac{3}{4}|2 - \mu|\right\}. \end{aligned}$$

This proves the result of Theorem 4.1. □

5 Coefficient functional associated with $\frac{\xi}{h(\xi)}$

In this section, we determine Fekete-Szeg$\ddot{o}$ coefficient functional bounds associated with the function $M(\xi)$ defined as

$$M(\xi) = \frac{\xi}{h(\xi)} = 1 + \sum_{n=2}^{\infty} u_n \xi^n \quad (\xi \in \Delta) \tag{5.1}$$

where the function h is in the class R_{tanh}.

Theorem 5.1. *Let $h \in R_{tanh}$ and $M(\xi) = \frac{\xi}{h(\xi)} = 1 + \sum_{n=2}^{\infty} u_n \xi^n$. Then for any complex number μ, we have*

$$|u_2 - \mu u_1^2| \leq \frac{1}{3} max\{1, \frac{3}{4}|1-\mu|\}. \tag{5.2}$$

Proof. By simple calculation, it can be shown that

$$M(\xi) = \frac{\xi}{h(\xi)} = 1 - h_2\xi + (h_2^2 - h_3)\xi^2 + \cdots . \tag{5.3}$$

From (5.1) and (5.3) we obtain

$$u_1 = -h_2, \tag{5.4}$$

and

$$u_2 = h_2^2 - h_3. \tag{5.5}$$

Using (3.11) and (3.12) in (5.4) and (5.5) we get

$$u_1 = -\frac{q_1}{4},$$

and

$$u_2 = \frac{q_1^2}{16} - \left(\frac{q_2}{6} - \frac{q_1^2}{12}\right) = \frac{7}{48}q_1^2 - \frac{q_2}{6}.$$

Thus for any complex number μ we have

$$|u_2 - \mu u_1^2| = \frac{1}{6}\left|q_2 - \frac{7-3\mu}{8}q_1^2\right|.$$

Application of Lemma 2.2 we get

$$|u_2 - \mu u_1^2| \leq \frac{1}{3} max\left\{1, \left|\frac{7-3\mu}{4} - 1\right|\right\} = \frac{1}{3} max\left\{1, \frac{3}{4}|1-\mu|\right\}.$$

The prove of Theorem 5.1 is thus completed. □

6 Hankel determinant for the class R_{tanh}

Theorem 6.1. *If the function h of the form (1.1) belongs to the class R_{tanh}, then*

$$|H_{2,2}(h)| = |h_2h_4 - h_3^2| \leq \frac{1}{4}. \tag{6.1}$$

Proof. Using (3.11)-(3.13) we have

$$H_{2,2}(h) = h_2h_4 - h_3^2 = \frac{q_1}{16}\left[\frac{q_1^3}{12} - \frac{q_1q_2}{2} + \frac{q_3}{2}\right] - \left(\frac{q_2}{6} - \frac{q_1^2}{12}\right)^2$$
$$= -\frac{1}{576}q_1^4 - \frac{q_1^2q_2}{288} + \frac{q_1q_3}{32} - \frac{q_2^2}{36}$$
$$= q_1\left(-\frac{1}{576}q_1^3 - \frac{q_1q_2}{288} + \frac{q_3}{32}\right) - \frac{q_2^2}{36}.$$

Applications of triangle inequality and followed by Lemma 2.3 give

$$|H_{2,2}(h)| = |h_2h_4 - h_3^2| \le |q_1|\left|-\frac{1}{576}q_1^3 - \frac{q_1q_2}{288} + \frac{q_3}{32}\right| + \frac{|q_2|^2}{36}$$
$$= 4\left(\frac{1}{576} + \frac{1}{144} + \frac{15}{576}\right) + \frac{1}{9}$$
$$= \frac{1}{4}.$$

This proves the Theorem 6.1. □

Theorem 6.2. *Let $h(\xi) \in R_{tanh}$ be of the form (1.1). Then*

$$|h_2h_3 - h_4| \le \frac{1}{4}. \tag{6.2}$$

The result is sharp for the function h_3 defined in (3.20).

Proof. From (3.11) to (3.13) we obtain

$$h_2h_3 - h_4 = \frac{q_1^3}{4}\left(\frac{q_2}{6} - \frac{q_1^2}{12}\right) - \left(\frac{q_1^3}{48} - \frac{q_1q_2}{8} + \frac{q_3}{8}\right)$$
$$= \frac{q_1q_2}{6} - \frac{q_1^3}{24} - \frac{q_3}{8}. \tag{6.3}$$

An application of Lemma 2.3 to (6.3) gives required estimate.
For the sharpness we take the function $h_3(\xi)$ from (3.20) as

$$h_3(\xi) = \xi + \frac{\xi^4}{4} - \frac{\xi^{10}}{30} + \cdots.$$

Here $h_2 = h_3 = 0$ and $h_4 = \frac{1}{4}$ so that

$$|h_2h_3 - h_4| = |-\frac{1}{4}| = \frac{1}{4}.$$

The proof of Theorem 6.2 is complete. □

Theorem 6.3. *If $h \in \mathcal{B}$ belongs to the class R_{tanh}, then*

$$|H_{3,1}(h)| \le \frac{83}{240}. \tag{6.4}$$

Proof. Applying triangle inequality to relation (1.3) yields

$$|H_{3,1}(h)| \le |h_3||h_2h_4 - h_3^2| + |h_4||h_4 - h_2h_3| + |h_5||h_3 - h_2^2|. \tag{6.5}$$

Implementing the results obtained in Theorem 3.1, Corollary 3.3, Theorem 6.1, 6.2 in (6.5) we obtain the desired estimate mentioned in (6.4). This proves the Theorem 6.3. □

7 Zalcman Functional

One of the classical conjecture in the filed of Geometric Function Theory introduced by Lawrence Zalcman in 1960 is that the coefficient of the functional class $\mathcal{S}$ satisfy the relation

$$|h_n^2 - h_{2n-1}| \leq (n-1)^2. \tag{7.1}$$

Equality holds for the well-known Koebe function $h(z) = \frac{z}{(1-z)^2}$ or its rotation. Many researchers have investigated the coefficient estimate for Zalcman conjecture for different subclasses of $\mathcal{S}$ in different direction and their results are available in literature.

Theorem 7.1. *If $h(\xi) = \xi + \sum_{n=2}^{\infty} h_n \xi^n$ is in the class R_{tanh}, then*

$$|h_3^2 - h_5| \leq \frac{3}{5}. \tag{7.2}$$

Proof. From (3.12) and (3.14), it follows that

$$\begin{aligned} |h_3^2 - h_5| &= \left| \left(\frac{q_2}{6} - \frac{q_1^2}{12} \right)^2 - \left(\frac{q_4}{10} - \frac{q_1 q_3}{10} + \frac{q_1^2 q_2}{20} - \frac{q_2^2}{20} \right) \right| \\ &= \left| q_1 \left(\frac{q_1^3}{144} - \frac{7}{90} q_1 q_2 + \frac{q_3}{10} \right) - \frac{1}{10} \left(q_4 - \frac{7}{90} q_2^2 \right) \right|. \end{aligned}$$

Application of triangle inequality and followed by Lemma 2.3 give

$$\begin{aligned} |h_3^2 - h_5| &\leq |q_1| \left| \frac{q_1^3}{144} - \frac{7}{90} q_1 q_2 + \frac{q_3}{10} \right| + \frac{1}{10} \left| q_4 - \frac{7}{90} q_2^2 \right| \\ &\leq 4 \left[\frac{1}{144} + \left| \frac{7}{90} - \frac{1}{72} \right| + \left| \frac{1}{144} - \frac{7}{90} + \frac{1}{10} \right| \right] + \frac{1}{5} \\ &= \frac{3}{5}. \end{aligned}$$

The proof of Theorem 7.1 is complete. □

8 Krushkal Inequality

For the class of univalent functions $\mathcal{S}$, Krushkal (see [13]) introduced and proved the inequality

$$|h_n^p - h_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p. \tag{8.1}$$

The following theorems prove the inequality (8.1) for the cases $n = 4,\ p = 1$ and $n = 5,\ p = 1$ for the class R_{tanh}.

Theorem 8.1. *Let $h \in \mathcal{B}$ be in the class R_{tanh}. Then*

$$|h_4 - h_2^3| \leq \frac{1}{4}. \tag{8.2}$$

The result is sharp for the function $h_3(\xi)$ defined in (3.20).

Proof. From (3.11) and (3.13) we get

$$|h_4 - h_2^3| = \left| \frac{1}{192} q_1^3 - \frac{q_1 q_2}{8} + \frac{q_3}{8} \right|. \tag{8.3}$$

An application of Lemma 2.3 to (8.3) gives required estimate as stated in the theorem. This completes the proof of Theorem 8.1. □

Theorem 8.2. *Let $h \in \mathcal{B}$ be in the function class R_{tanh}. Then*

$$|h_5 - h_2^4| \leq \frac{3}{5}. \tag{8.4}$$

Proof. Putting the values from (3.11) and (3.14) in the coefficient functional $(h_5 - h_2^4)$ we get

$$h_5 - h_2^4 = -q_1 \left(\frac{q_1^3}{256} - \frac{1}{20} q_1 q_2 + \frac{q_3}{10} \right) + \frac{1}{10} \left(q_4 - \frac{q_2^2}{2} \right). \tag{8.5}$$

An application of triangle inequality and followed by Lemma 2.1, 2.3 to the relation (8.5) gives the desired estimate. This proves the result of Theorem 8.2. □

Concluding Remark: In the present paper, the authors have introduced the family of bounded turning functions related to tan hyperbolic function and obtained sharp upper bounds of some of the initial coefficients, Fekete-Szeg*ö* inequality, Hankel determinant of order two and three, Zaclman conjecture and Krushkal inequality. One can use (p, q)-calculus $0 < |q| < p \leq 1$ to define the functional class and results in this paper can be generalized which give new direction to the research.

References

[1] A. Alotaibi, M. Arif, M. A, Alghamdi and S. Hussain, *Starlikeness associated with cosine hyperbolic function*, Mathematics, **8**(2020), 1118. https://doi.org/10.3390/math 8071118.

[2] M. Arif, M. Raza, H. Tang, S. Hussain and H. Khan, *Hankel determinant of order three for familiar subsets of analytic functions related with sine function*, Open Math., **17**(2019), 1615-1630, doi: 10.1515/math-2019-0132.

[3] K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent function*, Inequal. Theory Appl., **6** (2010), 1-7.

[4] D. Bansal, *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett., **26** (2013), 103-107.

[5] N. E. Cho, V. Kumar, S. S. Kumar and V. Ravichandran, *Radius problems for starlike functions associated with sine functions*, Bull. Iran. Math. Soc., **45**(2019), 213-232.

[6] P. Goel and S. S. Kumar, *Certain class of starlike functions associated with modified sigmoid function*, Bull. Malays. Math. Sci. Soc.,**43**(2020), 957-991.

[7] W. Janowski, *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polonic Math, **23**(1971), 159-177.

[8] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal Pure Appl. Math., **7**(2)(2006), Art. 50, 5 pages.

[9] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal., **1**(13)(2007), 619-625.

[10] W. Keopf, *On the Fekete-Szegö problem for close-to convex functions*, Proc. Amer. Math. Soc., **101**(1987), 89-95.

[11] F. Keough and E. Merkes, *A coefficient inequality for certain subclasses of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8-12.

[12] M. G. Khan, N. E. Cho, T. G. Shaba, B. Ahemad and W. K. Mashwani, *Coefficient functional for a class of bounded turning functions related to modified sigmoid function*, AIMS Mathematics, **7**(2)(2021), 3133-3149.

[13] S. K. Krushkal, *A short geometric proof of the Zalcman and Bieberach conjectures*, arXiv., https://arxiv.org/abs/1408.1948.

[14] S. S. Kumar and K. Arora, *Starlike functions associated with a petal shaped domain*, 2020. https://arxiv.org/abs/2010.10072.

[15] S. K. Lee, V. Ravichandran and S. Supramaniam, *Bounds for the second Hankel determinants of certain univalent functions*, J. Inequal. Appl., **2013**, 281(2013).

[16] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent function*, In : Proceeding of the Conference on Complex Analysis, (Tianjin, 1992), Conf. Proc. Lecture Notes Anal I, Int. Press Cambridge, MA, USA, (1994), pp. 157-169.

[17] R. Mendiratta, S. Nagpal and V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. , **38** (2015), 365-386.

[18] R. N. Mohapatra and T. Panigrahi, *Second Hankel determinant for a class of analytic functions defined by Komatu integral operator*, Rend. Mat. Appl., **41**(1)(2020), 51-58 .

[19] A. Naik and T. Panigrahi, *Upper bound on Hankel determinant for bounded turning function associated with Salagean-difference operator*, Survey Math Appl., **15**(2020), 525-543.

[20] T. Panigrahi and G. Murugusunadarmoorthy *The Fekete-Szegö inequality for subclass of analytic functions of complex order*, Adv. Stud. Cont. Math., **24**(1) 2014, 67-75.

[21] C. Pommerenke, *On the Hankel determinant of univalent function* , Mathematika, **14** (1967), 108-112.

[22] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, Germany (1975).

[23] R. K. Raina and J. Sokol, *On coefficient estimates for a certain class of starlike functions*, Hacet. J. Math Stat., **44**(2015), 1427-1433.

[24] M. Shafiq, H. M. Srivastava, N. Khan, Q. Z. Ahmad, M. Darus and S. Kiran, *An upper bounds of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers*, Symmetry, **12**(2020), 1043. doi:10.3390/sym 12061043.

[25] K. Sharma, N. K. Jain and V. Ravichandran, *Starlike functions associated with a cardioid*, Afr. Mat., **27**(2016), 923-939. https://doi.org/10.1007/s13370-015-0387-7.

[26] L. Shi, I. Ali, M. Arif, N. E. Cho, S. Hussain and H. Khan, *A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain*, Mathematics, **7**(2019), 418. https://doi.org/10.3390/math 7050418.

[27] J. Sokòl and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike function*, Zeszyty Naukowei oficyna Wydawnicza al. Powstańcòw Warszawy, **19**(1996), 101-105.

[28] H. M. Srivastava, Q. Z. Ahemad, M. Darus, B. Khan, N. Zaman, *Upper bounds of the third Hankel determinant for a subclass of close-to-convex functions associated with the leminiscate of Bernoulli* , Mathematics, **7**(2019), 848. doi:10.3390/math 7090848.

[29] L. A. Wani and A. Swaminathan, *Starlike and convex functions associated with a Nephroid domain*, Bull Malays. Math. Sci. Soc., **44**(2021),79-104. https://doi.org/10.1007/s40840-020-00935-6.

[30] P. Zaprawa, *Thrid Hankel determinant for subclasses of univalent functions*, Mediterr. J . Math., **14**(2017), Art. No. 19, 10 pages. https:// doi.org/10.1007/s00009-016-0829-y.

Author information

T. Panigrahi, Institute of Mathematics and Applications, Andharua, Bhubaneswar-751029, Odisha, India.
E-mail: trailokyap6@gmail.com

B. B. Mishra, Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University, Bhubaneswar-751024, Odisha, India.
E-mail: bmishrafma@kiit.ac.in

A. Naik, Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University, Bhubaneswar-751024, Odisha, India.
E-mail: avayanaik@gmail.com

Received: 2022-02-27.
Accepted: 2022-07-13.