Palestine Journal of Mathematics

Vol. 12(2)(2023) , 68–73

A new general recursive divisibility test by prime numbers via generalization of the famous criterion for divisibility by 3

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11A07; Secondary 11A41.

Keywords and phrases: Divisibility, congruence, prime, test.

Acknowledgement: The authors express their gratitude to the anonyms reviewers and editors of PJM.

Abstract In this work, we present a new general recursive divisibility test by prime numbers, involving a sequence based on the digits of the decimal base, we generalize the famous criterion for divisibility by 3. Finally, as an example of applications, we note that the some specific known cases remain valid.

1 Introduction

Prime numbers are used very often in nature and in life. Lots of people are interested to the rules of divisibility by prime numbers. Prime numbers have attracted the attention of many mathematicians around the world for many years, see for example the history of number theory in [9, 10, 15]. There are a number of well-known open-ended questions regarding prime numbers. The creation of a real rule, trick, method or test of divisibility by a prime numbers, is a very difficult elementary problem, there currently exists different algorithms used for divisibility tests, we can for example refer to [2, 4, 5, 12, 13, 14, 16, 18, 19], Some of these tests were simple and easy to used in practice or in theory. we note that among these articles, there is some authors who have used base ten to justify proof of their rules see for example [5, 18]. The aim of this article is to present a simple divisibility test which is applicable and has a very good theoretical interest. This article, in particular using the base of the decimal system, we study a new recursive divisibility test by prime numbers. Finally, as an example of applications, we discuss that the few known specific cases remain valid.

2 Preliminaries and basic notations

In this section, we state some basic definitions and properties of prime numbers to be used in this article.

Definition 2.1 (see [8]). Let n, s, b and $a_i \in \mathbb{N}$.

$$n = [a_s...a_0]_b \Leftrightarrow n = \sum_{i=0}^{s} a_i b^i = a_0 b^0 + ... + a_s b^s, \; 0 \leq a_i < b.$$

This equality is called the only writing of n in numeral basic system b, and a_i is the i-th digit.

Definition 2.2 (see [7]). Let a, b and $m \in \mathbb{N}$. We say that a is congruent to b modulo m ($a \equiv b[m]$), if m divided $(/)$ $a - b$.

Lemma 2.3 (see [14]). *let p be a prime number other than 2 and 5. if $p \equiv b_0[10]$ then $b_0 \in \{1, 3, 7, 9\}$.*

Throughout this paper, we denote by $\mathcal{P}$ the set of prime numbers other than 2 and 5. Let $\mathcal{P} \ni p \equiv b_0[10]$ where $b_0 \in \{1, 3, 7, 9\}$.

3 Main result

Proposition 3.1. *Let $p \in \mathcal{P}$ such that $p \equiv b_0[10]$, and $R_p = \{r_j \in \mathbb{N}^*/pr_j + 1 \equiv 0[10]\}$. Then*

(i) $R_p \neq \emptyset$, *and* $r_0 = \min R_p = \begin{cases} b_0, & \text{if } b_0 \in \{3,7\}, \\ 10 - b_0, & \text{if } b_0 \in \{1,9\}. \end{cases}$

(ii) $r_j \in R_p \Leftrightarrow r_j \equiv r_0[10]$.

Proof. (i) From Lemma 2.3, one has $b_0 \in \{1,3,7,9\}$. Then for $b_0 \in \{3,7\}$, we obtain $pb_0 + 1 \equiv (b_0^2 + 1)[10] \equiv 0[10]$, then $b_0 \in R_p$. For $b_0 \in \{1,9\}$, we have $p(10 - b_0) + 1 \equiv (10b_0 - b_0^2 + 1)[10] \equiv 0[10]$, then $10 - b_0 \in R_p$. Which give $R_p \neq \emptyset$.
Now Let $r_0 = \min R_p$, then $r_0 \in R_p$ and $pr_0 + 1 \equiv 0[10]$.
In the case when $b_0 \in \{3,7\}$, one has $b_0 \in R_p$ and $p(b_0 - r_0) \equiv 0[10]$. Since p is a prime number other than 2 and 5, and $p \wedge 10 = 1$, we get by Gauss's Lemma that $10/(b_0 - r_0)$. Moreover $0 < r_0 \leq b_0 \leq 7$ which implies $0 \leq b_0 - r_0 \leq 7$, then $b_0 - r_0 = 0$.
Now, let us consider the case when $b_0 \in \{1,9\}$, the argument is the same as above, we get that $(10 - b_0) \in R_p$ and $10/(10 - b_0 - r_0)$. Moreover $0 < r_0 \leq 10 - b_0 \leq 9$, which implies $0 \leq 10 - b_0 - r_0 \leq 9$, then $10 - b_0 - r_0 = 0$.

(ii) Let $r_j \in R_p$, we have $p(r_j - r_0) \equiv 0[10]$, and since p is a prime number other than 2 and 5 and $p \wedge 10 = 1$, by Gauss's Lemma we get $r_j - r_0 \equiv 0[10]$, which implies $r_j \equiv r_0[10]$. □

Proposition 3.2. *Let $p \in \mathcal{P}$ such that $p \equiv b_0[10]$, and $\widehat{p}_j = \frac{pr_j+1}{10}$ where $r_j \in R_p$. Then*

(i) $\widehat{p}_j \equiv \widehat{p}_0[p]$, $\forall j \in \mathbb{N}$.

(ii) $1 \leq \widehat{p}_0 < p < \widehat{p}_j$, $\forall j \geq 1$.

(iii) $p \wedge \widehat{p}_0 = 1$.

Proof. (i) Let $j \in \mathbb{N}$, due to the proposition 3.1, $r_j \equiv r_0[p]$, $\forall j \in \mathbb{N}$. Then, there exists $k \in \mathbb{N}$ such that

$$\widehat{p}_j = \frac{pr_j + 1}{10} = \frac{10pk + pr_0 + 1}{10} = pk + \widehat{p}_0.$$

So $\widehat{p}_j \equiv \widehat{p}_0[p]$.

(ii) We have $p - \widehat{p}_0 = \frac{p(10-r_0)+1}{10}$. Moreover by the proposition 3.1 $r_0 \in \{b_0, 10 - b_0\} < 10$ then $\widehat{p}_0 < p$. On the other hand $pr_0 \neq 0$ for all p a prime number in $\mathbb{N}$, then $\widehat{p}_0 \geq 1$. Now, using the fact that $\widehat{p}_j - p = p(k-1) + \widehat{p}_0$, and $k > 1$ for all $j > 0$. So $p < \widehat{p}_j$, $\forall j \geq 1$.

(iii) Since $1 \leq \widehat{p}_0 < p$ and p is a prime number, then $p \wedge \widehat{p}_0 = 1$. □

Example 3.3. The table of prime numbers $p < 50$, $p \neq 2, 5$ with the corresponding value $\widehat{p}_0$.

Table 1.

p	3	7	11	13	17	19	23	29	31	37	41	43	47
$\widehat{p}_0$	1	5	10	4	12	2	7	3	28	26	37	13	33

Proposition 3.4. *Let $p, q \in \mathcal{P}$. Then*

$$p \neq q \Leftrightarrow \widehat{p}_0 \neq \widehat{q}_0,$$

where $\widehat{p}_0$ and $\widehat{q}_0$ are defined in the proposition 3.2.

Proof. Let $p, q \in \mathcal{P}$ such that $p \equiv b_0[10]$ and $q \equiv c_0[10]$ and $\widehat{p}_0 = \frac{pr_0+1}{10}$, $\widehat{q}_0 = \frac{qr'_0+1}{10}$, where r_0, r'_0 are defined as in the proposition 3.1. In the first, we assume that $p \neq q$, we can assume for example that $p < q$ then we have three cases:

- Case 1: $p < q < 10$. Since p and q are two primes numbers other than 2 and 5, then $p = 3$ and $q = 7$, which give $\widehat{p}_0 = 1 \neq \widehat{q}_0 = 5$.

- Case 2: $p < 10 < q$. We assume that $\widehat{p_0} = \widehat{q_0}$, then, by the proposition 3.2, $pr_0 = qr'_0$ which implies q/pr_0. Since $r_0 < 10 < q$, then q/p which is absurd.
- Case3: $10 < p < q$. We assume that $\widehat{p_0} = \widehat{q_0}$, by the proposition 3.2, we get $pr_0 = qr'_0$ which implies q/pr_0 and p/qr'_0. Since $r_0 < 10 < q$, $r'_0 < 10 < p$ then q/p and p/q, so $p = q$ which is absurd.

For the other implication, take $\widehat{p_0} \neq \widehat{q_0}$ and assume that $p = q$, then $r_0 = r'_0$ and $pr_0 = qr'_0$, which give $\widehat{p_0} = \widehat{q_0}$, which is absurd. □

Let $p \in \mathcal{P}$ such that $p \equiv b_0[10]$, $n = [a_s ... a_0]_{10} \in \mathbb{N}$, and the increasing sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ defined by

$$\mathcal{A}_0 = a_0, \quad \mathcal{A}_{k+1} = \widehat{p_0}\mathcal{A}_k + a_{k+1}. \tag{3.1}$$

Proposition 3.5. *Let $\mathcal{P} \ni p \equiv b_0[10]$, and $n = [a_s ... a_0]_{10}$. Then*

(i) $\mathcal{A}_k = \sum_{i=0}^{k} \widehat{p_0}^{\,k-i} a_i, \forall k \geq 0.$

(ii) $\mathcal{A}_k = \widehat{p_0}^{\,k-s}\mathcal{A}_s, \forall k \geq s.$

Proof. (i) For $k = 0$, we have $\sum_{i=0}^{k=0} \widehat{p_0}^{\,k-i} a_i = a_0 = \mathcal{A}_0$. Now, we assume that $\mathcal{A}_k = \sum_{i=0}^{k} \widehat{p_0}^{\,k-i} a_i, k \geq 1$. From (3.1) we get

$$\begin{aligned} \mathcal{A}_{k+1} &= \widehat{p_0}\mathcal{A}_k + a_{k+1} = \widehat{p_0}\sum_{i=0}^{k}\widehat{p_0}^{\,k-i}a_i + a_{k+1} \\ &= \sum_{i=0}^{k}\widehat{p_0}^{\,k+1-i}a_i + a_{k+1} = \sum_{i=0}^{k+1}\widehat{p_0}^{\,k+1-i}a_i. \end{aligned}$$

(ii) For $k = s$, we have $\widehat{p_0}^{\,k-s}\mathcal{A}_s = \mathcal{A}_s$. We assume that $\mathcal{A}_k = \widehat{p_0}^{\,k-s}\mathcal{A}_s, k \geq s$. By using also (3.1), we obtain $\mathcal{A}_{k+1} = \widehat{p_0}\mathcal{A}_k + a_{k+1}$. Since $k + 1 > s$ we have $a_{k+1} = 0$. Which give

$$\mathcal{A}_{k+1} = \widehat{p_0}\mathcal{A}_k = \widehat{p_0}\widehat{p_0}^{\,k-s}\mathcal{A}_s = \widehat{p_0}^{\,k+1-s}\mathcal{A}_s.$$

□

The aim of the paper is to prove the following result concerning the divisibility by a prime numbers.

Theorem 3.6. *Let $p \in \mathcal{P}$ such that $p \equiv b_0[10]$, and $n = [a_s ... a_0]_{10} \in \mathbb{N}$. Then*

$$n \equiv 0[p] \Leftrightarrow \mathcal{A}_s \equiv 0[p],$$

where $\mathcal{A}_s = \sum_{i=0}^{s} \widehat{p_0}^{\,s-i} a_i$.

Proof. According to proposition 3.5, we have

$$\begin{aligned} \mathcal{A}_s = \sum_{i=0}^{s}\widehat{p_0}^{\,s-i}a_i &= \widehat{p_0}^{\,s}a_0 + \sum_{i=1}^{s}\widehat{p_0}^{\,s-i}a_i = \widehat{p_0}^{\,s}\big(n - \sum_{i=1}^{s}a_i10^i\big) + \sum_{i=1}^{s}\widehat{p_0}^{\,s-i}a_i \\ &= \widehat{p_0}^{\,s}n - \sum_{i=1}^{s}\widehat{p_0}^{\,s-i}a_i\big((10\widehat{p_0})^i - 1\big) \\ &= \widehat{p_0}^{\,s}n - \sum_{i=1}^{s}\widehat{p_0}^{\,s-i}a_i(10\widehat{p_0} - 1)\mathrm{Q}(10\widehat{p_0}). \end{aligned}$$

where Q is a polynomial of degree $i - 1$. Then, from $10\widehat{p_0} - 1 = pr_0$, we get

$$\mathcal{A}_s = \widehat{p_0}^{\,s}n - pr_0\sum_{i=1}^{s}\widehat{p_0}^{\,s-i}a_i\mathrm{Q}(10\widehat{p_0}). \tag{3.2}$$

In view of proposition 3.2, $\widehat{p_0} \wedge p = 1$. So, taking account of (3.2), we obtain

$$\begin{aligned} n \equiv 0[p] \quad &\Leftrightarrow \quad \widehat{p_0}^s n - pr_0 \sum_{i=1}^{s} \widehat{p_0}^{s-i} a_i \mathrm{Q}(10\widehat{p_0}) \equiv 0[p] \\ &\Leftrightarrow \quad \mathcal{A}_s \equiv 0[p]. \end{aligned}$$

□

The application of the theorem 3.6 is the following test of divisibility by a prime numbers.

Corollary 3.7. *Assume that the positive integer $n = [a_s ... a_0]_{10}$ is given, let $p \in \mathcal{P}$ such that $p \equiv b_0[10]$. Then*

$$p/[a_s ... a_0]_{10} \Leftrightarrow p/\Big(\sum_{i=0}^{s} (\widehat{p_0})^{s-i} a_i\Big).$$

where $\widehat{p_0} = \begin{cases} \frac{pb_0+1}{10}, & \text{if } b_0 \in \{3,7\}, \\ \frac{p(10-b_0)+1}{10}, & \text{if } b_0 \in \{1,9\}. \end{cases}$

We present two examples of divisibility by the prime numbers $7, 13$ and 29.

Example 3.8. (i) Let $p = 7$ and $n = 861 = 1 \times 10^0 + 6 \times 10^1 + 8 \times 10^2$. We have $b_0 = 7$, and thus $\widehat{p_0} = \frac{pb_0+1}{10} = 5$. Then

$$\mathcal{A}_s = \sum_{i=0}^{2} \widehat{p_0}^{2-i} a_i \quad = \quad 1 \times 5^2 + 6 \times 5^1 + 8 \times 5^0 = 63,$$

and clearly $7|63$. So, we conclude that $7|861$.

(ii) Let $p = 13$ and $n = 861 = 1 \times 10^0 + 6 \times 10^1 + 8 \times 10^2$. We have $b_0 = 3$, and thus $\widehat{p_0} = \frac{pb_0+1}{10} = 4$. Then

$$\mathcal{A}_s = \sum_{i=0}^{2} \widehat{p_0}^{2-i} a_i \quad = \quad 1 \times 4^2 + 6 \times 4^1 + 8 \times 4^0 = 48,$$

and clearly $13 \nmid 48$. So, we conclude that $13 \nmid 861$.

(iii) Let $p = 29$ and $n = 253141$. We have $b_0 = 9$, and thus $\widehat{p_0} = \frac{p(10-b_0)+1}{10} = 3$. Then

$$\mathcal{A}_s = \sum_{i=0}^{5} \widehat{p_0}^{5-i} a_i \quad = \quad 1 \times 3^5 + 4 \times 3^4 + 1 \times 3^3 + 3 \times 3^2 + 5 \times 3^1 + 2 \times 3^0 = 638,$$

Similarly, for $n' = 638$, we get $\mathcal{A}_s = \sum_{i=0}^{2} \widehat{p_0}^{2-i} a_i = 8 \times 3^2 + 3 \times 3^1 + 6 \times 3^0 = 87$, and clearly $29|87$, hence $29|638$. So, we conclude that $29|253141$.

Remark 3.9. Let $\mathcal{P} \ni p \equiv b_0[10]$. Then

(i)
- If $p \equiv 9[10]$ and $p > 89 \Rightarrow \widehat{p_0} \geq 10$,
- If $p \equiv 3[10]$ and $p > 23 \Rightarrow \widehat{p_0} \geq 10$,
- If $p \equiv 7[10]$ and $p > 17 \Rightarrow \widehat{p_0} \geq 10$,
- If $p \equiv 1[10]$ and $p > 11 \Rightarrow \widehat{p_0} \geq 10$.

(ii) We assume that $\widehat{p_0} \geq 10$. We have from theorem 3.6 that

$$\begin{aligned} n = [a_s ... a_0]_{10} \equiv 0[p] \quad &\Leftrightarrow \quad \sum_{i=0}^{s} \widehat{p_0}^{s-i} a_i \equiv 0[p] \\ &\Leftrightarrow \quad [a_0 ... a_s]_{\widehat{p_0}} \equiv 0[p]. \end{aligned}$$

which gives us a change of base without losing the divisibility by p.

Corollary 3.10. *Let* $\mathcal{P} \ni p \equiv b_0[10]$, *and* $n = [a_s...a_0]_{10} \in \mathbb{N}$. *Then*

$$n \equiv 0[p] \Leftrightarrow \mathcal{A}_\alpha \equiv 0[p], \forall \alpha \geq s.$$

Proof. According to proposition 3.5, one has $\mathcal{A}_\alpha = \widehat{p_0}^{\alpha-s}\mathcal{A}_s, \forall \alpha \geq s$. we obtain via assertion (iii) of proposition 3.2 that

$$\begin{aligned} n \equiv 0[p] &\Leftrightarrow \mathcal{A}_s \equiv 0[p] \\ &\Leftrightarrow \widehat{p_0}^{\alpha-s}\mathcal{A}_s \equiv 0[p], \quad \forall \alpha \geq s, \\ &\Leftrightarrow \mathcal{A}_\alpha \equiv 0[p], \quad \forall \alpha \geq s. \end{aligned}$$

□

In the following, as an example of application of previous result, we investigate the divisibility by the prime numbers 3 and 11.

Example 3.11. Let $n = [a_s...a_0]_{10} \in \mathbb{N}$. In this first, we present a very old result of the division on the prime number $p = 3$. We have $\widehat{p_0} = 1$ and

$$\begin{aligned} [a_s...a_0]_{10} \equiv 0[3] &\Leftrightarrow \sum_{i=0}^{s} \widehat{p_0}^{s-i} a_i \equiv 0[3] \\ &\Leftrightarrow \sum_{i=0}^{s} a_i \equiv 0[3]. \end{aligned}$$

Which gives that n is a multiple of 3 if and only if the sum of these digits is a multiple of 3.

In this second, we give a result of the divisibility by the prime number $p = 11$. One has $\widehat{p_0} = 10$ and

$$\begin{aligned} [a_s...a_0]_{10} \equiv 0[11] &\Leftrightarrow \sum_{i=0}^{s} 10^{s-i} a_i \equiv 0[11]. \\ &\Leftrightarrow [a_0...a_s]_{10} \equiv 0[11]. \end{aligned}$$

Which gives that n is a multiple of 11 if and only if the reverse of the order of these digits is a multiple of 11, for example

$$863764 \equiv 0[11] \Leftrightarrow 467368 \equiv 0[11].$$

Conclusion

in this work, we have developed a simple and efficient rule and test of divisibility by prime numbers by generalizing the case of 3. It remains to compare it by other tests, as well as to use the structure of the function defined in $\mathcal{P}$ to IN by $p \to \widehat{p_0}$, in depth to see the relationship between the distribution of prime numbers in IN and the distribution of their corresponding $\widehat{p_0}$.

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Received: 2022-01-14
Accepted: 2022-02-03