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CONSTRUCTION A TOPOLOGIES ON THE EDGES SET OF UNDIRECTED GRAPHS

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Abstract In this work, we investigate certain tie-ups between the theory of undirected graphs and topology. We define a topology on undirected graph by associating a topology with the set of edges of any undirected graph. Fundamental properties of undirected graph by their corresponding topology will be given. Furthermore, we prove some interesting theorems about connectedness and base of topological space (E, τ_E) such as if we have two isomorphic undirected graphs then their topologies are symmetric, but the converse is not true in general.

1 Introduction

The concept of topology and graph theory is widely used in many practical and theoretical aspects. Some studies which related to topology and graphing are studied separately and these studies found a link between topology structure and graph theory.

Researchers have done some work on the applications of combinatorial topological methods to graph theory and on the relations between graphs and topological surfaces.

In 1968, Bhargve and Ahllborn [5], in 1972, Lieberman [6] considered and analyzed the topological spaces with digraphs, In 2013, Hamza and Al-khafaji [2] constructed a topology on finite undirected graphs. In 2018, Abdu and Kilicman [1] presented another methodology of applying the topology on digraphs by associate two topologies with the set of edges of any directed graph, in 2020, Al'Dzhabri, Mahdi and Eissa [4] and in 2021, Zomam, Othman and Dammak [7] are constructed each digraph to topology.

Throughout this paper $(V_G,\ E_G)$ is a graph consists of a nonempty set V_G of vertices (or nodes) and a set E_G of edges. Each edge has two vertices associated with it and a path in graph G is a sequence of vertices and edges that begins and ends at two different vertices also G is connected graph if for every pair of distinct vertices $x_1, x_2\in V_G$, the graph G has a path between x_1, x_2 and denoted by $x_1\sim x_2$. Otherwise, we say the graph is disconnected.

2 Main Results

In 2013, Hamza and Al-khafaji [2] have constructed a topology on an undirected graph and a topology on subgraphs and in definition (2.2):

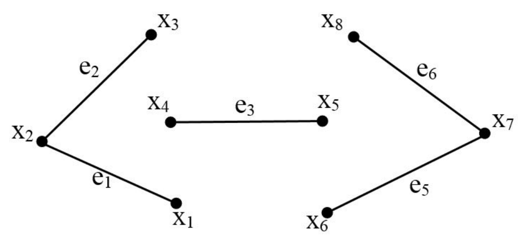
Let $G_\theta = (V_G,\ E_G)$ be a graph. We considered that each an isolated vertex as an edge and we called it an isolated edge. Put $E{=}E_G \cup S$ where S denotes the set of all isolated edges of the graph $(V_G,\ E_G)$. We have defined τ_E on E as follows:

$A\in\tau_E$ if, and only if, $A\in P(E)$, where $P(E)$ is the power set of E such that there is no a vertex $x_0\notin e_i, \forall\, e_i\in A$ such that $x_0\in e_j$ for some $e_j\in A$.

This definition is not topology, we can see that by the following example.

Example 2.1. By their definition $\{ e_3\}$ is not open set but $\{ e_1,\ e_2 ,\ e_3\}$ and $\{ e_3,\ e_5 ,\ e_6\}$ are open sets. We know that the $\{ e_1,\ e_2 ,\ e_3\} \cap \{ e_3,\ e_5 ,\ e_6\} = \{ e_3\}$ should be open set.

So, by this example we can see their definition is not correct, therefore all their results in paper [1] should review, then we will try to put my relentless efforts to this direction.



Definition 2.2. Let $(V_G,\ E_G)$ be a graph and consider that each an isolated vertex as an edge called isolated edge. Let $E = E_G \cup S$, where S denotes the set of all isolated edges of the graph $(V_G,\ E_G)$.

Define a collection τ_G on E by $\tau_G = \{\emptyset\} \cup \{A \subseteq E\ : e_i \in A$ and $e_i \cap e_j \neq \emptyset \Rightarrow e_j \in A\}$.

Theorem 2.3. *Let $(V_G,\ E_G)$ be any graph. Then, the pair $(E,\ \tau_E)$ is topological space.*

Proof.

(1) $\emptyset \in \tau_E$ by Definition. If $e_i \in E$ and $e_i \cap e_j \neq \emptyset$, then directly $e_j \in E$. That is, $E \in \tau_E$.

(2) Let $A,\ B \in \tau_E$. Let $e_i \in A \cap B$ and $e_i \cap e_j \neq \emptyset$. Then, $e_i \in A$ and $e_i \in B$ with $e_i \cap e_j \neq \emptyset$. Since $A,\ B \in \tau_E$, then $e_j \in A$ and $e_j \in B$. Hence $e_j \in A \cap B$. That is, $A \cap B \in \tau_E$.

(3) Let $G_\lambda \in \tau_E\ \forall\ \lambda \in \Delta$. Let $e_i \in \cup_{\lambda\in\Delta} G_\lambda$ and $e_i \cap e_j \neq \emptyset$. Then, there is at least $\lambda_0 \in \Delta$ such that $e_i \in G_{\lambda_0}$. Since $G_{\lambda_0} \in \tau_E$, then $e_j \in G_{\lambda_0} \subseteq \cup_{\lambda\in\Delta} G_\lambda$. That is, $\cup_{\lambda\in\Delta} G_\lambda \in \tau_E$. Therefore, $(E,\ \tau_E)$ is topological space.

Theorem 2.4. *Let $(V_G,\ E_G)$ be any graph. If $e_i \cap e_j \neq \emptyset$, $\forall\ e_i \neq e_j \in E$, then $(E,\ \tau_E)$ is trivial topological space.*

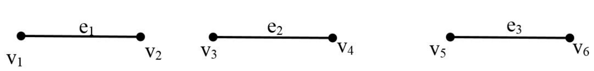
Proof. Let $A \in \tau_E$ and $A \neq \emptyset$, for $e_i \in A$, $e_i \cap e_j \neq \emptyset\ \forall j$. So $e_j \in A$. Therefore, $A = E_G = E$ and no isolated edge. That is $(E,\ \tau_E)$ is trivial topological space.

Theorem 2.5. *Let $(V_G,\ E_G)$ be any graph. If $e_i \cap e_j = \emptyset$, $\forall\ e_i \neq e_j \in E$, then $(E,\ \tau_E)$ is discrete topological space.*

Proof. Let $e_i \in E$. Since $e_i \cap e_j \neq \emptyset$, $\forall\ e_i \neq e_j \in E$. Then, only $e_i \cap e_i \neq \emptyset$.

That is, $\{e_i\} \in \tau_E$. Since e_i is arbitrary element in E, then $\{e_i\} \in \tau_E\ \forall\ e_i \in E$. That is $(E,\ \tau_E)$ is discrete topological space.

Example 2.6. If $E = \{e_1,\ e_2,\ e_3\}$.



Then, $\tau_E = \{\phi,\ E,\ \{e_1\},\ \{e_2\},\ \{e_3\},\ \{e_1,\ e_2\},\ \{e_1,\ e_3\},\ \{e_2,\ e_3\}\}$ is discrete topological space.

Definition 2.7. Let $(V_G,\ E_G)$ be any graph.

(1) A subset $V \subseteq V_G$ is called path-connected set in the graph $(V_G,\ E_G)$ if $\forall\ a,\ b \in V$, there is path between them and denoted by $a \sim b$. $E(V)$ denotes the set of all edges between element of V. That is, $E(V) \subseteq E$.

(2) A subset $V \subseteq V_G$ is called maximal path-connected set in $(V_G,\ E_G)$

If there is no path-connected set V' such that $V \subset V' \subset V_G$.

Remark 2.8. For any graph $(V_G, \ E_G)$, let $MP(V_G)$ denotes the set of all maximal path-connected sets in $(V_G, \ E_G)$. That is,

$$MP\,(V_G) = \{\ V\ \subseteq\ V_G\ : V \text{ is maximal path} - \text{connected set}\}.$$

$$EMP\,(V_G) = \{E(V)\ \subseteq\ E\ : V \text{ is maximal path} - \text{connected set}\}.$$

Lemma 2.9. *For any graph $(V_G, \ E_G)$,*

(1) The family $MP\,(V_G)$ forms partition of V_G.

(2) The family $EMP\,(V_G)$ forms partition of E.

Proof. For partition $MP\,(V_G)$ of V_G:

(1) (a) Let $V, \ V^{'} \in\ MP\,(V_G)$ and $V \cap\ V^{'}\ \neq \emptyset$. Then there is at least one vertex $a\ \in V \cap\ V^{'}$, then $a\ \in V$ and $a\ \in\ V^{'}$.

For all $x, \ y\ \in V \cap\ V^{'}$, we have one of the following:

(i) $x, \ y\ \in V$

(ii) $x, \ y \in\ V^{'}$

(iii) $x \in V$ and $\ y \in\ V^{'}$

(iv) $y \in V$ and $\ x \in\ V^{'}$

In case (i), $x \sim\ y$ since V is path-connected.

In case (ii), $x \sim\ y$ since $\ V^{'}$ is path-connected.

In case (iii), $x \sim\ a$ and $a \sim\ y$, hence $x \sim\ y$.

In case (iv), $x \sim\ a$ and $a \sim\ y$, hence $x \sim\ y$.

Hence, $V \cap\ V^{'}$ is path-connected set. Since $V \subset (V \cup V^{'}) \subset V_G$ and $V^{'} \subset (V\ \cup\ V^{'})\ \subset\ V_G$, then V and $\ V^{'}$ not maximal path connected sets. This contradiction. That is, $V \cap\ V^{'}\ =\ \emptyset$.
b- It is clear that

$$\cup\{V : V \in MP\,(V_G)\} \subseteq V_G \tag{2.1}$$

Let $x \in V_G$. Then, x is isolated point or not isolated point.
If x is isolated point, then $\{x\}$ is Maximal path connected set.
Hence, $x \in \{x\} \subseteq \cup\{V : V \in MP\,(V_G)\}$.
That is

$$V_G \subseteq U\,\{V : V \in MP\,(V_G)\} \tag{2.2}$$

From 2.1 and 2.2 we get that $\cup\{V : V \in MP\,(V_G)\} = V_G$.
Hence, $MP\,(V_G)$ from partition of V_G.
2- For partition $EMP\,(V_G)$ of E similar part(1).

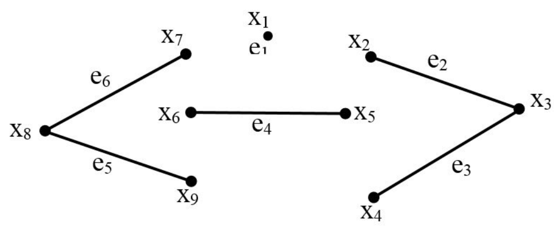
Lemma 2.10. *Let $(V_G, \ E_G)$ be any graph. Then the following statement are equivalent:*

(1) $(V_G, \ E_G)$ is connected graph.

(2) $MP\,(V_G) = \{V_G\}$.

(3) $EMP\,(V_G) = \{E\}$.

Proof. (1)⇒(2) If $(V_G, \ E_G)$ is connected graph, then $\forall\ x, \ y \in V_G\ , \ x \sim\ y$. That is V_G is maximal path-connected set. Hence $MP\,(V_G) = \{V_G\}$.
(2)⇒(1) is trivial.
(2)⇔(3) is trivial.

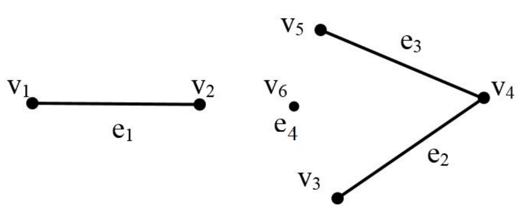


Example 2.11. $MP(V_G) = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_6, x_5\}, \{x_7, x_8, x_9\}\}$.
$EMP(V_G) = \{\{e_1\}, \{e_4\}, \{e_2, e_3\}, \{e_6, e_5\}\}$.

Lemma 2.12. *Let $(V_G,\ E_G)$ be any graph. For any maximal path-connected set V in $(V_G,\ E_G)$, then $E(V)$ is open set in $(E,\ \tau_E)$.*

Proof. Let V be any maximal path-connected set V in $(V_G,\ E_G)$.
Let $e_i \in E(V)$ and $e_i \cap e_j \neq \emptyset$ suppose that $e_j \notin E(V)$.
Since $e_i \cap e_j \neq \emptyset$, then there exists vertex $x \in e_i$ and $x \in e_j$.
Let $e_i :\ a \sim x$ and $e_j :\ x \sim b$, since $e_j \notin E(V)$, then $b \notin V$.
Since $x \sim b$ and $x \in V$, then $V \cup \{b\}$ is path-connected set.
Since $V \subset V \cup \{b\} \subset V_G$, then V is not maximal path-connected set. This is $e_j \in E(V)$. Hence, $E(V)$ is open set in $(E,\ \tau_E)$.

Remark 2.13. The converse of Lemma 2.12 no need to be true, for example, in the following graph.



Example 2.14. The set $E(V) = \{e_2,\ e_3,\ e_4\}$ is open set in $(E,\ \tau_E)$ but it is not edge set of maximal path-connected, since $V = \{v_3,\ v_4,\ v_5,\ v_6\}$ is not path-connected.

Theorem 2.15. *Let $(V_G,\ E_G)$ be a graph. Then, $(V_G,\ E_G)$ is connected graph if and only if $(E,\ \tau_E)$ is connected space.*

Proof. Suppose that $(V_G,\ E_G)$ is connected graph and $(E,\ \tau_E)$ is disconnected space. Then, there is a non-empty proper subset A of E such that A is both open and closed set in $(E,\ \tau_E)$.
Since A is proper subset non-empty and closed set, then A^c is proper subset of E and non-empty open set. Since $A \neq \phi$ and $A^c \neq \phi$, then there exists $e_i \in A$ and $e_j \in A^c$.
Let $e_i :\ x \sim y$ and $e_j :\ a \sim b$. Since $A \cap A^c = \emptyset$, then $e_i \cap e_j = \emptyset$.
Since A and A^c are open sets, then $x \nsim a$. That is $(V_G,\ E_G)$ is disconnected graph. This is contradiction.
Hence, $(E,\ \tau_E)$ is connected space.
Conversely, suppose that $(E,\ \tau_E)$ is connected space and $(V_G,\ E_G)$ disconnected graph.
Then, there are at least two vertexes x and y such that $x \nsim y$.
By lemma 2.9, part(1), there are two maximal path-connected sets V_x and V_y such that $x \in V_x$ and $y \in V_y$.
Take $A = E(V_x)$. Then, By lemma 2.12, A is open set in $(E,\ \tau_E)$.
Since $x \in V_x$, then $A \neq \phi$. Since $y \in V_y$ and $V_x \bigcap V_y = \emptyset$, then A is proper subset of E.
By lemma 2.9, part(2),
$A^c = E - E(V_x) = \cup\{\ E\ (V_\lambda) :\ V_\lambda$ is maximal path-connected set and $\lambda \in \Delta\}$.
So A is open set in $(E,\ \tau_E)$ By lemma 2.12.
Then, A is non-empty proper open and closed subset of E, that is $(E,\ \tau_E)$ is disconnected space this is contradiction. Hence, $(V_G,\ E_G)$ is connected graph.

Theorem 2.16. *Let* $(V_G,\ \ E_G)$ *be any graph. The family* $EMP\,(V_G)$ *is a base of topological space* $(E,\ \tau_E)\,.$

Proof. Firstly, we observe by lemma 2.12, that $E(V)$ is open set in $(E,\ \tau_E)$ for every maximal

path-connected set $V.$

That is, $E\,(V) \in\ \tau_E\ \ \forall\, V \in MP\,(V_G)\,.$ Hence, $EMP\,(V_G) \subseteq \tau_E.$

Secondly, let $e \in E$ be any point. It is enough to prove that the smallest open set containing e has maximal path-connected vertices set. That is, let A be the smallest open set containing e and let $V(A)$ be the set of all vertices of element of A. We will prove that $V\,(A) \in MP\,(V_G)\,.$

Let $e\ : x \sim y$ be the edge between $x,\ y\ \in V\,(E)\,.$

Suppose that $V\,(A)\ \notin\ MP\,(V_G)\,.$ Since $V\,(A)$ is path-connected set, then there is $V\ \in MP\,(V_G)$ such that $V\,(A) \subseteq V \subseteq V_G.$

Hence, there is at least

$$v^{'}\ \in V \text{and} v^{'}\ \notin V\,(A) \tag{2.3}$$

Since $v^{'}\ \in V,\ x \in V\,(A) \subset V$ and V is path-connected, then $v^{'}\ \sim x$ by edge, say $e^{'}$, that is, $e^{'}:\ v^{'}\ \sim x.$

Since $,e^{'}\ \in A$ and $e^{'}\ \cap e = \{x\} \neq \phi$, then $e^{'}\ \in A$. Hence $v^{'}\ \in V\,(A)\,.$

But this is contradiction with (2.3). Hence, $V\,(A) \in MP\,(V_G)\,.$

Therefore $EMP\,(V_G)$ forms a basis of $\tau_E.$

Definition 2.17. Let $(X,\ \tau_1)$ and $(X,\ \tau_2)$ be two finite topological spaces. We say that τ_1 and τ_2 are symmetric, denoted by $\tau_1 \sim \tau_2$ if and only if they satisfy the following conditions: $|\tau_1| \sim |\tau_2|$, and If $W_1 \in \tau_1$, then $\exists\, W_2 \in \tau_2$ such that $|W_1| =\ |W_2|$ and conversely.

Remark 2.18. (i) For any graph $(V_G,\ \ E_G)$, we mean by $e_{vv'}$, the edge e between the vertices v and v'.

(ii) For any two graphs $(V_{G_1},\ \ E_{G_1})$ and $(V_{G_2},\ \ E_{G_2})$,

we mean by $G_1 \cong G_2$ there are two bijective $f:\ V_{G_1}\ \longmapsto V_{G_2}$ and $g\ :\ E_{G_1}\ \longmapsto E_{G_2}$ such that $g\,(e_{vv'}) =\ e_{f(v)\ f(v')}.$

Theorem 2.19. *Let* $(V_{G_1},\ \ E_{G_1})$ *and* $(V_{G_2},\ \ E_{G_2})$ *be two graphs. If* $G_1 \cong G_2$*, then* $\tau_{E_1}\ \sim\ \tau_{E_2}.$

Proof. Define a function $H_\tau\ :\ \tau_{E_1} \rightarrow \tau_{E_2}$ by:

$$H(E) = \begin{cases} \emptyset & \text{if} \quad E = \emptyset \quad \text{for all } E \in_E \\ \left\{g\,(e_{vv'}) : e_{vv' \in E}\right\} & \text{if} \quad E_{E \neq \emptyset} \end{cases}$$

Firstly, we will show that H is well-defined function. That is we will prove that $H\,(E) \in \tau_{E_2},\ \forall\, E \in\ \tau_{E_1}.$

Let E be any open subset of E_{G_2}.

Let $g\,(e_{vv'})$ be any element in $H(E)$ and $e_{uu'}$ be any element in E_{G_2} such that $g\,(e_{vv'}) \cap e_{uu' \neq \phi}.$

Since g is one to one (surjective) function, then there exists $e_{ww'}$ in E_{G_1} such that $g\,(e_{ww'}) = e_{uu'}$. Then $g\,(e_{vv'}) \cap g\,(e_{ww'}) \neq \phi$

$\Rightarrow e_{f(v)f\left(v'\right)} \cap e_{f(w)f\left(w'\right)} \neq \phi$. Then, say $f(v) = f(w)$.

Since f is injective, then $v = w$.

Since E is open set in (E_{G_2}, τ_{E_2}) and $e_{ww'} \cap e_{vv' = \{v\} \neq \phi}$, then $e_{ww' \in E}$. Then $g\,(e_{ww'}) \in H(E)$. That is, $e_{uu'} \in H(E)$. Then $H(E)$ is open set in (E_2, τ_{E_2}).

Therefore $H(E)$ is well-defined function.

Secondly,since f and g bijective, then H is bijective function.

Hence, $|\tau_{E_1}| = |\ \tau_{E_2}|$. For $E \in \tau_{E_1},\ |E| = |H\,(E)|$ and conversely,

$\forall\, E \in \tau_{E_2},\ |E| = \left|H^{-1}(E)\right|$. That is $\tau_{E_1}\ \sim\ \tau_{E_2}.$

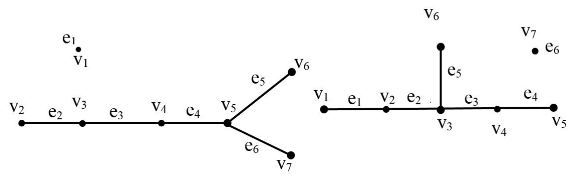
Remark 2.20. The converse of the theorem above no need to be true, for example.

Example 2.21.

$$\tau_{E_1} = \{\phi,\ E_1,\ \{e_1\}\,,\ \{e_2,\ e_3,\ e_4,\ e_5,\ e_6\}\}.$$

$$\tau_{E_2} = \{\phi,\ E_2,\ \{e_6\}\,,\ \{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}\}.$$

I can see that, $\tau_{E_1}\ \sim\ \tau_{E_2}$ but $G_1 \ncong\ G_2$.



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