Palestine Journal of Mathematics

Vol. 12(2)(2023) , 89–100

KOBAYASHI PSEUDO-METRIC AND DEFORMATIONS OF RIEMANN SURFACES

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 32G05, 32F45 ; Secondary 30F20, 32Q45.

Keywords and phrases: Kobayashi pseudo-metric . Deformations. Riemann surfaces

Abstract This paper deals with some properties of the infinitesimal form of the Kobayashi pseudo-distance. This form is shown to be a continuous function in the parameters of a differentiable deformation of each Riemann surface.
It is also shown in this paper that, each non elliptic Riemann surface has a non-trivial differentiable deformation.

NOTATIONS
Let a be a complex number and r, R, s strictly positive real numbers with $r < s$. Set :
$\Delta_R(a) = \{z \in \mathbb{C} : |z - a| < R\}$.
$\Delta_R = \Delta_R(0)$.
$\Delta = \Delta_1$: the open unit disc of $\mathbb{C}$.
$\overline{\Delta} = \{z \in \mathbb{C} : |z| \leqslant 1\}$: the closed unit disc of $\mathbb{C}$.
$S^1 = \{z \in \mathbb{C} : |z| = 1\}$: the unit circle of $\mathbb{C}$.
$A(r, s) = \{z \in \mathbb{C} : r < |z| < s\}$: the open annulus of $\mathbb{C}$ with centre 0, radius r and s.
For $t \in \overline{\Delta}$, $M_t = (M, J_t)$ where M is a complex manifold and J_t a complex structure on M.

1 Introduction

Let M be a complex manifold and B a differentiable manifold.
Let $\{J_t : t \in B\}$ be a differentiable family of complex structures on the underlying differentiable manifold of M and for each complex structure J_t, let M_t the corresponding complex manifold. Let F_{M_t} be the Kobayashi-Royden pseudo-metric on TM_t.
MARCUS WRIGHT has shown in [8] that F_{M_t} is upper semi-continuons in t. It is also mentioned in this article that the lower semi-continuity of F_{M_t} in t still remained an open question, even if M is compact. However, when M is a compact hyperbolic complex manifold, with a result of ROBERT BRODY in [9] and MARCUS WRIGHT quoted above, F_{M_t} is continuous in t, for all t in the parameters space.
In this document, we show that, if M is a Riemann surface with certain conditions on the family of complex structures, F_{M_t} is lower semi-continuous in t and thus continuous in t when we take the MARCUS WRIGHT result.
With this result we also show in this document that, each non elliptic Riemann surface has a non-trivial differentiable deformation.

The following theorems are proved :

Theorem A
Let (M, J) be a Riemann surface. Let Ω be a nonempty connected relatively compact open subset of M. Assume that there is a differentiable family of complex structures J_t $(t \in \overline{\Delta})$ on M such that $J_0 = J$ and all the complex structures J_t agree outside Ω. Then the map $F : TM \times \overline{\Delta} \longrightarrow \mathbb{R}^+$ given by the Kobayashi-Royden pseudo-metric on TM with respect to the family of complex structures J_t $(t \in \overline{\Delta})$ is continuous on $TM \times \overline{\Delta}$.

Theorem B
*Let (M, J) be a Riemann surface which is not biholomorphic to $\mathbb{P}_1$.
Then there exists a differentiable family of complex structures J_t on M parametrized by $t \in \overline{\Delta}$ such that $J_0 = J$ and (M, J_1) is not biholomorphic to (M, J).*

2 DEFORMATIONS OF COMPLEX STRUCTURES.

This section is founded on the fundamental and fruitful idea of Kunihiko Kodaira and D.C. Spencer which is the basis of their theory of deformation of complex structures. The following definition is accorded to them :

Definition 2.1. (Deformations of a complex structure).
A deformation of the complex structure on a complex manifold M is another complex structure on the underlying differentiable manifold of M obtained by using the same local complex coordinate charts on M but with different transition functions.

Remark 2.2. Let M be a complex manifold. When we put a new complex structure on the underlying differentiable manifold of M, the complex manifold obtained is a deformation of the complex manifold M.

Presentation (of deformations of a complex manifold).
A complex manifold M of dimension n is obtained by glueing domains
$V_1, V_2, ..., V_j, ...$ of $\mathbb{C}^n$ and by identifying $z_j \in V_j$ with $z_k = \varphi_{jk}(z_j) \in V_k$ via the transition functions $\{\varphi_{jk}\}$ given by an atlas $\{(U_j, z^j) : j \in \mathbb{N}^*\}$ of M such that $z^j(U_j) = V_j$ and $\varphi_{jk} = z^k \circ (z^j)^{-1}$. Hence $M = \cup_j U_j$ is identified with $\cup_j V_j$ (a countable and locally finite cover because M is paracompact).
These transition functions $\varphi_{jk} : z_j \longmapsto \varphi_{jk}(z_j) = z_k = (z_k^1, ..., z_k^n)$ are biholomorphic maps of $V_{jk} = z^j(U_j \cap U_k) \subset V_j$ onto $V_{kj} = z^k(U_j \cap U_k) \subset V_k$.

A deformation of M is considered to be a complex manifold obtained by glueing the same domains which cover M via different transition functions.
In other words, we replace the transition functions φ_{jk} by the transition functions $(\varphi_{jk})_t$ such that

$$(\varphi_{jk})_t(z_j) = \varphi_{jk}(z_j, t) = \varphi_{jk}(z_j, t_1, ..., t_m),$$

of variables (z_j, t) where the parameter $t = (t_1, ..., t_m) \in B$ (the parameters space B is a connected differentiable or complex manifold of dimension m) with the initial conditions $\varphi_{jk}(z_j, 0) = \varphi_{jk}(z_j)$.
Then the deformations M_t of $M = M_0$ are obtained by glueing the same domains $V_1, ..., V_j, ...$ which cover M and by identifying $z_j \in V_j$ with $z_k = \varphi_{jk}(z_j, t) \in V_k$, where in clear $\varphi_{jk}(z_j, t) = (z_k, t')$ and $t' \in B$ is identified with t in B.

Definition 2.3. Let M be a complex manifold and $\forall t \in B,\ M_t$ the deformations of M.
Let J_t be the complex structure of the complex manifold M_t for $t \in B$.
Consider the notations of the previous presentation.

- A deformation of the complex structure of M is said to be continuous or there exists a continuous family of complex structures $\{J$ on the underlying differentiable manifold of M, when the transition functions $(\varphi_{jk})_t$ are continuous maps with respect to the parameter t.

- A deformation of the complex structure of M is said to be differentiable or there exists a differentiable family of complex structur on the underlying differentiable manifold of M, when the transition functions $(\varphi_{jk})_t$ are differentiable maps with respect to the parameter t.

- A deformation of the complex structure of M is said to be holomorphic or there exists a holomorphic family of complex structures on the underlying differentiable manifold of M, when the transition functions $(\varphi_{jk})_t$ are holomorphic maps with respect to the parameter t.

Remark 2.4. With the definition of the transition functions given in the previous presentation, if the chart maps are homeomorphic (resp. diffeomorphic or biholomorphic) with respect to the parameter t, then the transition functions are continuous (resp. differentiable or holomorphic) maps with respect to the parameter t.

If the complex structure of the complex manifold M is given by an atlas with only one chart, then it is enough for the chart map to be continuous (resp. differentiable or holomorphic) with respect to the parameter t to have in order each of the above deformations listed.

Notation and terminology.
- In all cases of above deformation of the complex structure of M, the deformations are noted $M_t = (M, J_t)$ and $M = M_0 = (M, J_0)$.
- When the deformation of the complex structure of M is continuous (resp.
differentiable or holomorphic), the deformations M_t are called continuous (resp. differentiable or holomorphic) deformations of M.

Definition 2.5. (Trivial deformation of complex structures).
Let M be a complex manifold. A deformation of the complex structure of M is said to be *trivial* when all the deformations M_t are biholomorphic to $M = M_0$.

We recall these two following theorems to make a remark on theorem B.

Theorem 2.6. *(Uniformization theorem).*
The set of Riemann surfaces can be divided in three subsets : the elliptic Riemann surfaces, the parabolic Riemann surfaces and the hyperbolic Riemann surfaces.
- *The elliptic Riemann surfaces : These are the Riemann surfaces M isomorphic (i.e. biholomorphic) to the Riemann sphere $\mathbb{P}_1$.*

They are characterized by the fact that there is a non-constant holomorphic map $\mathbb{P}_1 \longrightarrow M$.
• *The parabolic Riemann surfaces : These are the Riemann surfaces* M *isomorphic to* $\mathbb{C}$ *or to* $\mathbb{C}^*$ *or to an elliptic curve* $\mathbb{C}/\Lambda$ *(*Λ *is a lattice of* $\mathbb{C}$*). They are characterized by the fact that they are not elliptic and that there is a non-constant holomorphic map* $\mathbb{C} \longrightarrow M$.
• *The hyperbolic Riemann surfaces : These are the Riemann surfaces* M *isomorphic to a quotient* $\mathbb{H}/\Gamma$ *where* Γ *is a discrete subgroup of* $PSL(2,\mathbb{R})$ *which acts on* $\mathbb{H}$ *in a properly discontinuous manner without fixed point. They are characterized by the fact that there is no non-constant holomorphic map* $\mathbb{C} \longrightarrow M$.

Proof. See [4], chapter 1. □

Theorem 2.7. *(Calabi-Vesentini).*
Let M *be a locally symmetric compact hermitian manifold of dimension* $n \geq 2$ *obtained as a quotient* $M = B^n/\Gamma$ *of the unit ball* B^n *of* $\mathbb{C}^n$ *by a discrete subgroup* Γ *of the group* $PU(1,n)$ *of biholomorphic isometries of the ball* B^n *with the Bergmann metric, which is properly discontinuous and fixed point free on* B^n.
Then the first cohomology group of M *with values in the holomorphic tangent bundle* TM *is nil. What is written*

$$H^1(M,TM) = 0.$$

This implies that there is no deformation of the complex structure of M.

Proof. See [3]. □

Remark 2.8. (Remark on theorem B).
In theorem B, the elliptic Riemann surfaces are excluded because, according to the uniformization theorem, the elliptic Riemann surfaces are biholomorphic to the Riemann sphere $\mathbb{P}_1$. Due to KODAIRA in [7], page 216, $H^1(\mathbb{P}_n, T\mathbb{P}_n) = 0, \forall n \in \mathbb{N}^*$. This implies that there is no deformation of the complex structure of $\mathbb{P}_n, \forall n \in \mathbb{N}^*$ (Theorem 2.7).

3 SPECIAL CASES OF THEOREM B.

(i) When (M,J) is a parabolic Riemann surface, the result is given in [6].

(ii) Let's show that for $M = \Delta$ and J the canonical complex structure of Δ, theorem B is fulfilled.
Let $t \in \overline{\Delta}$, set $\forall z \in \Delta$

$$\phi_t(z) = \frac{z}{1-|z|^2}(1-|z|^2 + |t|^2 tan(\frac{\pi}{2}|z|^2))$$

By definition, $\phi_t(z)$ is a differentiable map with respect to t.
• For $t = 0$, $\phi_0 = id_\Delta$. Thus ϕ_0 is a diffeomorphism of Δ onto itself .
• For $t \neq 0$, let's show that ϕ_t is a diffeomorphism of Δ onto $\mathbb{C}$.
- Let's show that ϕ_t is surjective.
It is clear that ϕ_t is continuous on Δ. So $|\phi_t|$ is continuous on Δ.
We also have $|\phi_t(0)| = 0$ and $\lim\limits_{|z|\to 1} |\phi_t(z)| = +\infty$.
Since $|\phi_t|$ is continuous, then $\forall z \in \Delta$, $|\phi_t(z)|$ takes all values in $[0,+\infty[$ and $\phi_t(z)$ takes all possibles arguments. Thus $Im\phi_t = \mathbb{C}$. Hence ϕ_t is surjective.
- Let's show that ϕ_t is injective.
Let $z_1, z_2 \in \Delta$ such that $\phi_t(z_1) = \phi_t(z_2)$, that is

$$(*) \quad \frac{z_1}{1-|z_1|^2}(1-|z_1|^2 + |t|^2 tan(\frac{\pi}{2}|z_1|^2)) = \frac{z_2}{1-|z_2|^2}(1-|z_2|^2 + |t|^2 tan(\frac{\pi}{2}|z_2|^2))$$

(a) If $|z_1| = |z_2|$, then

$$\frac{1}{1-|z_1|^2}(1-|z_1|^2 + |t|^2 tan(\frac{\pi}{2}|z_1|^2)) = \frac{1}{1-|z_2|^2}(1-|z_2|^2 + |t|^2 tan(\frac{\pi}{2}|z_2|^2))$$

and the equality $(*)$ gives $z_1 = z_2$.
(b) If $|z_1| \neq |z_2|$ with $0 < |z_1| < |z_2|$, then the equality $(*)$ gives :

$$\frac{|z_1|}{1-|z_1|^2}(1-|z_1|^2 + |t|^2 tan(\frac{\pi}{2}|z_1|^2)) = \frac{|z_2|}{1-|z_2|^2}(1-|z_2|^2 + |t|^2 tan(\frac{\pi}{2}|z_2|^2))$$

$$\frac{|z_1|(1-|z_2|^2)(1-|z_1|^2 + |t|^2 tan(\frac{\pi}{2}|z_1|^2))}{|z_2|(1-|z_1|^2)(1-|z_2|^2 + |t|^2 tan(\frac{\pi}{2}|z_2|^2))} = 1$$

$$\frac{(1-|z_2|^2)(1-|z_1|^2+|t|^2 tan(\frac{\pi}{2}|z_1|^2))}{(1-|z_1|^2)(1-|z_2|^2+|t|^2 tan(\frac{\pi}{2}|z_2|^2))} = \frac{|z_2|}{|z_1|} > 1.$$

Hence

$$(1-|z_2|^2)(1-|z_1|^2+|t|^2 tan(\frac{\pi}{2}|z_1|^2)) > (1-|z_1|^2)(1-|z_2|^2+|t|^2 tan(\frac{\pi}{2}|z_2|^2)).$$

$(1-|z_2|^2)(1-|z_1|^2)+(1-|z_2|^2)(|t|^2 tan(\frac{\pi}{2}|z_1|^2)) > (1-|z_1|^2)(1-|z_2|^2)+(1-|z_1|^2)(|t|^2 tan(\frac{\pi}{2}|z_2|^2)).$

Thus

$$(1-|z_2|^2)(|t|^2 tan(\frac{\pi}{2}|z_1|^2)) > (1-|z_1|^2)(|t|^2 tan(\frac{\pi}{2}|z_2|^2)).$$

That is absurd because $|z_1| < |z_2|$ and $(1-|z_2|^2)(|t|^2 tan(\frac{\pi}{2}|z_1|^2)) < (1-|z_1|^2)(|t|^2 tan(\frac{\pi}{2}|z_2|^2))$. So $|z_1| = |z_2|$. This gives $z_1 = z_2$. And then ϕ_t $(t \in \overline{\Delta}^*)$ is injective. Therefore ϕ_t $(t \in \overline{\Delta}^*)$ is a bijective map of Δ onto $\mathbb{C}$.

- Let's show that ϕ_t $(t \in \overline{\Delta}^*)$ is of class C^1 and its derivative is invertible at each point.

Recall : (Inverse function theorem).
Let Ω be an open set of $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^n$ a injective map of class C^1.
If $\mathcal{M}_J(f)(a)$ the Jacobian matrix of f in a is invertible for each $a \in \Omega$ then $\mathcal{U} = f(\Omega)$ is an open subset of $\mathbb{R}^n$ and f is a C^1-diffeomorphism of Ω onto $\mathcal{U}$.

Identify Δ with $B^2 = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| < 1\}$ and also $\mathbb{C}$ with $\mathbb{R}^2$.
Then $\phi_t : B^2 \longrightarrow \mathbb{R}^2$ such that $\forall (x,y) \in B^2, \quad \phi_t(x,y) = (\phi_t^1(x,y), \phi_t^2(x,y)$ with

$$\phi_t^1(x,y) = \frac{x}{1-(x^2+y^2)}(1-(x^2+y^2)+|t|^2 tan(\frac{\pi}{2}(x^2+y^2)))$$

$$\phi_t^2(x,y) = \frac{y}{1-(x^2+y^2)}(1-(x^2+y^2)+|t|^2 tan(\frac{\pi}{2}(x^2+y^2)))$$

Hence ϕ_t is a differentiable map. Then the partial derivatives of each component of ϕ_t exist.
Let $(x,y) \in B^2$, set

$$\mathcal{M}_J(\phi_t)(x,y) = \begin{pmatrix} \frac{\partial \phi_t^1}{\partial x}(x,y) & \frac{\partial \phi_t^1}{\partial y}(x,y) \\ \frac{\partial \phi_t^2}{\partial x}(x,y) & \frac{\partial \phi_t^2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the Jacobian matrix of ϕ_t in (x,y).
Set $r = x^2+y^2$, we have :

$A = \frac{\partial \phi_t^1}{\partial x}(x,y) =$

$$\frac{[(1-r+|t|^2 tan\frac{\pi}{2}r) + x^2(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))][1-r] + 2x^2(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$A = \frac{1-r+|t|^2 tan\frac{\pi}{2}r}{1-r} + \frac{x^2(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))}{1-r} + \frac{2x^2(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$B = \frac{\partial \phi_t^1}{\partial y}(x,y) = \frac{xy(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))(1-r) + 2xy(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$B = \frac{xy(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))}{1-r} + \frac{2xy(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$C = \frac{\partial \phi_t^2}{\partial x}(x,y) = \frac{xy(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))(1-r) + 2xy(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$C = \frac{xy(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))}{1-r} + \frac{2xy(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$D = \frac{\partial \phi_t^2}{\partial y}(x,y) =$

$$\frac{[(1-r+|t|^2 tan\frac{\pi}{2}r) + y^2(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))][1-r] + 2y^2(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

$$D = \frac{1-r+|t|^2 tan\frac{\pi}{2}r}{1-r} + \frac{y^2(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))}{1-r} + \frac{2y^2(1-r+|t|^2 tan\frac{\pi}{2}r)}{(1-r)^2}$$

Since $\forall (x,y) \in B^2,\ r = x^2+y^2 < 1$, then each partial derivative of ϕ_t is continuous. Therefore ϕ_t is of class C^1 on B^2 (and thus on Δ).
We have

$$det\mathcal{M}_J(\phi_t)(x,y) = A \times D - B \times C.$$

Let's show that $det\mathcal{M}_J(\phi_t)(x,y) \neq 0$. We simply write $\mathcal{M}_J(\phi_t)(x,y) = \mathcal{M}_J(\phi_t)$.
$det\mathcal{M}_J(\phi_t) = \frac{1}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)^2 + \frac{x^2+y^2}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r)) + \frac{2x^2+2y^2}{(1-r)^3}(1-r+|t|^2tan\frac{\pi}{2}r)^2 + \frac{x^2y^2}{(1-r)^2}(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))^2 +$

$$\frac{2x^2y^2+2x^2y^2}{(1-r)^3}(1-r+|t|^2tan\frac{\pi}{2}r)(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r)) + \frac{4x^2y^2}{(1-r)^4}(1-r+|t|^2tan\frac{\pi}{2}r)^2$$

$-\frac{x^2y^2}{(1-r)^2}(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r))^2 - \frac{2x^2y^2+2x^2y^2}{(1-r)^3}(1-r+|t|^2tan\frac{\pi}{2}r)(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r)) - \frac{4x^2y^2}{(1-r)^4}(1-r+|t|^2tan\frac{\pi}{2}r)^2$

$det\mathcal{M}_J(\phi_t) = \frac{1}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)^2 + \frac{r}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)(-2+\pi|t|^2(1+tan^2\frac{\pi}{2}r)) + \frac{2r}{(1-r)^3}(1-r+|t|^2tan\frac{\pi}{2}r)^2.$

Assume that $det\mathcal{M}_J(\phi_t) = 0$, then

$$\frac{1}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)^2 + \frac{r}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)(\pi|t|^2(1+tan^2\frac{\pi}{2}r))$$

$$+\frac{2r}{(1-r)^3}(1-r+|t|^2tan\frac{\pi}{2}r)^2 = \frac{2r}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)$$

Set

$$K = \frac{1}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)^2 + \frac{r}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)(\pi|t|^2(1+tan^2\frac{\pi}{2}r))$$

then $K > 0$. Hence

$$K + \frac{2r}{(1-r)^3}((1-r)^2 + 2(1-r)|t|^2tan\frac{\pi}{2}r + (|t|^2tan\frac{\pi}{2}r)^2) = \frac{2r}{(1-r)^2}(1-r+|t|^2tan\frac{\pi}{2}r)$$

$$K + \frac{2r}{1-r} + \frac{4r}{(1-r)^2}|t|^2tan\frac{\pi}{2}r + \frac{2r}{(1-r)^3}(|t|^2tan\frac{\pi}{2}r)^2 = \frac{2r}{1-r} + \frac{2r}{(1-r)^2}|t|^2tan\frac{\pi}{2}r$$

Thus

$$K + \frac{2r}{(1-r)^2}|t|^2tan\frac{\pi}{2}r + \frac{2r}{(1-r)^3}(|t|^2tan\frac{\pi}{2}r)^2 = 0.$$

That is absurd because $K > 0$. Hence $det\mathcal{M}_J(\phi_t)(x,y) \neq 0,\ \forall (x,y) \in B^2$.
So the derivative of ϕ_t at each point $(x,y) \in B^2$ is invertible.
Therefore ϕ_t $(t \in \overline{\Delta}^*)$ is a diffeomorphism of class C^1 of Δ onto $\mathbb{C}$. (According to the inverse function theorem).
Then :
For $t = 0$. Let J be the canonical complex structure of $\Delta \cong B^2$. Set $J_0 = J$.
Then (Δ, J_0) is biholomorphic to (Δ, J) with the atlas $\{(\Delta, \phi_0 = Id_\Delta)\}$.
For $t \neq 0$. Let J' be the canonical complex structure of $\mathbb{C} \cong \mathbb{R}^2$.
Set $J_t = (\phi_t^{-1})^T \circ J' \circ (\phi_t)^T$. Then $\phi_t : (\Delta, J_t) \longrightarrow (\mathbb{C}, J')$ is a biholomorphism map with the atlas of $M = \Delta$ given by $\{(\Delta, \phi_t)\}$.

Hence, since ϕ_t is differentiable in t, then there exists a differentiable family of complex structures J_t on Δ parametrized by $t \in \overline{\Delta}$ such that $J_0 = J$ (J is the canonical complex structure of Δ).
However (Δ, J_0) is not biholomorphic to (Δ, J_1). If so, since (Δ, J_1) is biholomorphic to $(\mathbb{C}, J')$, then (Δ, J_0) would be biholomorphic to $(\mathbb{C}, J')$. According to Liouville's theorem this is impossible.
Then in this case, the Riemann surface $\mathbb{C}$ is a differentiable deformation of the hyperbolic Riemann surface Δ. Thus theorem B is fulfilled for $(M, J) = \Delta$.

NB : In [6] (section 3), a map ϕ_t^* allows to have similarly to (ii) that the Riemann surface Δ is a differentiable deformation of the Riemann surface $\mathbb{C}$. It is also shown in this article that $\Delta^* = \Delta \setminus \{0\}$ is a differentiable deformation of the Riemann surface $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(iii) When the map ϕ_t is restricted to Δ^* or to $\Delta \setminus \{0; \frac{1}{2}\}$, we obtain the same results as in (ii) and thus theorem B is fulfilled for $(M, J) = \Delta^* = \Delta \setminus \{0\}$ and also for $(M, J) = \Delta \setminus \{0; \frac{1}{2}\}$.

When the map ϕ_t^* in [6] (section 3, point (i)) is restricted to $\Delta \setminus \{0; \frac{1}{2}\}$ then theorem B is fulfilled for $(M, J) = \mathbb{P}_1 \setminus \{0; 1; +\infty\} \equiv \mathbb{C} \setminus \{0; 1\}$.

Hence the Riemann surface $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a differentiable deformation of the hyperbolic Riemann surface Δ^*.
Moreover the hyperbolic Riemann surface $\mathbb{P}_1 \setminus \{0; 1; +\infty\} \equiv \mathbb{C} \setminus \{0; 1\}$ is a differentiable deformation of the hyperbolic Riemann surface $\Delta \setminus \{0; \frac{1}{2}\}$ and vice versa.

Finally, the following proposition has been proved :

Proposition 3.1. *Let (M, J) be a Riemann surface biholomorphic to Δ, Δ^*, $\Delta \setminus \{0, \frac{1}{2}\}$ or to $\mathbb{P}_1 \setminus \{0, 1, \infty\}$.
Then there exists a differentiable family of complex structures J_t on M parametrized by $t \in \overline{\Delta}$ such that $J_0 = J$ and (M, J_1) is not biholomorphic to (M, J).*

According to section 3, we still have to see the hyperbolic Riemann surfaces in their generality so that theorem B is fulfilled for all Riemann surfaces except $\mathbb{P}_1$. For the remaining Riemann surfaces, we will use other concepts to achieve this. Hence the following orientation :

4 HYPERBOLIC LENGTH

On the unit disc, the Poincaré metric is considered to be the complete Riemannian metric defined by

$$ds^2 = \frac{dzd\overline{z}}{(1 - |z|^2)^2}$$

The Poincaré distance ρ is the distance on Δ defined by the Poincaré metric.

Definition 4.1. (Kobayashi pseudo-distance).
Let M be a connected complex manifold. The *Kobayashi pseudo-distance* on M is the map defined as follows : Let $p, q \in M$, choose a finite sequence of points $p = p_0, ..., p_n = q$ in M and the holomorphic maps $f_i : \Delta \longrightarrow M$ with $p_{i-1}, p_i \in f_i(\Delta), i = 1, ..., n$. Then

$$d_M(p, q) = \inf_{\{p_i\},\{f_i\}} \sum_{i=1}^{n} \rho(f_i^{-1}(p_{i-1}), f_i^{-1}(p_i)).$$

The map d_M is a pseudo-distance on M. If d_M is a distance, M is said to be a *hyperbolic manifold*. In particular the hyperbolic Riemann surfaces defined in uniformization theorem 2.6 are hyperbolic manifolds.

Proposition 4.2. *[10]*
(1) If M and N are connected complex manifolds, then

$$\forall f \in Hol(M, N) \text{ and } \forall p, q \in M,\ d_N(f(p), f(q)) \leq d_M(p, q).$$

(2) For the unit disc Δ, the pseudo-distance d_Δ coincides with ρ (i.e. $d_\Delta = \rho$).

Definition 4.3. (Kobayashi-Royden Pseudo-metric).
Let M be a connected complex manifold.
The (infinitesimal) *Kobayashi-Royden pseudo-metric* is the real positive values function $F_M : TM \longrightarrow \mathbb{R}^+$ defined by $\forall (x, v) = v \in TM$

$$(F_M)(v) = \inf_{f \in Hol(\Delta, M)} \{\|u\| : u \in T\Delta \ et \ f^T(u) = v\}$$

where $\|u\| = \|u\|_{ds^2}$.

Remark 4.4. - The map F_M restricted to each fiber T_pM is continuous.
- When $\Omega \subset M$ is open and $v = (x, v) \in TM$ with $x \in \Omega$, then we have :

$$(F_M)(v) = \inf_{f \in Hol(\Delta, \Omega)} \{\|u\| : u \in T\Delta \ et \ f^T(u) = v\}.$$

Definition 4.5. (Hyperbolic length).
Let M be a hyperbolic Riemann surface and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $\gamma : S^1 \longrightarrow M$ be a differentiable curve in M.
The *hyperbolic length* of γ is the real positive number

$$L_M(\gamma) = \int_0^1 (F_M)(\gamma'(t))\mathrm{d}t.$$

Proposition 4.6. *The hyperbolic length of γ above defined is stable by biholomorphism i.e. when $G : M \longrightarrow M'$ is a biholomorphic map then $L_M(\gamma) = L_{M'}(G(\gamma))$.*

Proof. Let $G : M \longrightarrow M'$ be a biholomorphic map. Then $G^T : TM \longrightarrow TM'$ is a diffeomorphism and $(G^{-1})^T = (G^T)^{-1}$. Hence $\forall v' \in TM', \exists!\, v \in TM$ such that $G^T(v) = v'$.

$$(F_{M'})(v') = \inf_{f\in Hol(\Delta,M')} \{\|u\| : u \in T\Delta \; et \; f^T(u) = v'\}.$$

$$(F_{M'})(v') = \inf_{f\in Hol(\Delta,M')} \{\|u\| : u \in T\Delta \; et \; f^T(u) = G^T(v)\}.$$

$$(F_{M'})(v') = \inf_{f\in Hol(\Delta,M')} \{\|u\| : u \in T\Delta \; et \; (G^T)^{-1} \circ f^T(u) = v\}.$$

$$(F_{M'})(v') = \inf_{(G^{-1}\circ f)\in Hol(\Delta,M)} \{\|u\| : u \in T\Delta \; et \; (G^{-1} \circ f)^T(u) = v\}.$$

Thus $(F_{M'})(v') \geq \inf_{g\in Hol(\Delta,M)} \{\|u\| : u \in T\Delta \; et \; g^T(u) = v\}$. Therefore

$$(F_{M'})(v') \geq (F_M)(v).$$

Similarly $(F_M)(v) \geq (F_{M'})(v')$. Hence $(F_M)(v) = (F_{M'})(v')$. □

Definition 4.7. (Simple curve).
Let M be a complex manifold.
A *simple curve* on M is an injective and immersive differentiable map $\gamma : S^1 \to M$.

Definition 4.8. (Hyperbolic length spectrum).
Let M be a hyperbolic Riemann surface and Γ_M the set of homotopy classes of simple curves in M. For every classe $[\gamma] \in \Gamma_M$, we call *stable hyperbolic length* of γ classe, the real number denoted by $\Lambda([\gamma])$ which is the infimum of the hyperbolic length of all simple curves in the homotopy classe of γ.
We call **Hyperbolic length spectrum** of M the set Σ_M of the stable hyperbolic lengths $\Lambda([\gamma])$ for every classe $[\gamma] \in \Gamma_M$.

When M is a complete hyperbolic manifold, then the Hyperbolic length spectrum $\Sigma_M = \{\Lambda([\gamma]) : [\gamma] \in \Gamma_M\}$ is countable.

NB : According to proposition 4.6, Σ_M is an invariant of the complex structure on M (*i.e.* if (M, J) is biholomorphic to (M, J'), then $\Sigma_{(M,J)} = \Sigma_{(M,J')}$).

Proposition 4.9. *Let M be a hyperbolic Riemann surface. Assume that every simple curve in M can be deformed to a curve of arbitrarily small hyperbolic length (i.e. the spectrum Σ_M is trivial ($\Sigma_M = \{0\}$)).*
Then M is biholomorphic to one of the following Riemann surface :
(i) the unit disc Δ ;
(ii) the punctured unit disc Δ^* ;
(iii) $\mathbb{P}_1 \setminus \{0, 1, \infty\}$.

Proof. See [5], page 8. □

Lemma 4.10. *Let $r, \alpha \in \mathbb{R}_+^*$ such that $Logr > \alpha > 2\pi$. Then there is a holomorphic map from the unit disc Δ to the annulus $A(\frac{1}{r}, r) = \{z \in \mathbb{C} : \frac{1}{r} < |z| < r\}$ mapping the real interval $[0, \frac{2\pi}{\alpha}]$ onto $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. Take the map $z \longmapsto exp(i\alpha z)$ and note that $|z| < 1$ induces $|Im(z)| < 1$. Since $Logr > \alpha$, then $\frac{1}{r} < |exp(i\alpha z)| < r$. □

Corollary 4.11. *For $r > 1$, let $\mu(r)$ denote the hyperbolic length of S^1 as a curve in the Riemann surface $A(\frac{1}{r}, r)$. Then $\lim_{r\to+\infty} \mu(r) = 0$.*

Proof. According to lemma 4.10 $\mu(r)$ is bounded from above by the hyperbolic length of $[0, \frac{2\pi}{\alpha}]$, for $Logr > \alpha > 2\pi$. When $\alpha \to +\infty$ our assertion is fulfilled. □

Lemma 4.12. *Let γ be a simple curve in the hyperbolic Riemann surface (M, J). Let $\epsilon > 0$ and $c \in \overline{\Delta}$ with $|c| = 1$. Then there exists a differentiable family of complex structures J_t ($t \in \overline{\Delta}$) such that :*
(1) The hyperbolic length of γ with respect to the complex structure J_c is less than ϵ ;
(2) J_0 equals the given complex structure J ;
(3) There is a nonempty connected relatively compact subset Ω of M such that $(J_t)_x = (J_s)_x$ for all $x \notin \Omega$ and for all $s, t \in \overline{\Delta}$, i.e. all the complex structures J_t agree outside Ω.

Proof. We choose real constants r and r' such that $r > r' > 1$. Then there is an open neighbourhood V of γ diffeomorphic to the annulus $A(\frac{1}{r}, r)$ taking γ to the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let ψ be this diffeomorphism. Since $A(\frac{1}{r'}, r') \subset A(\frac{1}{r}, r)$, define $W = \psi^{-1}(A(\frac{1}{r'}, r'))$. Choose a differentable real function $\theta : M \to [0, 1]$ such that :
(1) $\theta_{|\overline{W}} \equiv 1$,
(2) θ is the constant map zero in some open neighbourhood of $M \setminus V$.
Next we choose a hermitian metric h on M and a hermitian metric τ on $A(\frac{1}{r}, r)$. We define $H_t = (1 - |t|^2\theta)h + |t|^2\theta\psi^*\tau$ (where $(\psi^*\tau)(x) = \tau(\psi(x)), \forall x \in V$). H_t is a Riemannian metric on M which determines a complex structure J_t on M with J_0 being the original complex structure on M. By construction (W, J_c) is biholomorphic to $A(\frac{1}{r'}, r')$. Since the choice of the constants $r > r' > 1$ was arbitrary, the value of r' may be as large as desired. Then the hyperbolic length of γ with respect to (W, J_c) becomes as small as desired (Corollary 4.11). Since the injection of (W, J_c) into (M, J_c) is distance-decreasing, because it is a holomorphic map (Proposition 4.2), then we obtain our required assertion if we set $\Omega = V$. □

5 CONTINUITY OF KOBAYASHI-ROYDEN PSEUDO-METRIC.

Definition 5.1. (Equicontinuity).
Let X be a topological space and Y a metric space with distance function d.
Let $C(X, Y)$ be the set of continuous functions of X into Y with the compact-open topology. Let $\mathcal{F} \subset C(X, Y)$ and $x \in X$.
The family $\mathcal{F}$ is said to be equicontinuous at x if for every $\epsilon > 0$, there exists a neighbourhood U of x such that $d(f(x), f(x')) < \epsilon$, $\forall x' \in U$ and $\forall f \in \mathcal{F}$.

Theorem 5.2. *(Arzela-Ascoli theorem).*
Let X be a locally compact, separable space (i.e. X contains a countable, dense subset) and Y a locally compact metric space. Then a family $\mathcal{F} \subset C(X, Y)$ is relatively compact in $C(X, Y)$ if and only if :
(a) $\mathcal{F}$ is equicontinuous at every point $x \in X$;
(b) for every $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is relatively compact in Y.

Proof. See [10], page 8. □

Remark on Arzela-Ascoli theorem.
If a family $\mathcal{F} \subset C(X, Y)$ is relatively compact in $C(X, Y)$ with the compact-open topology, then, according to [10] (page 8), every sequence of maps $f_n \in \mathcal{F}$ contains a subsequence which converges to some map $f \in C(X, Y)$ uniformly on every compact subset of X.

Theorem A
Let (M, J) be a Riemann surface. Let Ω be a nonempty connected relatively compact open subset of M. Assume that there is a differentiable family of complex structures J_t $(t \in \overline{\Delta})$ on M such that $J_0 = J$ and all the complex structures J_t agree outside Ω. Then the map $F : TM \times \overline{\Delta} \longrightarrow \mathbb{R}^+$ given by the Kobayashi-Royden pseudo-metric on TM with respect to the family of complex structures J_t $(t \in \overline{\Delta})$ is continuous on $TM \times \overline{\Delta}$.

We note that $\forall((x, v), t) = (v, t) \in TM \times \overline{\Delta}$,

$$F(v, t) = F_{(M,J_t)}(v) = F_{M_t}(v) = \inf_{f \in Hol(\Delta, M_t)} \{\|u\| : u \in T\Delta \; et \; f^T(u) = v\}.$$

Remark on theorem A.
It is important that we deform the complex structure only inside some fixed nonempty relatively compact subset of M. Without this assumption the statement is not true. For example, we have seen that there is a family of complex structures J_t $(t \in \overline{\Delta})$ on Δ such that (M, J_0) is biholomorphic to the unit disc while (M, J_t) for every $t \neq 0$ is biholomorphic to $\mathbb{C}$. Evidently for this family F is not continuous, since it vanishes for $t \neq 0$ and is non-zero for $t = 0$.

For theorem A's proof, we shall use the two following lemmas :

In this section, for a complex manifold M, we assume the existence of a hermitian metric h on M and denote $\| \, \|_h$ the norm defined by the metric h.
For all map of class C^∞, $f : \Delta \longrightarrow M$, we define $\|f'(z_0)\|_h := \|d_{z_0} f(\frac{\partial}{\partial z}_{|0})\|_h$.

Lemma 5.3. *Let M be a complex manifold of dimension 1 and $\Omega \subset M$ a relatively compact open subset of M. Let h be a hermitian metric on M.*
If M is hyperbolic then $\sup_{f \in Hol(\Delta, \Omega)} \|f'(0)\|_h < +\infty$.

Proof. Let M be a hyperbolic complex manifold of dimension 1.
Assume that $\sup_{f \in Hol(\Delta, \Omega)} \|f'(0)\|_h = +\infty$.

Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of maps in $Hol(\Delta,\Omega)$ such that $\|f_n'(0)\|_h$ is monotonic increasing and converges to $+\infty$. Since Ω is relatively compact in M, then $Hol(\Delta,\Omega)$ is relatively compact in $Hol(\Delta,M)$ with the compact-open topology. Hence one can assume that $f_n \longrightarrow f \in Hol(\Delta,M)$. Therefore $f_n(0) \longrightarrow f(0) = x \in M$. Choose a chart $\varphi : U \longrightarrow \mathbb{C}^n$ where U is an open neighbourhood of x with $\varphi(U) \supset \overline{B(0,s)}$ and $\varphi(x) = 0$. Let $r < 1$. According to Cauchy's integral formula and to the analycity of holomorphic function, we have

$$(\varphi\circ f_n)'(0) = \frac{1}{2\pi i}\int_{C_r}\frac{(\varphi\circ f_n)(w)}{(w-0)^2}dw$$

where $C_r = \{w \in \mathbb{C} : |w| = r\} = \{re^{it} \in \mathbb{C} : -\pi \le t \le \pi\}$.
Then, for $\|.\|_{euc}$ the Euclidian norm on $\mathbb{C}$ (*i.e.* $\forall z \in \mathbb{C}, \|z\|_{euc} = |z|$) and for n such that $(\varphi\circ f_n)(\Delta_r) \subset B(0,s)$, we obtain :

$$\|(\varphi\circ f_n)'(0)\|_{euc} \le |\frac{1}{2\pi i}|\int_{C_r}|\frac{(\varphi\circ f_n)(w)}{w^2}dw|$$

$$\|(\varphi\circ f_n)'(0)\|_{euc} \le \frac{1}{2\pi}\frac{s}{r^2}\int_{C_r}|dw| = \frac{1}{2\pi}\frac{s}{r^2}\int_{-\pi}^{\pi}|d(re^{it})|$$

$$\|(\varphi\circ f_n)'(0)\|_{euc} \le \frac{1}{2\pi}\frac{s}{r^2}\int_{-\pi}^{\pi}|rie^{it}dt| = \frac{1}{2\pi}\frac{s}{r}\int_{-\pi}^{\pi}|e^{it}|dt$$

$$\|(\varphi\circ f_n)'(0)\|_{euc} \le \frac{1}{2\pi}\frac{s}{r}\int_{-\pi}^{\pi}dt = \frac{1}{2\pi}\frac{s}{r}(\pi+\pi) = \frac{s}{r}$$

$$\|(\varphi\circ f_n)'(0)\|_{euc} \le \frac{s}{r}.$$

Hence $\|(\varphi\circ f_n)'(0)\|_{euc} \le \frac{s}{r}$ for every n such that $\varphi\circ f_n(\Delta_r) \subset B(0,s)$. This implies $\|\varphi'(x)f_n'(0)\|_{euc} \le \frac{s}{r}$ and by compacity of $\varphi^{-1}(\overline{B}(0,s)$ there exists $A > 0$ such that $\|f_n'(0)\|_h \le \frac{As}{r}$. (Moreover the two norms $\|.\|_h$ and $\|.\|_{\varphi^*euc}$ are equivalent).
When $\|f_n'(0)\|_h \longrightarrow +\infty$, then $r = r(n) \to 0$. Thus $\forall m \in \mathbb{N}^*$, there exists n sufficiently large such that $\varphi\circ f_n(\Delta_{\frac{1}{m}})$ is not contained in $B(0,s)$ and $\varphi\circ f_n(\Delta_{\frac{1}{m}})\cap\partial B(0,s) \ne \emptyset$. Therefore there is a sequence (x_m) where $x_m \in f_n(\Delta_{\frac{1}{m}})$ and $\varphi(x_m) \in \partial B(0,s)$. That induces $d_M(f_n(0), x_m) \le \rho(0, q_m) \to 0$ when $m \to +\infty$ ($q_m \in \Delta_{\frac{1}{m}}$). Since d_M is continuous and $\varphi^{-1}(\partial B(0,s)$ is compact, then $x_m \to y \in \varphi^{-1}(\partial B(0,s)$.
Hence $d_M(x,y) = 0$. So M is not hyperbolic (absurd).
Therefore $\sup_{f\in Hol(\Delta,\Omega)}\|f'(0)\|_h < +\infty$. □

Lemma 5.4. *(Parametrization of Brody).*
Let M be a complex manifold with Hermitian metric h. Let $f \in Hol(\Delta_r, M)$, ($r \in \mathbb{R}_+^*$) with $\|f'(0)\|_h \ge c > 0$.
Then there exists $\widetilde{f} \in Hol(\Delta_r, M)$ such that

$$\sup_{z\in\Delta_r}\|\widetilde{f}'(z)\|_h(\frac{r^2-|z|^2}{r^2}) = \|\widetilde{f}'(0)\|_h = c$$

Proof. See [9], page 3. □

Remark 5.5. Let M be a complex manifold with Hermitian metric h and $c \in \mathbb{R}_+^*$.
Let Ω^1 and Ω^2 be two nonempty relatively compact subsets of M such that $\overline{\Omega^1} \subset \Omega^2$.
Due to lemma 5.4, for every map $f \in Hol(\Delta,\Omega^1)$ with $\|f'(0)\|_h \ge c$, there exists $\widetilde{f} \in Hol(\Delta,\Omega^1)$ such that $\sup_{z\in\Delta}\|\widetilde{f}'(z)\|_h < +\infty$, because
$\sup_{z\in\Delta}\|\widetilde{f}'(z)\|_h(1-|z|^2) = \|\widetilde{f}'(0)\|_h = c < +\infty$ and $\forall z \in \Delta,\ 0 < (1-|z|^2) \le 1$.
Hence every family $\mathcal{F}$ of maps of class C^∞ of Δ into Ω^1 which fulfills lemma 5.4's conditions, for fixed number c, is relatively compact in the compact-open topology of $C(\Delta,\Omega^2)$. (According to Arzela-Ascoli theorem).
Indeed :
- The family $\mathcal{F}$ is equicontinuous because for every compact subset $K \subset \Delta$, there is a uniform bound on $\|\widetilde{f}'(z)\|_h$ for all $\widetilde{f} \in \mathcal{F}$ and all $z \in K$;
- For all $x \in \Delta$, the set $\{\widetilde{f}(x) : \widetilde{f} \in \mathcal{F}\}$ is relatively compact in Ω^2, because $\{\widetilde{f}(x) : \widetilde{f} \in \mathcal{F}\} \subset \Omega^1$ and Ω^1 is relatively compact in Ω^2.
Then all sequence of maps in $\mathcal{F}$ contains a subsequence which converges in $C(\Delta,\Omega^2)$ uniformly on every compact subset of Δ for the compact-open topology of $C(\Delta,\Omega^2)$.

Proof of Theorem A.
We will need the result only for the case where M is hyperbolic. However, the statement holds if M is not hyperbolic. Indeed M is not hyperbolic if and only if one of the following conditions are fulfilled :
(1) M is compact and $b_1(M) \leqslant 2$.
(2) There exists a complex analytic compactification $M \hookrightarrow \widetilde{M}$ such that $\widetilde{M}$ is simply-connected and $\widetilde{M} \setminus M$ contains at most two points.
In this formulation it is clear that the property of not being hyperbolic can not be changed by modifying the complex structure only inside some fixed nonempty relatively compact subset of M. Then $F_{(M,J_t)}$ vanish for all t if $F_{(M,J_0)}$ vanishes (due to uniformization theorem). In this case the Kobayashi-Royden pseudo-metric depends continuously on t, because it is constantly zero.

So, we will proove theorem A for (M, J_0) a hyperbolic Riemann surface.
Let Ω^1, Ω^2 be nonempty connected relatively compact open subsets of M such that $\overline{\Omega} \subset \Omega^1$ and $\overline{\Omega^1} \subset \Omega^2$.
• Let $(x, v) \in TM$ with $x \notin \Omega$ and $t \in \overline{\Delta}$.
Since all the complex structures are equal on $M \setminus \Omega$ then $\forall (x, v) \in TM$ with $x \notin \Omega$ and $\forall t \in \overline{\Delta}$, $F(v, t)$ is continuous in (v, t), because $F(v, t) = F_{(M,J)}(v)$ and the result is given by [10], proposition 3.5.38, page 98.
• Let $(x, v) \in TM$ with $x \in \Omega^1$ and $t \in \overline{\Delta}$.
Let

$$D_j(t) = \sup_{f\in Hol(\Delta,\Omega_t^j)} \|f'(0)\|_h \ , \quad j = 1, 2.$$

According to lemma 5.3, $D_j(t) < +\infty, \forall t \in \overline{\Delta}$ and $\forall j = 1, 2$.
Let's show that for $(x, v) \in TM$ with $x \in \Omega^1$ and $\forall t \in \overline{\Delta}$, $F(v, t) = \frac{\|v\|_h}{D_2(t)}$.
In this case $F(v, t)$ can be defined by

$$F(v, t) = \inf_{f\in Hol(\Delta,\Omega_t^2)} \{\|u\| : u \in T_0\Delta,\ f(0) = x\ et\ f^T_{|T_0\Delta}(u) = v\}.$$

Let $u \in T_0\Delta$, $t \in \overline{\Delta}$, $f \in Hol(\Delta, \Omega_t^2)$ with $f^T(u) = v$ and $\|.\|$ the Poincaré norm on Δ. We have :

$$\|u\| = \|(\|u\|(\frac{\partial}{\partial z}_{|0}))\|$$

and

$$f^T_{|T_0\Delta}(\|u\|(\frac{\partial}{\partial z}_{|0})) = \|u\| f^T_{|T_0\Delta}(\frac{\partial}{\partial z}_{|0})$$

because $f^T_{|T_0\Delta} = d_0 f$ is linear. Then

$$\|d_0f(\|u\|(\frac{\partial}{\partial z}_{|0}))\|_h = \|\|u\|d_0f(\frac{\partial}{\partial z}_{|0})\|_h = \|v\|_h.$$

Thus

$$\|u\|\|d_0f(\frac{\partial}{\partial z}_{|0})\|_h = \|v\|_h \ \text{ with } \ \|d_0f(\frac{\partial}{\partial z}_{|0})\|_h = \|f'(0)\|_h > 0.$$

So

$$\|u\| = \frac{\|v\|_h}{\|f'(0)\|_h}.$$

Hence

$$F(v, t) = \inf_{f\in Hol(\Delta,\Omega_t^2)} \{\|u\| : u \in T_0\Delta, f(0) = x\ et\ f^T_{|T_0\Delta}(u) = v\} = \frac{\|v\|_h}{\sup\limits_{f\in Hol(\Delta,\Omega_t^2)} \|f'(0)\|_h}.$$

Therefore $F(v, t) = \frac{\|v\|_h}{D_2(t)}$.

Then $F(v, t) = F_1(v) \times F_2(t)$ with $F_1(v) = \|v\|_h$ and $F_2(t) = \frac{1}{D_2(t)}$.
Thus $F = (F_1 \circ pr_1) \times (F_2 \circ pr_2)$ where $pr_i,\ i = 1, 2$ are the canonical projections.

The map F_1 and the canonical projections are continuous.

Let's show that D_2 is continuous.
According to Marcus Wright in [8], D_2 is upper semi-continuous.
Let's show the lower semi-continuity of D_2.
Let $c \in \mathbb{R}_+^*$. Assume given a sequence $t_i \to t_0$ with $D_1(t_i) \geq c > 0$. This also give $D_2(t_i) \geq c > 0$.

Hence by definition there is a sequence of maps $f_i \in Hol(\Delta, \Omega^1_{t_i})$ with $\|f_i'(0)\|_h \geq c$.
Applying lemma 5.4, we obtain a sequence of maps $\widetilde{f_i} \in Hol(\Delta, \Omega^1_{t_i})$ such that $\widetilde{f_i}$ fulfills

$$\sup_{z\in\Delta} \|\widetilde{f_i}'(z)\|_h(1-|z|^2) = \|\widetilde{f_i}'(0)\|_h = c.$$

According to remark 5.5, the sequence $(\widetilde{f_i})$ contains a subsequence which uniformly converges on every compact set ; the limit map $\widetilde{f_0} \in Hol(\Delta, \Omega^2_{t_0})$ with $\|\widetilde{f_0'}(0)\|_h = c$ and thus $D_2(t_0) \geq c$.
Since c was arbitrary, then D_2 is lower semi-continuous on $\overline{\Delta}$.
Thus D_2 is continuous. Furthermore $D_2(t) > 0,\ \forall t \in \overline{\Delta}$. Therefore F_2 is continuous.

Finally $F = (F_1 \circ pr_1) \times (F_2 \circ pr_2)$ is continuous.
Which achieves the proof of theorem A. □

Proposition 5.6. *Let (M, J) be a hyperbolic Riemann surface and J_t $(t \in \overline{\Delta})$ a differentiable family of complex structures on M which agree outside a nonempty relatively compact open subset Ω of M such that $J = J_0$. Let γ be a simple curve in M and for each t define $\lambda(t)$ as the infimum of the hyperbolic length (with respect to the complex structure J_t) of simple curves homotopic to γ.*
Then the function $t \longmapsto \lambda(t)$ is continuous.

Proof. We have $\lambda : \overline{\Delta} \longrightarrow \mathbb{R}$ and $\forall t \in \overline{\Delta}, \lambda(t) = \inf\limits_{\xi\in[\gamma]} L_t(\xi)$ where ξ is a simple curve homotopic to γ and $L_t(\xi) = L_{(M,J_t)}(\xi) = \int_0^1 (F_{(M,J_t)})(\xi'(s))\mathrm{d}s$.
According to theorem A, F is continuous on $TM \times \overline{\Delta}$. Then the map $(s,t) \in [0,1] \times \overline{\Delta} \mapsto (F_{(M,J_t)})_{\xi(s)}(\xi'(s))$, where $\xi \in [\gamma]$, is continuous. Thus the map $t \in \overline{\Delta} \mapsto \int_0^1 (F_{(M,J_t)})_{\xi(s)}(\xi'(s))\mathrm{d}s$ is continuous on $\overline{\Delta}$.
So $\forall t_0 \in \overline{\Delta}$ and $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall t \in \Delta_\delta(t_0)$,

$$\Big|\int_0^1 (F_{(M,J_t)})_{\xi(s)}(\xi'(s))\mathrm{d}s - \int_0^1 (F_{(M,J_{t_0})})_{\xi(s)}(\xi'(s))\mathrm{d}s\Big| \leq \epsilon.$$

Then

$$\int_0^1 (F_{(M,J_{t_0})})_{\xi(s)}(\xi'(s))\mathrm{d}s - \epsilon \leq \int_0^1 (F_{(M,J_t)})_{\xi(s)}(\xi'(s))\mathrm{d}s \leq \int_0^1 (F_{(M,J_{t_0})})_{\xi(s)}(\xi'(s))\mathrm{d}s + \epsilon.$$

Hence

$$\int_0^1 (F_{(M,J_t)})_{\xi(s)}(\xi'(s))\mathrm{d}s \leq \int_0^1 (F_{(M,J_{t_0})})_{\xi(s)}(\xi'(s))\mathrm{d}s + \epsilon$$

$$L_t(\xi) \leq L_{t_0}(\xi) + \epsilon.$$

Therefore

$$\inf_{\xi\in[\gamma]} L_t(\xi) \leq \inf_{\xi\in[\gamma]} L_{t_0}(\xi) + \epsilon$$

$$\lambda(t) \leq \lambda(t_0) + \epsilon, \forall t \in \Delta_\delta(t_0).$$

Thus λ is upper semi-continuous.
Similary we show that λ is lower semi-continuous always using the continuity of F, but with the following inequality $\int_0^1 (F_{(M,J_{t_0})})_{\xi(s)}(\xi'(s))\mathrm{d}s - \epsilon \leq \int_0^1 (F_{(M,J_t)})_{\xi(s)}(\xi'(s))\mathrm{d}s$. □

6 PROOF OF THEOREM B.

Proof. Let M be a non elliptic Riemann surface. We distinguish the following cases :
(1) M is parabolic.
(2) M is hyperbolic, but the hyperbolic length spectrum is trivial.
(3) M is hyperbolic and the hyperbolic length spectrum is not trivial.

Let's show that theorem B is fulfilled for case (1).
Since M is parabolic then the result is given in [6] and recalled in section 3 at (i).

Let's show that theorem B is fulfilled for case (2).
For this case, according to proposition 4.9, M is biholomorphic to Δ or to Δ^* or to $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ and the result is given by

proposition 3.1.

Let's show that theorem B is fulfilled for case (3).
In this case $M = (M, J)$ is hyperbolic and there exists a simple curve γ and a constant $a > 0$ such that every simple curve homotopic to γ has a hyperbolic length at least a with respect to the complex structure J of M. Fix such a curve γ and let $\lambda_\gamma(0) = a$ its stable hyperbolic length, *i.e.* the infimum of the hyperbolic length of all simple curves homotopic to γ.
Lemma 4.12 implies that there is a nonempty relatively compact open subset $\Omega \subset M$ and a differentiable family of complex structures J_t $(t \in \overline{\Delta})$ on M such that all these complex structures agree outside Ω, $J = J_0$ and such that the hyperbolic length $L_{(M,J_c)}(\gamma)$ of γ with respect to the complex structure J_c fulfills the inequality $L_{(M,J_c)}(\gamma) < a$ $(c \in \overline{\Delta}, |c| = 1)$. For each $t \in \overline{\Delta}$ we define $\lambda(t)$ as the infimum of the hyperbolic length (with respect to the complex structure J_t) of simple curves homotopic to γ. Now the function $t \longmapsto \lambda(t)$ is continuous due to proposition 5.6 and furthermore non-constant by construction $(\lambda(0) = a > \lambda(c))$.
With the definition of the hyperbolic lenght spectrum given in section 4, $\Sigma_t = \Sigma_{(M,J_t)}$ is a countable subset of $\mathbb{R}^+$ for every $t \in \overline{\Delta}$. Since the function λ is continuous and non-constant, then there exists a parameter $s \in \overline{\Delta}$ such that $\lambda(s) \notin \Sigma_0$. Hence $\Sigma_s \neq \Sigma_0$ and therefore (M, J_s) is not biholomorphic to (M, J_0).
In particular $\Sigma_c \neq \Sigma_0$ and (M, J_c) is not biholomorphic to (M, J_0). □

References

[1] BENNETT CHOW, SUN-CHIN CHU, ... ,LEI NI, *The Ricci Flow: Techniques and Applications*. Part IV : Long-Time Solutions and Related Topics . 2010. American Mathematical Society.

[2] DANIEL HUYBRECHTS,*Complex geometry, an introduction*, Springer, 2000.

[3] EUGENIO CALABI and EDOARDO VESENTINI. *On Compact Locally Symetric Kähler Manifolds*, Annals of Mathematics, 1960.

[4] HENRI PAUL DE SAINT-GERVAIS, *Uniformisation des surfaces de Riemann*, ENS Edition 2010.

[5] JÖRG WINKELMAN, *Deformation of Riemann Surfaces*, Lehrstuhl Analysis II, Mathematisches Institut, NA 4/73, Ruhr-Universität Bochum, 44780 Bochum, Germany, 2014.

[6] KOFFI B. ZOUZOU et TOUSSAINT SOHOU, *Déformation des surfaces de Riemann paraboliques*, Annales Mathématiques Africaines volume 9 (2021) pp. 51-64.

[7] KUNIHIKO KODAIRA, *Complex Manifolds and Deformation of Complex Structures*, Springer, 1986.

[8] MARCUS WRIGHT, *The Kobayashi Pseudometric on Algebraic Manifold of General type and Deformation of Complex Manifolds*, American Mathematical Society. Volume 232, 1977.

[9] ROBERT BRODY, *Compact Manifolds and Hyperbolicity*, American Mathematical Society. Volume 235, January 1978.

[10] SHOSHICHI KOBAYASHI, *Hyperbolic Complex Space*, Springer, 1998.

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Received: 2022-01-18
Accepted: 2022-03-30