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A STUDY OF PELL-NARAYANA SEQUENCE: PROPERTIES AND CIRCULANT MATRIX ASPECTS

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Abstract In this paper, we introduce the Pell-Narayana sequence. We investigate and derive Binet-like-formula, partial sum, generating function related to the Pell-Narayana sequence. We derive some new identities involving the Pell-Narayana sequence. Also, eigenvalues and determinant of circulant matrix involving Pell-Narayana sequence are computed.

1 Introduction

In the last decades, the study of number has been substantially developed having emphasis on Fibonacci and other equally important numbers. With concern to the development of the Fibonacci number, Falcon and Plaza [6] investigated the properties of the k-Fibonacci number via matrix approach. Bolat and Kose [4] derived some new identities involving k-Fibonacci numbers and derived its generating function. On the other hand, Pell sequence (P_n) [9] is defined by the recursive relation

$$P_n = 2P_{n-1} + P_{n-2},$$

with the initial values $P_0 = 0,\ P_1 = 1$. The first values of (P_n) are

$$0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ 5741.$$

The other basic properties of Pell numbers are investigated in [9]. A different approach called matrix algebra has been used to determine the Fibonacci number by Kalman [10] and the Pell numbers by Bicknell [3].
In the 14th century, while studying the following problem of a herd of cows and calves: A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years? (e.g. [1]). This problem can be solved in the same way that Fibonacci solved its problem about rabbits (e.g. [11]).

The Narayana sequence (N_n) [1] is defined by the recursion relation

$$N_{n+3} = N_{n+2} + N_n \quad (\forall n \geq 3,)$$

With initial values $N_0 = 2,\ \ N_1 = 3,\ N_2 = 4$. The first values of (N_n) are

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ \ 189,\ 277.$$

In paper [16], the authors investigated the eigenvalues and determinant of special circulant matrix involving (k,h)-Jacobsthal sequence and (k,h)-Jacobsthal-like sequence. In this paper firstly the Pell-Narayana sequence is introduced. The Binet -like-formula, partial sum, generating function related to this sequence are represented. Some identities with some examples about this sequence are given. Finally the eigenvalues and determinant of circulant matrix involving Pell-Narayana sequence are represented.

For more information about Pell sequence, Pell Lucas sequence, Narayana sequence and some generalizations of these sequences we refer to [5, 7, 8, 12, 14, 15, 16, 18].

2 Pell-Narayana Sequence

Definition 2.1. We define the Pell-Narayana sequence (PN_r) by the recursive relation

$$PN_r = 2PN_{r-1} + PN_{r-3} \tag{2.1}$$

with the initial values $PN_0 = 0,\ PN_1 = 1, PN_2 = 1$. The first values of the Pell-Narayana sequence are

0, 1, 1, 2, 5, 11, 24, 53, 117, 258, 569, 1255, 2768, 6105, 13465, 29698, 65501, 144967.

Remark 2.2. Pell-Narayana sequence (PN_r) has characteristic equation $x^3 - 2x^2 - 1 = 0$. From the Cardano's formula for the cubic equation $x^3 - 2x^2 - 1 = 0$, we can see that this equation has one real root α and two complex roots β,γ where

$$\alpha = \sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \left(\frac{-2}{3}\right) \approx 2.205569904302,$$

$$\begin{aligned}\beta &= \frac{-\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)}{2} + \left(\frac{2}{3}\right) \\ &\quad + \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right) i \\ &\approx -0.102784715 + (0.665469511)\, i\,,\end{aligned}$$

and

$$\begin{aligned}\gamma &= \frac{-\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)}{2} + \left(\frac{2}{3}\right) \\ &\quad - \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right) i \\ &\approx -0.102784715 - (0.665469511)\, i,\end{aligned}$$

where $i = \sqrt{-1}$.

Theorem 2.3. *The generating function for the Pell-Narayana sequence (PN_r) is*

$$\sum_{r=0}^{\infty} PN_r x^r = \frac{x - x^2}{1 - 2x - x^3}.$$

Proof. Suppose that the generating function for the Pell-Narayana sequence (PN_r) has the form

$$g(x) = \sum_{n=0}^{\infty} PN_r x^n = PN_0 + PN_1 x + PN_2 x^2 + PN_3 x^3 + \cdots + PN_r x^r + \cdots \quad .$$

Then, we have

$$2xg(x) = 2xPN_0 + 2PN_1 x^2 + 2PN_2 x^3 + 2PN_3 x^4 + \cdots + 2PN_r x^{r+1} + \cdots \quad ,$$

and

$$x^3 g(x) = PN_0 x^3 + PN_1 x^4 + PN_2 x^5 + PN_3 x^6 + \cdots + PN_r x^{r+3} + \cdots .$$

Thus, we obtain

$g(x) - 2xg(x) - x^3 g(x) = (PN_0 + PN_1 x + PN_2 x^2 + PN_3 x^3 + \cdots) - -(2xPN_0 + 2PN_1 x^2 + 2PN_2 x^3 + 2PN_3 x^4 + \cdots 2PN_r x^{r+1} + \cdots) - -(PN_0 x^3 + PN_1 x^4 + PN_2 x^5 + PN_3 x^6 + \cdots + PN_r x^{r+3} + \cdots)$
$= PN_0 + (PN_1 - 2PN_0)x + (PN_2 - 2PN_1)x^2 + (PN_3 - 2PN_2 - PN_0)x^3 + \cdots + (PN_r - 2PN_{r-1} - PN_{r-3})x^n + \cdots.$

Therefore, we get

$$g(x)\left(1 - 2x - x^3\right) = 0 + x + (1-2)x^2 + 0 = x - x^2.$$

Consequently

$$\sum_{r=0}^{\infty} PN_r x^r = \frac{x - x^2}{1 - 2x - x^3}.$$

Theorem 2.4. *Let $r \geq 0$ be an integer. Then the Binet-like formula for the Pell-Narayana sequence (PN_r) is*

$$PN_r = \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}\beta^r + \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r.$$

where α, β, γ are the roots of the equation $x^3 - 2x^2 - 1 = 0$.

Proof. From Remark 1, we see that the cubic equation $f(x) = x^3 - 2x^2 - 1 = 0$ has three distinct roots α, β, γ. Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots $h(x) = f\left(\frac{1}{x}\right) = 1 - 2x - x^3$. Finally, we have

$$h(x) = 1 - 2x - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x).$$

According to the generating function of Pell-Narayana sequence, we have

$$\begin{aligned} g(x) &= \frac{x - x^2}{1 - 2x - x^3} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} + \frac{C}{1-\gamma x} \\ &= A\sum_{r=0}^{\infty}(\alpha x)^r + B\sum_{r=0}^{\infty}(\beta x)^r + C\sum_{r=0}^{\infty}(\gamma x)^r. \end{aligned} \tag{2.2}$$

Thus, we have

$$g(x) = \frac{x - x^2}{1 - 2x - x^3} = \frac{A(1-\beta x)(1-\gamma x) + B(1-\alpha x)(1-\gamma x) + C(1-\alpha x)(1-\beta x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Therefore, by comparison of the left and right sides of this equality, we get that

$$x - x^2 = A(1-\beta x)(1-\gamma x) + B(1-\alpha x)(1-\gamma x) + C(1-\alpha x)(1-\beta x).$$

If we set $x = \frac{1}{\alpha}$, we find that

$$\frac{1}{\alpha} - \frac{1}{\alpha^2} = A\left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\gamma}{\alpha}\right).$$

Consequently, we get

$$A = \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we get

$$B = \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}, \quad C = \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}.$$

By equation (2.2) we obtain that

$$g(x) = \sum_{r=0}^{\infty} \frac{(\alpha-1)\,\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} x^r + \sum_{r=0}^{\infty} \frac{(\beta-1)\,\beta^r}{(\beta-\alpha)(\beta-\gamma)} x^r + \sum_{r=0}^{\infty} \frac{(\gamma-1)\,\gamma^r}{(\gamma-\alpha)(\gamma-\beta)} x^r$$

$$= \sum_{r=0}^{\infty} \left[\frac{(\alpha-1)\,\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\,\beta^r}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\,\gamma^r}{(\gamma-\alpha)(\gamma-\beta)} \right] x^r.$$

Consequently, we obtain

$$PN_r = \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^r + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)} \beta^r + \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^r.$$

Theorem 2.5. *Let* $r \geq 0$ *be an integer. Then*

(a) $$PN_{r+1} + PN_r = \frac{(\alpha^2-1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^r + \frac{(\beta^2-1)}{(\beta-\alpha)(\beta-\gamma)} \beta^r + \frac{(\gamma^2-1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^r,$$

(b) $$PN_{r+1} - PN_r = \frac{(\alpha-1)^2}{(\alpha-\beta)(\alpha-\gamma)} \alpha^r + \frac{(\beta-1)^2}{(\beta-\alpha)(\beta-\gamma)} \beta^r + \frac{(\gamma-1)^2}{(\gamma-\alpha)(\gamma-\beta)} \gamma^r.$$

Proof. They can be proved by direct calculations from Theorem 2.2.

Theorem 2.6. *Let* $r \geq 0$ *be an integer and* k *be an arbitrary integer. Then*

(a) $$PN_{r+k} + PN_{r-k} = \frac{(\alpha-1)(\alpha^{2k}+1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^{r-k} + \frac{(\beta-1)(\beta^{2k}+1)}{(\beta-\alpha)(\beta-\gamma)} \beta^{r-k} + \frac{(\gamma-1)(\gamma^{2k}+1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^{r-k},$$

(b) $$PN_{r+k} - PN_{r-k} = \frac{(\alpha-1)\left(\alpha^{2k}-1\right)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^{r-k} + \frac{(\beta-1)\left(\beta^{2k}-1\right)}{(\beta-\alpha)(\beta-\gamma)} \beta^{r-k} + \frac{(\gamma-1)\left(\gamma^{2k}-1\right)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^{r-k}.$$

Proof. They can be proved by direct calculations from Theorem 2.2.

Corollary 2.7. *From Theorem 2.4 for* $k = 1$*, we have*

(a) $$PN_{r+1} + PN_{r-1} = \frac{(\alpha-1)(\alpha^2+1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^{r-1} + \frac{(\beta-1)(\beta^2+1)}{(\beta-\alpha)(\beta-\gamma)} \beta^{r-1} + \frac{(\gamma-1)(\gamma^2+1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^{r-1},$$

(b) $$PN_{r+1} - PN_{r-1} = \frac{(\alpha-1)^2(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^{r-1} + \frac{(\beta-1)^2(\beta+1)}{(\beta-\alpha)(\beta-\gamma)} \beta^{r-1} + \frac{(\gamma-1)^2(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^{r-1}.$$

Lemma 2.8. *Let* $r \geq 0$ *be an integer. Then*

$$\sum_{r=0}^{n} PN_r = \frac{1}{2} \left[PN_{r+3} - PN_{r+1} - PN_{r+2} + 1 \right].$$

Proof. From the definition of Pell-Narayana sequence, we know that

$$PN_{r-3} = PN_r - 2PN_{r-1}.$$

Thus, we have

$$PN_0 = PN_3 - 2PN_2 = PN_3 - PN_2 - PN_2,$$

$$PN_1 = PN_4 - PN_3 - PN_3, \quad PN_2 = PN_5 - PN_4 - PN_4\ ,$$

$$\vdots$$

$$PN_{r-2} = PN_{r+1} - PN_r - PN_r, \quad PN_{r-1} = PN_{r+2} - PN_{r+1} - PN_{r+1},$$

$$PN_r = PN_{r+3} - PN_{r+2} - PN_{r+2}.$$

Therefore, we get

$$\begin{aligned}\sum_{r=0}^{n} PN_r &= -PN_2 - PN_3 - PN_4 - PN_5 - \cdots - PN_r - PN_{r+1} - PN_{r+2} + PN_{r+3} \\ &= -\sum_{r=0}^{n} PN_r + PN_0 + PN_1 - PN_r - PN_{r+1} - PN_{r+2} + PN_{r+3} \\ &= -\sum_{r=0}^{n} PN_r + 1 - PN_{r+1} - PN_{r+2} + PN_{r+3}.\end{aligned}$$

Thus

$$2\sum_{r=0}^{n} PN_r = PN_{r+3} - PN_{r+1} - PN_{r+2} + 1.$$

Consequently, we get

$$\sum_{r=0}^{n} PN_r = \frac{1}{2}\left[PN_{r+3} - PN_{r+1} - PN_{r+2} + 1\right].$$

3 More identities involving Pell-Narayana sequence

Theorem 3.1. *Let* $n \geq 0$ *be an integer and* $\varphi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. *Then*

$$\begin{bmatrix} PN_n \\ PN_{n+1} \\ PN_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \varphi^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Proof. We prove this theorem by mathematical induction on n.
For $n = 1$, we have

$$\varphi^1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} PN_1 \\ PN_2 \\ PN_3 \end{bmatrix}.$$

Thus, the result is true for $n = 1$. Now, suppose that the result is true for $n = k$. Hence we have

$$\begin{bmatrix} PN_k \\ PN_{k+1} \\ PN_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \varphi^k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then, we have

$$\varphi^{k+1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \varphi\varphi^k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \varphi \begin{bmatrix} PN_k \\ PN_{k+1} \\ PN_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} PN_k \\ PN_{k+1} \\ PN_{k+2} \end{bmatrix} = \begin{bmatrix} PN_{k+1} \\ PN_{k+2} \\ PN_{k+3} \end{bmatrix}.$$

Therefore, the result is true for $n = k$. Consequently, by the induction, the result is true for every n. This proves the theorem.

Remark 3.2. As we know the characteristic polynomial of the recursive relation $PN_r = 2PN_{r-1} + PN_{r-3}$ is $p(x) = x^3 - 2x^2 - 1 = 0$. This polynomial can be written as $p(x) = \det(xI - \varphi) = 0$, where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\varphi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.
From the well-known Cayley Hamilton theorem in matrix algebra, we have $p(\varphi) = 0$. Thus, we have

$$\varphi^3 - 2\varphi^2 - I = 0. \tag{3.1}$$

Theorem 3.3. *Let* $= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, *then*

$$I = \varphi^3 - 2\varphi^2 = \varphi^3 - 2\varphi^5 + 4\varphi^4,$$

and

$$\varphi^n = \varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4}.$$

Proof. According to the Remark 2, we have

$$\begin{aligned} I &= \varphi^3 - 2\varphi^2 = \varphi^2(\varphi - 2I) \\ &= \varphi^2\left(\varphi - 2\left(\varphi^3 - 2\varphi^2\right)\right) = \varphi^2\left(\varphi - 2\varphi^3 + 4\varphi^2\right) = \varphi^3 - 2\varphi^5 + 4\varphi^4. \end{aligned}$$

Thus

$$I = \varphi^3 - 2\varphi^5 + 4\varphi^4.$$

This proves the first equality. Multiplying both sides of the above equality by φ^n, we obtain

$$\varphi^n = \varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4}. \tag{3.2}$$

Thus, the proof is completed.

Corollary 3.4. *Let* $r \geq 0$ *be an integer. Then*

$$\varphi^{n+5} = \frac{1}{2}\left[4\varphi^{n+4} + \varphi^{n+3} - \varphi^n\right]. \tag{3.3}$$

According to this corollary, we have the following interesting example and Theorem about the Pell-Narayana sequence (PN_r).

Example 3.5. From the first values of Pell-Narayana sequence (PN_r), we have

$$24 = \frac{1}{2}\left[4 \times 11 + 5 - 1\right].$$

In exact, we have

$$PN_{1+5} = \frac{1}{2}\left[4PN_5 + PN_4 - PN_1\right] = \frac{1}{2}\left[4PN_{1+4} + PN_{1+3} - PN_1\right].$$

Therefore, by induction, we have the following identity about the Pell-Narayana sequence (PN_r).

Theorem 3.6. *Let* $r \geq 0$ *be an integer. Then*

$$PN_{r+5} = \frac{1}{2}\left[4PN_{r+4} + PN_{r+3} - PN_r\right].$$

Proof. We prove this theorem by mathematical induction on n. According to the last example, we see that

$$PN_{1+5} = \frac{1}{2}\left[4PN_{1+4} + PN_{1+3} - PN_1\right].$$

Then if we assume that $PN_{t+5} = \frac{1}{2}\left[4PN_{t+4} + PN_{t+3} - PN_r\right]$ for all $t < n$. Then we have

$$\begin{aligned} PN_{n+5} &= 2PN_{n+4} + PN_{n+2} \\ &= 2 \times \frac{1}{2}\left[4PN_{n+3} + PN_{n+2} - PN_{n-1}\right] + \frac{1}{2}\left[4PN_{n+1} + PN_n - PN_{n-3}\right] \\ &= 4PN_{n+3} + PN_{n+2} - PN_{n-1} + 2PN_{n+1} + \frac{1}{2}PN_n - \frac{1}{2}PN_{n-3} \\ &= \frac{1}{2}\left(4\left[2PN_{n+3} + PN_{n+1}\right]\right) + \frac{1}{2}\left(2PN_{n+2} + PN_n\right) - \frac{1}{2}\left(2PN_{n-1} + PN_{n-3}\right) \\ &= \frac{1}{2}\left[4PN_{n+4} + PN_{n+3} - PN_n\right]. \end{aligned}$$

Thus, the result is true for all n.

Theorem 3.7. *Let $r, n \geq 0$ be integer. Then*

$$\varphi^{n+r} = \varphi^{n+r+6} + 8\varphi^{n+r+7} + 12\varphi^{n+r+8} - 16\varphi^{n+r+9} + 4\varphi^{n+r+10}.$$

Proof. By Theorem 3.2 , we have $\varphi^n = \varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4}$. Hence

$$\begin{aligned} \varphi^{n+r} &= \varphi^n\varphi^r = \left(\varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4}\right)\left(\varphi^{r+3} - 2\varphi^{r+5} + 4\varphi^{r+4}\right) \\ &= \varphi^{n+r+6} - 2\varphi^{n+r+8} + 4\varphi^{n+r+7} - 2\varphi^{n+r+8} + 4\varphi^{n+r+10} - 8\varphi^{n+r+9} \\ &\quad + 4\varphi^{n+r+7} - 8\varphi^{n+r+9} + 16\varphi^{n+r+8} \\ &= \varphi^{n+r+6} + 8\varphi^{n+r+7} + 12\varphi^{n+r+8} - 16\varphi^{n+r+9} + 4\varphi^{n+r+10}. \end{aligned}$$

Thus, the proof is completed.

Corollary 3.8. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} \textbf{\textit{(a)}}\ \varphi^{2n} &= \varphi^{2n+6} + 8\varphi^{2n+7} + 12\varphi^{2n+8} - 16\varphi^{2n+9} + 4\varphi^{2n+10}, \\ \textbf{\textit{(b)}}\ \varphi^{2n+9} &= \frac{1}{16}\left(\varphi^{2n+6} + 8\varphi^{2n+7} + 12\varphi^{2n+8} + 4\varphi^{2n+10} - \varphi^{2n}\right). \end{aligned}$$

Proof. On setting $r = n$ in Theorem 3.4, we get the desired results.

Example 3.9. From the first values of Pell-Narayana sequence (PN_r) we have

$$1255 = \frac{1}{16}\ \left[117 + 8\,(258) + 12\,(569) + 4\,(2768) - 1\right].$$

Exactly, we have

$$PN_{11} = \frac{1}{16}\ \left[PN_8 + 8PN_9 + 12PN_{10} + 4PN_{12} - PN_2\right],$$

Or, equivalently, we have

$$PN_{2\times1+9} = \frac{1}{16}\left[PN_{2\times1+6} + 8PN_{2\times1+7} + 12PN_{2\times1+8} + 4PN_{2\times1+10} - PN_{2\times1}\right].$$

Theorem 3.10. *Let $r \geq 0$ be an integer. Then*

$$PN_{2r+9} = \frac{1}{16}\left[PN_{2r+6} + 8PN_{2r+7} + 12PN_{2r+8} + 4PN_{2r+10} - PN_{2r}\right].$$

Proof. Approaching the proof of the above result in similar manner of Theorem 3.3, we get the desired result. We omit the details.

4 Circulant matrix involving Pell-Narayana sequence

Definition 4.1. A matrix $C = [c_{i,j}] \in M_{n\times n}$ is called a Circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

A circulant matrix $C = [c_{i,j}]$ can be written by $C = Circ\,(c_0, c_1, \cdots, c_{n-1})$.

Lemma 4.2. *Let $C = Circ\,(c_0, c_1, \cdots, c_{n-1})$ be a $n \times n$ circulant matrix. Then we have*

$$\rho_j\,(C) = \sum_{k=0}^{n-1} c_k w^{-jk},$$

where ρ_j for $j = 0, 1, 2, \cdots, n-1$ is the eigenvalue of the circulant matrix C and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ (see [5]).

Remark 4.3. Since α, β, γ are the roots of the cubic equation $x^3 - 2x^2 - 1 = 0$. By direct calculation we can prove that

(a) $\alpha + \beta + \gamma = 2,$

(b) $\alpha\beta\gamma = 1,$

(c) $\alpha\beta + \alpha\gamma + \beta\gamma = 0.$

Lemma 4.4. *Suppose that α, β, γ are the roots of the equation $x^3 - 2x^2 - 1 = 0$ and*

$$k_1 = \frac{\alpha - 1}{(\alpha - \beta)(\alpha - \gamma)}\ , k_2 = \frac{\beta - 1}{(\beta - \alpha)(\beta - \gamma)}\ ,\ k_3 = \frac{\gamma - 1}{(\gamma - \alpha)(\gamma - \beta)}\ ,$$

Then we have

(a) $k_1 + k_2 + k_3 = 0,$

(b) $k_1 + k_2 = \dfrac{1-\gamma}{(\alpha-\gamma)(\beta-\gamma)}\ ,\ k_1 + k_3 = \dfrac{1-\beta}{(\alpha-\beta)(\gamma-\beta)}\ ,\ k_2 + k_3 = \dfrac{1-\alpha}{(\beta-\alpha)(\gamma-\alpha)},$

(c) $(k_1 + k_3)\beta + (k_2 + k_3)\,\alpha + (k_1 + k_2)\,\gamma = -1,$

(d) $\dfrac{k_1}{\alpha} + \dfrac{k_2}{\beta} + \dfrac{k_3}{\gamma} = -1.$

Proof. They can be proved directly by some computations according to the definition of $k_1, k_2,\ k_3$ and properties of α, β, γ.

Theorem 4.5. *Let $C = Cir\,(PN_0, PN_1, \cdots, PN_{n-1})$ be a $n \times n$ circulant matrix whose entries are the Pell-Narayana sequence (PN_n). Then the eigenvalues of C are*

$$\rho_j\,(\mathrm{C}) = \frac{(PN_{n-1} + 1)\,w^{-2j} + (PN_{n-2} - 1)\,w^{-j} + PN_n}{w^{-3j} + w^{-2j} - 1},\ (\text{ for } j = 0, 1, 2, \cdots, n-1)$$

where $i = \sqrt{-1}$, and $w = e^{\frac{2\pi i}{n}}$.

Proof. By lemma 4.1, for the eigenvalues of circulant matrix $C = Cir\,(PN_0, PN_1, \cdots, PN_{n-1})$

we have

$$\begin{aligned}\rho_j(C) &= \sum_{k=0}^{n-1} PN_k w^{-jk}\\ &= \sum_{k=0}^{n-1}\left[\frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{k}+\frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}\beta^k+\frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^k\right]w^{-jk}\\ &= \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}\sum_{k=0}^{n-1}\alpha^k w^{-jk}+\frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}\sum_{k=0}^{n-1}\beta^k w^{-jk}\\ &+\frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}\sum_{k=0}^{n-1}\gamma^k w^{-jk},\end{aligned}$$

Therefore, by taking $k_1=\frac{\alpha-1}{(\alpha-\beta)(\alpha-\gamma)}$, $k_2=\frac{\beta-1}{(\beta-\alpha)(\beta-\gamma)}$, $k_3=\frac{\gamma-1}{(\gamma-\alpha)(\gamma-\beta)}$, we have

$$\begin{aligned}\rho_j(\mathrm{C}) &= k_1\left(\frac{(\alpha w^{-j})^n-1}{\alpha w^{-j}-1}\right)+k_2\left(\frac{(\beta w^{-j})^n-1}{\beta w^{-j}-1}\right)+k_3\left(\frac{(\gamma w^{-j})^n-1}{\gamma w^{-j}-1}\right)\\ &= k_1\left(\frac{\alpha^n-1}{\alpha w^{-j}-1}\right)+k_2\left(\frac{\beta^n-1}{\beta w^{-j}-1}\right)+k_3\left(\frac{\gamma^n-1}{\gamma w^{-j}-1}\right)\\ &= \frac{k_1(\alpha^n-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)+k_2(\beta^n-1)(\alpha w^{-j}-1)(\gamma w^{-j}-1)}{(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)}\\ &+\frac{k_3(\gamma^n-1)(\alpha w^{-j}-1)(\beta w^{-j}-1)}{(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)}.\end{aligned}$$

Therefore, we get

$$\rho_j(\mathrm{C}) = \frac{-(k_1+k_2+k_3)+(k_1\alpha^n+k_2\beta^n+k_3\gamma^n)+(k_1\alpha^n\beta\gamma+k_2\beta^n\alpha\,\gamma+k_3\gamma^n\alpha\beta)\,w^{-2j}}{(\alpha\beta\gamma)\,w^{-3j}-(\alpha\beta+\alpha\gamma+\beta\gamma)\,w^{-2j}+(\alpha+\beta+\gamma)w^{-j}-1}$$

$$\begin{aligned}&+\frac{-(k_1\alpha^n\beta+k_2\beta^n\alpha+k_3\gamma^n\alpha)\,w^{-j}-(k_1\alpha^n\gamma+k_2\beta^n\gamma+k_3\gamma^n\beta)\,w^{-j}}{(\alpha\beta\gamma)\,w^{-3j}-(\alpha\beta+\alpha\gamma+\beta\gamma)\,w^{-2j}+(\alpha+\beta+\gamma)w^{-j}-1}\\ &+\frac{-(k_1\beta\gamma+k_2\alpha\gamma+k_3\alpha\beta)\,w^{-2j}+(k_1\beta+k_2\alpha+k_3\alpha+k_1\gamma+k_2\gamma+k_3\beta)w^{-j}}{(\alpha\beta\gamma)\,w^{-3j}-(\alpha\beta+\alpha\gamma+\beta\gamma)\,w^{-2j}+(\alpha+\beta+\gamma)w^{-j}-1}.\end{aligned}$$

According to the Remark 3 and Lemma 4.2, after some computations, we get

$$\begin{aligned}\rho_j(\mathrm{C}) &= \frac{(PN_{n-1}+1)\,w^{-2j}+(PN_{n+1}-2PN_n-1)\,w^{-j}+PN_n}{w^{-3j}+w^{-2j}-1}\\ &= \frac{(PN_{n-1}+1)\,w^{-2j}+(PN_{n-2}-1)\,w^{-j}+PN_n}{w^{-3j}+w^{-2j}-1}.\end{aligned}$$

Thus the proof is completed.

Example 4.6. The following table represents the eigenvalues of $C = Cir(PN_0,\ PN_1,\ \cdots,\ PN_{n-1})$ for some values of n.

n	Eigenvalues of C=$Cir(PN_0,\ PN_1, \cdots, PN_{n-1})$
2	-1 1
3	4 -2 $-1+i$, $-1-i$
4	9 $0.572949-4.39201i$ $0.572949+4.39201i$ $-3.92705+1.40008i$ $-3.92705-1.40008i$
5	20 $1+12.1244i$ $1-12.1244i$ $-7+5.19615i$ $-7-5.19615i$ -8
6	44 $6.61021+29.0331i$ $6.61021-29.0331i$ $-12.0102+15.739i$ $-12.0102-15.739i$ $16.6+5.08579i$ $16.6-5.08579i$
7	97 $23.9914+66.1335i$ $23.9914-66.1335i$ $-20+43i$ $-20-43i$ $-33.9914+20.1335i$ $-33.9914-20.1335i$ -37

Lemma 4.7. *Let* x, y, z *be real numbers and* $n>0$ *be an integer. Then*

$$\prod_{k=0}^{n-1}\left(x-yw^{-k}+zw^{-2k}\right) = x^n\left(1-\left(\frac{y-\sqrt{y^2-4xz}}{2x}\right)^n-\left(\frac{y+\sqrt{y^2-4xz}}{2x}\right)^n+\left(\frac{z}{x}\right)^n\right)$$
$$= x^n+z^n-\left[\left(\frac{y-\sqrt{y^2-4xz}}{2}\right)^n+\left(\frac{y+\sqrt{y^2-4xz}}{2}\right)^n\right],$$

where $w=e^{\frac{2\pi i}{n}}$.

Proof. One can derive the above result in similar manner as given in [2]. We omit the details.

Lemma 4.8. *Let* $n>0$ *be an integer. Then*

$$\prod_{j=0}^{n-1}\left(w^{-3j}+w^{-2j}-1\right)=(-1)^n\left(-Q_{-n}-Q_n\right),$$

where (Q_n) *is the Perrin sequence that is defined by the recursive relation* $Q_{n+3}=Q_{n+1}+Q_n$ *with initial values* $Q_0=3, Q_1=0, Q_2=2$ *and sequence* (Q_{-n}) *is defined by recursive relation* $Q_{-n}=Q_{-(n-1)}+Q_{-(n-3)}$.

Proof. One can derive the above result in similar manner as given in [2]. We omit the details.

Theorem 4.9. *Let $C = Cir\left(PN_0, PN_1, \cdots, PN_{n-1}\right)$ be a $n \times n$ circulant matrix whose entries are the Pell-Narayana sequence(PN_n). Then determinant of C is*

$$\begin{aligned}\det(C) &= \left(PN_n{}^r + (PN_{n-1}+1)^r\right.\\ &\quad - \left[\left(\frac{(PN_{n-2}-1) - \sqrt{(PN_{n-2}-1)^2 - 4PN_n\left(PN_{n-1}+1\right)}}{2}\right)^r\right.\\ &\quad \left.\left. + \left(\frac{(PN_{n-2}-1) + \sqrt{(PN_{n-2}-1)^2 - 4PN_n\left(PN_{n-1}+1\right)}}{2}\right)^r\right]\right)\\ &\quad \times \left(\frac{1}{(-1)^n(Q_{-n}-Q_n)}\right).\end{aligned}$$

Proof. Let $\rho_0, \rho_1, \cdots, \rho_{r-1}$ are the eigenvalues of circulant matrix C. From a basic theorem in matrix algebra about the determinant of a matrix, we have

$$\det(C) = \prod_{j=0}^{n-1} \rho_j$$

Therefore by Theorem 4.1, we get

$$\begin{aligned}\det(C) &= \prod_{j=0}^{n-1} \rho_j\\ &= \prod_{j=0}^{n-1} \frac{\left(PN_{n-1}+1\right)w^{-2j} + \left(PN_{n-2}-1\right)w^{-j} + PN_n}{w^{-3j}+w^{-2j}-1}\\ &= \prod_{j=0}^{n-1}\left[\left(PN_{n-1}+1\right)w^{-2j} + \left(PN_{n-2}-1\right)w^{-j} + PN_n\right] \times \left(\frac{1}{\prod_{j=0}^{r-1}\left(w^{-3j}+w^{-2j}-1\right)}\right).\end{aligned}$$

Therefore by Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned}\det(C) &= \left(PN_n{}^r + (PN_{n-1}+1)^r\right.\\ &\quad - \left[\left(\frac{(PN_{n-2}-1) - \sqrt{(PN_{n-2}-1)^2 - 4PN_n\left(PN_{n-1}+1\right)}}{2}\right)^r\right.\\ &\quad \left.\left. + \left(\frac{(PN_{n-2}-1) + \sqrt{(PN_{n-2}-1)^2 - 4PN_n\left(PN_{n-1}+1\right)}}{2}\right)^r\right]\right)\\ &\quad \times \left(\frac{1}{(-1)^n(Q_{-n}-Q_n)}\right).\end{aligned}$$

Example 4.10. The following table shows the determinant of $C = Cir\left(PN_0, PN_1, \cdots, PN_{n-1}\right)$ for some values of n.

n	**Determinant of** C=$Cir(PN_0, PN_1, \cdots, PN_{n-1})$
2	-2
3	2
4	-16
5	3069
6	-1799680
7	4609034012
8	-62350489778837

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