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SOME NEW RESULTS ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH q-DIFFERENTIAL OPERATOR AND SUBORDINATION

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Abstract In this work, we define a new subclass of analytic functions and investigate its geometric properties. The new class which is denoted by $\mathcal{R}_q(\vartheta, g)$ consists of analytic functions defined by means of a q-differential operator and subordination. Indeed, some of the investigated properties include: characterization, q-integral representation, coefficient estimate, q-differential subordination and Fekete-Szegö estimates associated with the mth-root transform of functions in $\mathcal{R}_q(\vartheta, g)$. Our results however, generalize many known and new ones.

1 Introduction

By the usual notation, let $\mathcal{A}$ denote the set of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad f(0) = f'(0) - 1 = 0 \tag{1.1}$$

such that $z \in \mathcal{E} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ be a subset of $\mathcal{A}$, such that $\mathcal{S}$ consists of functions that are both analytic and univalent in $\mathcal{E}$. Also, a function $f \in \mathcal{S}$ is a member of the class $\mathcal{C}$ of convex functions if it satisfies the condition $\Re\{z[f''(z)/f'(z)] + 1\} > 0$, for $z \in \mathcal{E}$.

In [9], the Hadamard product (or convolution) of two analytic functions:

$$f \quad \text{and} \quad \mathcal{F}(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \tag{1.2}$$

usually symbolized as $(f \star \mathcal{F})(z)$ is defined by

$$(f \star \mathcal{F})(z) = z + \sum_{k=2}^{\infty} a_k \alpha_k z^k = (\mathcal{F} \star f)(z) \in \mathcal{A}. \tag{1.3}$$

Recall that the class

$$\mathcal{W} := \left\{ b(z) = \sum_{k=1}^{\infty} b_k z^k : \ |b(z)| < 1, \ b(0) = 0, \ z \in \mathcal{E} \right\} \tag{1.4}$$

where $b(z)$ is analytic in $\mathcal{E}$, is known as the class of Schwarz functions. So in view of (1.2), f is subordinate to $\mathcal{F}$, symbolically expressed as $f \prec \mathcal{F}$, if there is a Schwarz function b (not necessarily univalent) defined in $\mathcal{E}$ such that

$$f(z) = \mathcal{F}(b(z)) \quad (z \in \mathcal{E}). \tag{1.5}$$

If $\mathcal{F}$ is univalent in $\mathcal{E}$, then

$$f(z) \prec \mathcal{F}(z) \iff f(0) = \mathcal{F}(0) \text{ and } f(\mathcal{E}) \subset \mathcal{F}(\mathcal{E}).$$

Jackson [11, 12] (see also [1, 3, 4, 14, 16, 17]) introduced the idea of q-calculus. For $q \in (0,1)$, the q-derivative of $h(z)$ is defined by

$$\left.\begin{array}{l} \mathcal{D}_q h(z) = \begin{cases} h'(0) & for \quad z = 0 \quad (\text{if it exists}) \\ \frac{h(z)-h(qz)}{(1-q)z} & for \quad z \neq 0 \end{cases} \\ and \\ \mathcal{D}_q^2 h(z) = \mathcal{D}_q(\mathcal{D}_q h(z)) \end{array}\right\} \tag{1.6}$$

so that using (1.6) in (1.1) gives

$$\mathcal{D}_q f(z) = 1 + \sum_{k=2}^{\infty}[k]_q a_k z^{k-1} \quad \text{and} \quad \mathcal{D}_q^2 f(z) = \sum_{k=2}^{\infty}[k]_q[k-1]_q a_k z^{k-2} \tag{1.7}$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $\lim\limits_{q\to 1^-}[k]_q = k$. Let $h(z)$ be integrable in the domain $\Xi \subset \mathbb{C}$, then for $q \in (0,1)$, the definite q-integral of $h(z)$ is defined by

$$\int_0^z h(t)d_q t = z(1-q)\sum_{k=0}^{\infty} h(zq^k)q^k \tag{1.8}$$

provided the series converges in Ξ. For example, if $h(z) = z^k$, then using (1.8) gives

$$\int_0^z t^k d_q t = z(1-q)\sum_{k=0}^{\infty}(zq^k)^k q^k = \frac{z^{k+1}}{[k+1]_q}, \tag{1.9}$$

$k \neq -1$, $[k+1]_q = \frac{1-q^{k+1}}{1-q}$ and observe that

$$\lim_{q\to 1^-}\int_0^z t^k d_q t = \lim_{q\to 1^-}\left(\frac{z^{k+1}}{[k+1]_q}\right) = \frac{z^{k+1}}{k+1} = \int_0^z t^k dt$$

where $\int_0^z t^k dt$ is the classical definite integral of $h(z) = z^k$. See [3, 4, 13, 14] for more details.

Remark 1.1 ([3, 4, 14]). Let $q \in (0,1)$, then the following q-series expansions hold.

(i) q-Binomial expansion:

$$\left.\begin{array}{l} (z-\kappa)_q^n = (z-\kappa)(z-q\kappa)(z-q^2\kappa)\cdots(z-q^{n-1}\kappa) \quad (\kappa, n \in \mathbb{R}), \\ (\kappa-z)_q^n = (-1)^n q^{n(n-1)/2}(z-\kappa q^{-n+1})_q^n \quad (\kappa, n \in \mathbb{R}), \end{array}\right\} \tag{1.10}$$

(ii) q-Taylor's expansion:

$$\frac{1}{(1-z)_q^n} = \begin{cases} 1 + \sum\limits_{k=1}^{\infty}\frac{[n]_q[n+1]_q[n+2]_q\cdots[n+k-1]_q}{[k]_q!}z^k & for \quad n \in \mathbb{N}, \\ 1 & for \quad n = 0, \end{cases} \tag{1.11}$$

(iii) q-Derivative:

$$\mathcal{D}_q(\kappa-z)_q^n = -[n]_q(\kappa-qz)_q^{n-1} \qquad (\kappa, n \in \mathbb{R}). \tag{1.12}$$

Remark 1.2. The following convolution properties can easily be verified by using (1.1), (1.3), (1.7) and (1.11).

(i) $$f(z) \star \frac{z}{(1-z)_q^2} = z\mathcal{D}_q f(z). \tag{1.13}$$

(ii) $$z\mathcal{D}_q(f_1(z) \star f_2(z)) = z\mathcal{D}_q f_1(z) \star f_2(z) = f_1(z) \star z\mathcal{D}_q f_2(z). \tag{1.14}$$

(iii) $$f_1(z) \star \kappa f_2(z) = \kappa(f_1(z) \star f_2(z)) \quad (\kappa \text{ is constant}). \tag{1.15}$$

(iv) $$f_1(z) \star (f_2(z) + f_3(z)) = f_1(z) \star f_2(z) + f_1(z) \star f_3(z). \tag{1.16}$$

2 Relevant Lemmas

Let $\mathcal{P}$ denote the well-known class of analytic functions of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \ (\Re p(z) > 0,\ z \in \mathcal{E}) \tag{2.1}$$

and let $\mathcal{P}(\alpha) \subseteq \mathcal{P}(0) =: \mathcal{P}$ denote the class of analytic functions of the form

$$p_\alpha(z) = 1 + (1-\alpha)p_1 z + (1-\alpha)p_2 z^2 + \cdots \ (\Re p(z) > \alpha,\ \alpha \in [0,1),\ z \in \mathcal{E}). \tag{2.2}$$

The following lemmas shall be needed to proof our theorems.

Lemma 2.1 ([9]). *Let $p \in \mathcal{P}$, then $|p_k| \leqslant 2,\ \forall k \in \mathbb{N}$. The inequality is sharp for the function $p_0(z) = (1+z)/(1-z)$.*

Lemma 2.2 ([9]). *Let $b \in \mathcal{W}$, then $|b_k| \leqslant 1,\ \forall k \in \mathbb{N}$. The inequality is sharp for the function $b(z) = e^{i\vartheta} z^k,\ \vartheta \in [0, 2\pi)$.*

Lemma 2.3 ([23]). *Let $p \in \mathcal{P}$, then $\left|p_2 - v{p_1}^2\right| \leqslant 2\max\left\{1, \left|2v-1\right|\right\},\ v \in \mathbb{C}$. The inequality is sharp for the function*

$$p(z) = \begin{cases} \frac{1+z^2}{1-z^2} & for \quad v \in [0,2], \\ \frac{1+z}{1-z} & for \quad v \in (-\infty, 0] \cup [2, \infty). \end{cases}$$

The next lemma is the q-analogous version of the cited work.

Lemma 2.4 ([10, 23]). *Let $w(z)$ be a convex function such that $w(0) = a$ and let $c \in \mathbb{C} \setminus \{0\}$ with $\Re c \geqslant 0$. If the function $u(z) = a + u_k z^k + u_{k+1} z^{k+1} + \cdots$ is analytic in $\mathcal{E}$ and*

$$u(z) + \frac{1}{c} z \mathcal{D}_q u(z) \prec w(z), \quad then \quad u(z) \prec v(z) \prec w(z) \quad (z \in \mathcal{E})$$

where

$$v(z) = \frac{c}{k z^{c/k}} \int_0^z w(t) t^{(c/k)-1} d_q t.$$

3 Main Results

The investigated class is defined as follows.

Definition 3.1. Let $\mathcal{G}$ be a class consisting of analytic functions of the form

$$g(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots \quad (z \in \mathcal{E}) \tag{3.1}$$

normalized by the condition $g(0) = 1$. Let $\vartheta \in (-\pi, \pi]$, $g \in \mathcal{G}$ and $\mathcal{D}_q f(z)$ be as defined in (1.7), then a function $f \in \mathcal{A}$ is a member of the class $\mathcal{R}_q(\vartheta, g)$ if it satisfies the q-differential subordination

$$\mathcal{D}_q f(z) + \frac{1 + e^{i\vartheta}}{2} z \mathcal{D}_q^2 f(z) \prec g(z) \quad (z \in \mathcal{E}). \tag{3.2}$$

Remark 3.2. Some special cases of condition (3.2) that are considered in some of our corollaries and remarks are presented as follows.

(i) A function $f \in \mathcal{A}$ is a member of the subclass $\mathcal{R}_q(0, g)$ of $\mathcal{R}_q(\vartheta, g)$ if it satisfies the q-differential subordination

$$\mathcal{D}_q f(z) + z \mathcal{D}_q^2 f(z) \prec g(z) \quad (z \in \mathcal{E}). \tag{3.3}$$

(ii) Likewise, a function $f \in \mathcal{A}$ is a member of the subclass $\mathcal{R}_q(\pi, g)$ of $\mathcal{R}_q(\vartheta, g)$ if it satisfies the q-differential subordination

$$\mathcal{D}_q f(z) \prec g(z) \quad (z \in \mathcal{E}). \tag{3.4}$$

Remark 3.3. Let $p(z)$, $p_\alpha(z)$ and $g(z)$ be as defined in (2.1), (2.2) and (3.1), respectively, then

(i) $\lim_{q\to 1^-} \mathcal{R}_q(0,p) = \mathcal{L}$, the class studied in [7];

(ii) $\lim_{q\to 1^-} \mathcal{R}_q(0,p_\alpha) = \mathcal{L}(\alpha)$, the class studied in [21];

(iii) $\lim_{q\to 1^-} \mathcal{R}_q(\vartheta,p) = \mathcal{R}(\vartheta)$, the class studied in [22];

(iv) $\lim_{q\to 1^-} \mathcal{R}_q(\pi,p) = \mathcal{B}$, the class studied in [6, 19];

(v) $\lim_{q\to 1^-} \mathcal{R}_q(\pi,p_\alpha) = \mathcal{B}(\alpha)$, the class studied in [15];

(vi) $\lim_{q\to 1^-} \mathcal{R}_q(\vartheta,p) = \mathcal{R}(\delta,\phi)$, the class studied in [24];

(vii) $\lim_{q\to 1^-} \mathcal{R}_q(\vartheta,g) = \mathcal{R}(\vartheta,g)$, the class studied in [23].

In this work, the q-differential operator and the subordination principle were employed to define a new subclass of analytic functions after which some geometric properties such as: characterization, q-integral representation, coefficient estimates, q-differential subordination, and Fekete-Szegö estimate for the new class were investigated. Some motivated works include those in [5, 20, 23, 24]. The main results are presented as follows.

Theorem 3.4 (Characterization Property). *Let $\vartheta \in (-\pi,\pi]$ and $g \in \mathcal{G}$, then a necessary and sufficient condition for function $f \in \mathcal{A}$ to be a member of the class $\mathcal{R}_q(\vartheta,g)$ is that*

$$\frac{1}{z}\left(f(z) \star \frac{z(1-z)_q^2\{2-[2]_q z(1-e^{i\vartheta})\} + [2]_q z^2(1-qz)\{2-z(1-e^{i\vartheta})\}}{2(1-z)_q^4}\right) \neq g(e^{i\phi}),$$

$z \in \mathcal{E}$, $\phi \in [0,2\pi)$ and g is defined in (3.1).

Proof. Suppose $f \in \mathcal{R}_q(\vartheta,g)$, then the LHS of (3.2) cannot take values on the boundary $g(|z| = 1)$ thus,

$$\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2} z\mathcal{D}_q^2 f(z) \neq g(e^{i\phi}) \quad (z \in \mathcal{E},\ \phi \in [0,2\pi)) \tag{3.5}$$

and equivalently

$$\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2} z\mathcal{D}_q^2 f(z) = \frac{1-e^{i\vartheta}}{2}\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2}\mathcal{D}_q(z\mathcal{D}_q f(z)) \neq g(e^{i\phi}). \tag{3.6}$$

But from (1.13) we have

$$f(z) = f(z) \star \frac{z}{(1-z)_q} \qquad \text{and} \qquad z\mathcal{D}_q f(z) = f(z) \star \frac{z}{(1-z)_q^2} \tag{3.7}$$

so that by putting (3.7) into (3.6) gives

$$\frac{1-e^{i\vartheta}}{2}\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2}\mathcal{D}_q(z\mathcal{D}_q f(z))$$
$$= \frac{1-e^{i\vartheta}}{2}\mathcal{D}_q\left(f(z) \star \frac{z}{(1-z)_q}\right) + \frac{1+e^{i\vartheta}}{2}\mathcal{D}_q\left(f(z) \star \frac{z}{(1-z)_q^2}\right) \neq g(e^{i\phi}).$$

Now using (1.15) yields

$$\mathcal{D}_q\left(f(z) \star \frac{1-e^{i\vartheta}}{2}\frac{z}{(1-z)_q} + f(z) \star \frac{1+e^{i\vartheta}}{2}\frac{z}{(1-z)_q^2}\right) \neq g(e^{i\phi}),$$

using (1.16) yields

$$\mathcal{D}_q\left(f(z) \star \left(\frac{1-e^{i\vartheta}}{2}\frac{z}{(1-z)_q} + \frac{1+e^{i\vartheta}}{2}\frac{z}{(1-z)_q^2}\right)\right) \neq g(e^{i\phi})$$

and by simplification we get

$$\mathcal{D}_q\left(f(z)\star\left(\frac{z(1-z)_q(1-e^{i\vartheta})+z(1+e^{i\vartheta})}{2(1-z)_q^2}\right)\right)\neq g(e^{i\phi}).$$

Further, using (1.10) yields

$$\mathcal{D}_q\left(f(z)\star\left(\frac{z(1-z)(1-e^{i\vartheta})+z(1+e^{i\vartheta})}{2(1-z)_q^2}\right)\right)\neq g(e^{i\phi}),$$

$$\mathcal{D}_q\left(f(z)\star\left(\frac{2z-z^2(1-e^{i\vartheta})}{2(1-z)_q^2}\right)\right)\neq g(e^{i\phi}) \tag{3.8}$$

and using (1.14) yields

$$\frac{1}{z}\left(f(z)\star z\mathcal{D}_q\left(\frac{2z-z^2(1-e^{i\vartheta})}{2(1-z)_q^2}\right)\right)\neq g(e^{i\phi}). \tag{3.9}$$

Consider the function

$$\delta(z)=\frac{2z-z^2(1-e^{i\vartheta})}{2(1-z)_q^2}$$

in (3.9) and using (1.12) and (1.10) yields

$$\mathcal{D}_q\delta(z)=\frac{[2(1-z)_q^2]\{2-[2]_qz(1-e^{i\vartheta})\}+[2z-z^2(1-e^{i\vartheta})]\{2[2]_q(1-qz)\}}{4(1-z)_q^4}.$$

Some simplifications give

$$z\mathcal{D}_q\delta(z)=\frac{z(1-z)_q^2\{2-[2]_qz(1-e^{i\vartheta})\}+[2]_qz^2(1-qz)\{2-z(1-e^{i\vartheta})\}}{2(1-z)_q^4} \tag{3.10}$$

so that by putting (3.10) into (3.9) gives Theorem 3.4. ■

Letting f satisfy condition (3.3) implies that $f\in\mathcal{R}_q(0,g)$. Thus, the next result follows from Theorem 3.4.

Corollary 3.5. *A function $f\in\mathcal{A}$ is a member of the class $\mathcal{R}_q(0,g)$ if and only if*

$$\frac{1}{z}\left(f(z)\star\frac{z(1-z)_q^2+[2]_qz^2(1-qz)}{(1-z)_q^4}\right)\neq g(e^{i\phi})\quad(\phi\in[0,2\pi))$$

and $f\in\mathcal{A}$ is a member of the class $\lim\limits_{q\to1^-}\mathcal{R}_q(0,g)$ if and only if

$$\frac{1}{z}\left(f(z)\star\frac{z(1+z)}{(1-z)^3}\right)\neq g(e^{i\phi}).$$

Letting f satisfy condition (3.4) implies that $f\in\mathcal{R}_q(\pi,g)$. Thus, the next result follows from Theorem 3.4.

Corollary 3.6. *A function $f\in\mathcal{A}$ is a member of the class $\mathcal{R}_q(\pi,g)$ if and only if*

$$\frac{1}{z}\left(f(z)\star\frac{z(1-z)_q^2(1-[2]_qz)+[2]_qz^2(1-qz)(1-z)}{(1-z)_q^4}\right)\neq g(e^{i\phi})\quad(\phi\in[0,2\pi))$$

and $f\in\mathcal{A}$ is a member of the class $\lim\limits_{q\to1^-}\mathcal{R}_q(\pi,g)$ if and only if

$$\frac{1}{z}\left(f(z)\star\frac{z}{(1-z)^2}\right)\neq g(e^{i\phi}).$$

Remark 3.7. If $q \to 1^-$,

(i) and $g(e^{i\phi}) = \zeta(t) := t \pm i\sqrt{2t-1}$ $(t \geqslant \frac{1}{2})$, then Theorem 3.4 becomes the result of Trojnar-Spelina [24];

(ii) then Theorem 3.4 becomes the result of Srivastava *et al.* [23].

Theorem 3.8 (q-Integral Representation). *If $\vartheta \in (-\pi,\pi)$ and $g \in \mathcal{G}$, then $f \in \mathcal{R}_q(\vartheta, g)$ if and only if there exists a function $b \in \mathcal{W}$ such that*

$$f(z) = c\int_0^z t^{-c}\left(\int_0^t \tau^{c-1} g(b(\tau))d_q\tau\right)d_qt + ([c-1]_q - (c-1))\int_0^t t^{-c}J(t)d_qt,$$

$$J(z) = \frac{z^c}{[c]_q} + \sum_{k=2}^{\infty}[k]_q a_k \frac{z^{k+c-1}}{[k+c-1]_q} \tag{3.11}$$

and

$$c = \frac{2}{1+e^{i\vartheta}} \quad (\vartheta \neq \pi) \tag{3.12}$$

for all $z \in \mathcal{E}$.

Proof. Let $f \in \mathcal{R}_q(\vartheta, g)$, then by the definition of subordination, (3.2) can be written as

$$\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2} z\mathcal{D}_q^2 f(z) = g(b(z)) \tag{3.13}$$

where $b \in \mathcal{W}$ (see (1.4)). Making use of (3.6) gives

$$\frac{1-e^{i\vartheta}}{2}\mathcal{D}_q f(z) + \frac{1+e^{i\vartheta}}{2}\mathcal{D}_q(z\mathcal{D}_q f(z)) = g(b(z)), \tag{3.14}$$

multiplying through by $\frac{2}{1+e^{i\vartheta}}$ gives

$$\left(\frac{1-e^{i\vartheta}}{1+e^{i\vartheta}}\right)\mathcal{D}_q f(z) + \mathcal{D}_q(z\mathcal{D}_q f(z)) = \left(\frac{2}{1+e^{i\vartheta}} - 1\right)\mathcal{D}_q f(z) + \mathcal{D}_q(z\mathcal{D}_q f(z)) = \frac{2}{1+e^{i\vartheta}} g(b(z))$$

and using c in (3.12) and multiplying through by z^{c-1} gives

$$(c-1)z^{c-1}\mathcal{D}_q f(z) + z^{c-1}\mathcal{D}_q(z\mathcal{D}_q f(z)) = cz^{c-1}g(b(z)). \tag{3.15}$$

Equivalently,

$$\mathcal{D}_q(z^{c-1}(z\mathcal{D}_q f(z))) + ((c-1) - [c-1]_q)z^{c-1}\mathcal{D}_q f(z) = cz^{c-1}g(b(z))$$

hence,

$$\mathcal{D}_q(z^{c-1}(z\mathcal{D}_q f(z))) + ((c-1) - [c-1]_q)\mathcal{D}_q J(z) = cz^{c-1}g(b(z))$$

for $J(z)$ in (3.11). Now

$$\mathcal{D}_q\{z^{c-1}(z\mathcal{D}_q f(z)) + ((c-1) - [c-1]_q)J(z)\} = cz^{c-1}g(b(z))$$

so that by q-integration we have

$$z^c\mathcal{D}_q f(z) + ((c-1) - [c-1]_q)J(z) = c\int_0^z \tau^{c-1}g(b(\tau))d_q\tau$$

or

$$\mathcal{D}_q f(z) = cz^{-c}\int_0^z \tau^{c-1}g(b(\tau))d_q\tau + ([c-1]_q - (c-1))z^{-c}J(z)$$

and another q-integration completes the proof. ∎

Corollary 3.9. *If $\vartheta \longrightarrow \pi$ and $g \in \mathcal{G}$, then for $b \in \mathcal{W}$, (3.13) becomes*

$$\mathcal{D}_q f(z) = g(b(z)) \quad (z \in \mathcal{E}) \tag{3.16}$$

which implies that as $\vartheta \longrightarrow \pi$, $f \in \mathcal{R}_q(\pi, g)$ if and only if

$$f(z) = \int_0^z g(b(\tau)) d_q\tau.$$

And if $q \longrightarrow 1^-$, then (3.16) becomes

$$f'(z) = g(b(z)) \quad (z \in \mathcal{E})$$

which implies that as $\vartheta \longrightarrow \pi$, $f \in \lim_{q\to 1^-} \mathcal{R}_q(\pi, g)$ if and only if

$$f(z) = \int_0^z g(b(\tau)) d\tau.$$

Letting f satisfy condition (3.3) implies that $f \in \mathcal{R}_q(0, g)$. Thus, the next result follows from Theorem 3.8.

Corollary 3.10. *If $f \in \mathcal{R}_q(0, g)$, then*

$$f(z) = \int_0^z t^{-1} \left(\int_0^t g(b(\tau)) d_q\tau \right) d_q t$$

and if $f \in \lim_{q\to 1^-} \mathcal{R}_q(0, g)$, then

$$f(z) = \int_0^z t^{-1} \left(\int_0^t g(b(\tau)) d\tau \right) dt.$$

Remark 3.11. If $q \to 1^-$,

(i) and $g(z) = \phi(z) := 1 + \frac{2}{\pi^2} \left(\ln \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2$, then Theorem 3.8 becomes the result of Trojnar-Spelina [24];

(ii) then Theorem 3.8 becomes the result of Srivastava *et al.* [23].

Theorem 3.12 (Coefficient Estimate). *Let $\vartheta \in (-\pi, \pi]$, $z \in \mathcal{E}$ and $g \in \mathcal{G}$. If $f \in \mathcal{R}_q(\vartheta, g)$, then*

$$|a_k| \leqslant \frac{2 \sum_{j=1}^{k-1} \binom{k-2}{j-1} |A_j|}{[k]_q |\Delta_k|} \quad (k = \{2, 3, \ldots\})$$

where

$$\left.\begin{aligned} \Delta_k &= 2 + [k-1]_q (1 + e^{i\vartheta}), \\ \text{and} \qquad |\Delta_k| &= \sqrt{2\{2 + [k-1]_q (2 + [k-1]_q)(1 + \cos\vartheta)\}} \geqslant 2. \end{aligned}\right\} \tag{3.17}$$

Proof. Putting (1.4) and (1.7) into (3.13) yields

$$1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} + \sum_{k=2}^{\infty} [k-1]_q [k]_q \left(\frac{1 + e^{i\vartheta}}{2} \right) a_k z^{k-1}$$
$$= 1 + A_1 b(z) + A_2 b^2(z) + A_3 b^3(z) + \cdots. \tag{3.18}$$

Using (3.17) and binomially expanding RHS of (3.18) give

$$1+\sum_{k=2}^{\infty}\Delta_k\frac{[k]_q}{2}a_kz^{k-1}=1+A_1b_1z+\{A_1b_2+A_2b_1^2\}z^2+\{A_1b_3+2A_2b_1b_2+A_3b_1^3\}z^3$$
$$+\{A_1b_4+A_2(2b_1b_3+b_2^2)+3A_3b_1^2b_2+A_4b_1^4\}z^4+\cdots.$$

Comparing the coefficients of both sides gives

$$\Delta_2\frac{[2]_q}{2}a_2=A_1b_1 \implies a_2=\frac{2A_1b_1}{[2]_q\Delta_2},$$
$$\Delta_3\frac{[3]_q}{2}a_3=A_1b_2+A_2b_1^2 \implies a_3=\frac{2(A_1b_2+A_2b_1^2)}{[3]_q\Delta_3},$$
$$\Delta_4\frac{[4]_q}{2}a_4=A_1b_3+2A_2b_1b_2+A_3b_1^3 \implies a_4=\frac{2(A_1b_3+2A_2b_1b_2+A_3b_1^3)}{[4]_q\Delta_4},$$

$$\Delta_5\frac{[5]_q}{2}a_5=A_1b_4+A_2(2b_1b_3+b_2^2)+3A_3b_1^2b_2+A_4b_1^4$$
$$\implies a_5=\frac{2\Big(A_1b_4+A_2(2b_1b_3+b_2^2)+3A_3b_1^2b_2+A_4b_1^4\Big)}{[5]_q\Delta_5},$$

$$\Delta_6\frac{[6]_q}{2}a_6=A_1b_5+A_2(2b_1b_4+2b_2b_3)+A_3(3b_1^2b_3+3b_1b_2^2)+4A_4b_1^3b_2+A_5b_1^5$$
$$\implies a_6=\frac{2\Big(A_1\omega_5+A_2(2b_1b_4+2b_2b_3)+A_3(3b_1^2b_3+3b_1b_2^2)+4A_4b_1^3b_2+A_5b_1^5\Big)}{[6]_q\Delta_6}$$

and

$$\Delta_7\frac{[7]_q}{2}a_7=A_1b_6+A_2(2b_1b_5+2b_2b_4+b_3^2)+A_3(3b_1^2b_4+6b_1b_2b_3+b_2^3)$$
$$+A_4(3b_1^3b_3+6b_1^2b_2^2)+5A_5b_1^4b_2+A_6b_1^6$$
$$\implies a_7=\frac{2\begin{pmatrix}A_1b_6+A_2(2b_1b_5+2b_2b_4+b_3^2)+A_3(3b_1^2b_4+6b_1b_2b_3+b_2^3)\\ +A_4(3b_1^3b_3+6b_1^2b_2^2)+5A_5b_1^4b_2+s_6b_1^6\end{pmatrix}}{[7]_q\Delta_7}.$$

Using Lemma 2.2 implies that

$$|a_2|\leqslant\frac{2|A_1|}{[2]_q|\Delta_2|},$$
$$|a_3|\leqslant\frac{2(|A_1|+|A_2|)}{[3]_q|\Delta_3|},$$
$$|a_4|\leqslant\frac{2(|A_1|+2|A_2|+|A_3|)}{[4]_q|\Delta_4|},$$
$$|a_5|\leqslant\frac{2(|A_1|+3|A_2|+3|A_3|+|A_4|)}{[5]_q|\Delta_5|},$$
$$|a_6|\leqslant\frac{2(|A_1|+4|A_2|+6|A_3|+4|A_4|+|A_5|)}{[6]_q|\Delta_6|},$$
$$|a_7|\leqslant\frac{2(|A_1|+5|A_2|+10|A_3|+10|A_4|+5|A_5|+|A_6|)}{[7]_q|\Delta_7|}$$

and in general, we have the result of the Theorem. ∎

Letting f satisfy condition (3.3) implies that $f \in \mathcal{R}_q(0, g)$. Thus, the next result follows from Theorem 3.12.

Corollary 3.13. *If $f \in \mathcal{R}_q(0, g)$, then*

$$|a_k| \leqslant \frac{\sum_{j=1}^{k-1} \binom{k-2}{j-1} |A_j|}{[k]_q \sqrt{1 + 2[k-1]_q + [k-1]_q^2}} \quad (k = \{2, 3, \dots\})$$

and if $f \in \lim_{q \to 1^-} \mathcal{R}_q(0, g)$, then

$$|a_k| \leqslant \frac{\sum_{j=1}^{k-1} \binom{k-2}{j-1} |A_j|}{k^2} \quad (k = \{2, 3, \dots\}).$$

Letting f satisfy condition (3.4) implies that $f \in \mathcal{R}_q(\pi, g)$. Thus, the next result follows from Theorem 3.12.

Corollary 3.14. *If $f \in \mathcal{R}_q(\pi, g)$, then*

$$|a_k| \leqslant \frac{\sum_{j=1}^{k-1} \binom{k-2}{j-1} |A_j|}{[k]_q} \quad (k = \{2, 3, \dots\})$$

and if $f \in \lim_{q \to 1^-} \mathcal{R}_q(\pi, g)$, then

$$|a_k| \leqslant \frac{\sum_{j=1}^{k-1} \binom{k-2}{j-1} |A_j|}{k} \quad (k = \{2, 3, \dots\}).$$

Theorem 3.15 (q-DIFFERENTIAL SUBORDINATION). *Let $\vartheta \in (-\pi, \pi)$, $g \in \mathcal{G}$ is a convex function and let c be as defined in (3.12). If $f \in \mathcal{R}_q(\vartheta, g)$, then*

$$\mathcal{D}_q f(z) \prec c \left[\frac{1}{c}\right]_q \int_0^1 g(zt^{1/c}) d_q t \prec g(z) \quad (z \in \mathcal{E}).$$

The result is sharp.

Proof. Let $f \in \mathcal{R}_q(\vartheta, g)$ and let $p(z) = \mathcal{D}_q f(z)$ so that for c in (3.12), (3.2) can be expressed as

$$p(z) + \frac{1}{c} z \mathcal{D}_q p(z) \prec g(z).$$

Now for a convex function g, $\vartheta \in (-\pi, \pi)$, $\Re c \geqslant 0$ and in view of Lemma 2.4 (for $k = 1$), we have

$$p(z) \prec \frac{c}{z^c} \int_0^z \eta^{c-1} g(\eta) d_q \eta \prec g(z). \tag{3.19}$$

Recall that $p(z) = \mathcal{D}_q f(z)$ and letting $\eta = zt^{1/c}$ implies that $d\eta = z \left[\frac{1}{c}\right]_q t^{(1/c)-1} d_q t$ $(t = \{0, 1\})$, so that from (3.19) we get

$$\mathcal{D}_q f(z) \prec \frac{c}{z^c} \int_0^1 (zt^{1/c})^{c-1} g(zt^{1/c}) z \left[\frac{1}{c}\right]_q t^{(1/c)-1} d_q t \prec g(z) \tag{3.20}$$

and by simpification

$$\mathcal{D}_q f(z) \prec c \left[\frac{1}{c}\right]_q \int_0^1 g(zt^{1/c}) d_q t \prec g(z) \quad (z \in \mathcal{E}). \tag{3.21}$$

Since the result of Lemma 2.4 is sharp, so also is this result and the proof is complete. ∎

Letting f satisfy condition (3.3) implies that $f \in \mathcal{R}_q(0,g)$. Thus, the next result follows from Theorem 3.15.

Corollary 3.16. *If $f \in \mathcal{R}_q(0,g)$, then*

$$\mathcal{D}_q f(z) \prec \int_0^1 g(zt)d_qt \prec g(z) \quad (z \in \mathcal{E})$$

and if $f \in \lim_{q\to 1^-} \mathcal{R}_q(0,g)$, then

$$f'(z) \prec \int_0^1 g(zt)dt \prec g(z) \quad (z \in \mathcal{E}).$$

Remark 3.17. Let $q \to 1^-$, then Theorem 3.15 becomes the result of Srivastava *et al.* [23].

The Fekete-Szegö Problem

Fekete-Szegö functional $\psi(f,\rho) = |a_3 - \rho a_2^2|$ for function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is well-known. It was reported by Fekete and Szegö [8] when they proved the claim of Littlewood-Parley conjecture to be false for odd univalent functions. The functional has received great attention particularly for many subclasses of analytic and univalent functions. The determination of sharp upper bounds for the non-linear functional $\psi(f,\rho)$ for any subclass $\tilde{\mathcal{A}} \subset \mathcal{A}$ is what has come to be known as Fekete-Szegö problem of class $\tilde{\mathcal{A}}$. Some recent studies of Fekete-Szegö problem include the works in [1, 5, 16, 17, 18, 20, 23].

In particular, Ali *et al.* [2] (see also [16]) investigated the Fekete-Szegö problem associated with the mth-root transform for some subclasses of $\mathcal{S}$. In [9], the mth-root transform for function $f \in \mathcal{S}$ was defined by

$$\begin{aligned}\mathcal{T}(z) = \sqrt[m]{f(z^m)} &= z + \frac{1}{m}a_2z^{m+1} + \left(\frac{1}{m}a_3 - \frac{m-1}{2m^2}a_2^2\right)z^{2m+1} + \cdots \\ &\equiv z + \sum_{k=1}^{\infty} d_{km+1}z^{km+1}. \end{aligned} \tag{3.22}$$

Theorem 3.18. *Let $f \in \mathcal{S}$ belongs to $\mathcal{R}_q(\vartheta,g)$. If $\vartheta \in (-\pi,\pi]$ and $g \in \mathcal{G}$, then for $\gamma \in \mathbb{C}$,*

$$|d_{2m+1} - \gamma d_{m+1}^2| \leqslant \frac{2|A_1|}{m[3]_q|\Delta_3|}\max\left\{1, \left|\frac{A_1[3]_q\Delta_3(2\gamma+m-1)}{m[2]_q^2\Delta_2^2} - \frac{A_2}{A_1}\right|\right\},$$

d_{km+1} is as defined in (3.22), $m \in \mathbb{N}$ and $|\Delta_k|$ $(k = \{2,3,\ldots\})$ is as defined in (3.17).

Proof. Firstly, it is well-known that (1.4) and (2.1) are related such that

$$b(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \cdots \tag{3.23}$$

and putting (3.23) into (3.1) gives

$$g(b(z)) = 1 + \frac{1}{2}A_1p_1z + \frac{1}{2}\left(\frac{1}{2}A_2p_1^2 + A_1\left(p_2 - \frac{1}{2}p_1^2\right)\right)z^2 + \cdots. \tag{3.24}$$

Putting (1.7) and (3.24) in (3.13) yields

$$\begin{aligned}1 + \sum_{k=2}^{\infty}\{2 + [k-1]_q(1+e^{i\vartheta})\}\frac{[k]_q}{2}a_kz^{k-1} \\ = 1 + \frac{1}{2}A_1p_1z + \frac{1}{2}\left(\frac{1}{2}A_2p_1^2 + A_1\left(p_2 - \frac{1}{2}p_1^2\right)\right)z^2 + \cdots,\end{aligned} \tag{3.25}$$

so that if we use (3.17) and we compare the coefficients in (3.25), then we get

$$\Delta_2 \frac{[2]_q}{2} a_2 = \frac{1}{2} A_1 p_1 \implies a_2 = \frac{A_1 p_1}{[2]_q \Delta_2} \tag{3.26}$$

and

$$\Delta_3 \frac{[3]_q}{2} a_3 = \frac{1}{2}\left(\frac{1}{2} A_2 p_1^2 + A_1\left(p_2 - \frac{1}{2} p_1^2\right)\right) \implies a_3 = \frac{A_2 p_1^2 + 2A_1\left(p_2 - \frac{1}{2}p_1^2\right)}{2[3]_q \Delta_3}. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.22) gives

$$d_{m+1} = \frac{1}{m} a_2 = \frac{A_1 p_1}{m[2]_q \Delta_2}$$

and

$$d_{2m+1} = \frac{1}{m}\left(a_3 - \frac{m-1}{2t} a_2^2\right) = \frac{A_2 p_1^2 + 2A_1(p_2 - \frac{1}{2}p_1^2)}{2m[3]_q \Delta_3} - \frac{(m-1)A_1^2 p_1^2}{2m^2 [2]_q^2 \Delta_2^2}$$

so that

$$d_{2m+1} - \gamma d_{m+1}^2 = \frac{A_1}{m[3]_q \Delta_3}\left\{p_2 - \left(\frac{1}{2} + \frac{(m-1)A_1[3]_q \Delta_3}{2m[2]_q^2 \Delta_2^2} + \frac{\gamma A_1 [3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} - \frac{A_2}{2A_1}\right) p_1^2\right\}$$

and

$$|d_{2m+1} - \gamma d_{m+1}^2| = \frac{|A_1|}{m[3]_q |\Delta_3|}\left|p_2 - \left(\frac{1}{2} + \frac{(m-1)A_1[3]_q \Delta_3}{2m[2]_q^2 \Delta_2^2} + \frac{\gamma A_1 [3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} - \frac{A_2}{2A_1}\right) p_1^2\right| \tag{3.28}$$

$$= \frac{|A_1|}{m[3]_q |\Delta_3|}\left|p_2 - v p_1^2\right| \tag{3.29}$$

where

$$v = \frac{1}{2}\left(1 + \frac{(m-1)A_1[3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} + \frac{2\gamma A_1 [3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} - \frac{A_2}{A_1}\right).$$

Using Lemma 2.3 in (3.29) gives

$$|2v - 1| = \left|\frac{(m-1)A_1[3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} + \frac{2\gamma A_1 [3]_q \Delta_3}{m[2]_q^2 \Delta_2^2} - \frac{A_2}{A_1}\right| \tag{3.30}$$

so that by putting (3.30) in (3.29) completes the proof. ■

Setting $m = 1$ in Theorem 3.18 gives the following results.

Corollary 3.19. *If $f \in \mathcal{R}_q(\vartheta, g)$, then*

$$|d_3 - \gamma d_2^2| = |a_3 - \gamma a_2^2| \leqslant \frac{2|A_1|}{[3]_q |\Delta_3|} \max\left\{1, \left|\frac{2\gamma A_1 [3]_q \Delta_3}{[2]_q^2 \Delta_2^2} - \frac{A_2}{A_1}\right|\right\}$$

and if $q \to 1^-$, then

$$|d_3 - \gamma d_2^2| = |a_3 - \gamma a_2^2| \leqslant \frac{|A_1|}{3\sqrt{5 + 4\cos\vartheta}} \max\left\{1, \left|\frac{3\vartheta A_1 (2 + e^{i\vartheta})}{(3 + e^{i\vartheta})^2} - \frac{A_2}{A_1}\right|\right\}.$$

These are the results for function $f \in \mathcal{R}_q(\vartheta, g)$ of the form (1.1).

Letting f satisfy condition (3.3) implies that $f \in \mathcal{R}_q(0, g)$. Thus, the next result follows from Theorem 3.18.

Corollary 3.20. *If $f \in \mathcal{R}_q(0,g)$, then for $\gamma \in \mathbb{C}$,*

$$|d_{2m+1}-\gamma d_{m+1}^2| \leqslant \frac{|A_1|}{m[3]_q\sqrt{1+[2]_q(2+[2]_q)}} \max\left\{1, \left|\frac{A_1[3]_q(2\gamma+m-1)(1+[2]_q)}{8m[2]_q^2} - \frac{A_2}{A_1}\right|\right\}.$$

and if $f \in \lim\limits_{q\to 1^-} \mathcal{R}_q(0,g)$, then

$$|d_{2m+1}-\gamma d_{m+1}^2| \leqslant \frac{|A_1|}{9m} \max\left\{1, \left|\frac{9A_1(2\gamma+m-1)}{32m} - \frac{A_2}{A_1}\right|\right\}.$$

Letting f satisfy condition (3.4) implies that $f \in \mathcal{R}_q(\pi,g)$. Thus, the next result follows from Theorem 3.18.

Corollary 3.21. *If $f \in \mathcal{R}_q(\pi,g)$, then for $\gamma \in \mathbb{C}$,*

$$|d_{2m+1}-\gamma d_{m+1}^2| \leqslant \frac{|A_1|}{[3]_q m} \max\left\{1, \left|\frac{A_1[3]_q(2\gamma+m-1)}{2m[2]_q^2} - \frac{A_2}{A_1}\right|\right\}$$

and if $f \in \lim\limits_{q\to 1^-} \mathcal{R}_q(\pi,g)$, then

$$|d_{2m+1}-\gamma d_{m+1}^2| \leqslant \frac{|A_1|}{3m} \max\left\{1, \left|\frac{3A_1(2\gamma+m-1)}{8m} - \frac{A_2}{A_1}\right|\right\}.$$

Remark 3.22. If $q \to 1^-$ in Theorem 3.18, then we get the result of Srivastava *et al.* [23].

4 Conclusion

The class $\mathcal{R}_q(\vartheta,g)$ studied in this work is the class of analytic functions defined by q-differential operator and subordination. Some of the geometric properties that were investigated include the characterization property, q-integral representation, coefficient estimate, q-differential subordination and the Fekete-Szegö functional. Our results however, generalized some of the known ones as evident in many of our corollaries and remarks.

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