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Note on 3-prime near-ring involving left generalized derivations

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Abstract. There is a large body of evidence showing that the existence of a suitably-constrained derivation on a 3-prime near-ring forces the near-ring to be a commutative ring. In the present paper, we investigate the notion of left generalized derivation satisfying certain algebraic identities in 3-prime near-ring N which forces N to be a noncommutative ring. Moreover, an example proving the necessity of the primeness of N is given.

1 Introduction

Throughout this paper N is a right near-ring and $Z(N)$ is the multiplicative center of N; and usually N will be a 3-prime, if it has the property that $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. A near ring N is called zero-symmetric if $x.0 = 0$ for all $x \in N$ (recall that right distributivity yields that $0.x = 0$). As usual $x \circ y = xy + yx$ will denote the well-known Jordan product. Recalling that N is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$. A derivation d on a near-ring N is a group endomorphism on $(N, +)$ which satisfies $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [9], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Let d be a derivation of N, an additive mapping $f : N \to N$ is said to be a right generalized derivation of N associated with d if $f(xy) = f(x)y + xd(y)$ for all $x, y \in N$ and f is said to be a left generalized derivation of N associated with d if $f(xy) = d(x)y + xf(y)$ for all $x, y \in N$, or equivalently $f(xy) = xf(y) + d(x)y$ holds for all pairs $x, y \in N$. Moreover, f is said to be a generalized derivation of N associated with d it is both a right and left generalized derivation of N associated with d. Clearly, in [8] the conclusion of Lemma 2 also remains valid if we replace the generalized derivation by left generalized derivation. So, in this paper, unless otherwise specified, we will use the word near-ring to mean zero-symmetric right near-ring.
Most of the works appeared in the literature from 1987 till now concerning the study of the structure of 3-prime near-rings admitting either a derivation, generalized derivation, semi-derivation, etc., is to verify the commutativity of these near-rings for example, see [1], [2], [3], [4], [5], and [10]). So, it is natural to ask what can we say about the structure of near-rings in which left generalized derivations satisfy certain identities?. As for terminologies used here without mention, we refer to Pilz (1983).

2 Identities with left generalized derivations

Our aim in this section is to prove that if a right near-ring satisfies suitable conditions, then the near-ring cannot be a commutative ring. To prove our main results, we need the following lemmas.

Lemma 2.1. ([6], Lemma 2) Let N be a 3-prime near-ring. If d is a derivation of N, then $d(x) \in Z(N)$ for all $x \in Z(N)$.

Lemma 2.2. ([2], Theorem 2) If a prime near-ring N admits a nontrivial derivation d for which $d(N) \subseteq Z(N)$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion-free, then N is a commutative ring.

Theorem 2.3. *Let N be a 2-torsion free 3-prime near-ring. If N admits a left generalized derivation f such that $f(x \circ y) = 0$ for all $x, y \in N$, then either $f = 0$ or $Z(N) = \{0\}$.*

Proof. Suppose that f is a left generalized derivation associated with a derivation d such that

$$f(x \circ y) = 0 \text{ for all } x, y \in N, \tag{2.1}$$

that is $f(xy + yx) = d(x)y + xf(y) + d(y)x + yf(x) = 0$ for all $x, y \in N$. Replacing x and y by $x \circ y$ and using (2.1) with 2-torsion freeness of N, we get

$$d(x \circ y)(x \circ y) = 0 \text{ for all } x, y \in N. \tag{2.2}$$

On the other hand, substituting xy for x in (2.1), because of $(xy) \circ y = (x \circ y)y$, we obtain

$$d(x \circ y)y + (x \circ y)f(y) = 0 \text{ for all } x, y \in N. \tag{2.3}$$

Taking $y = u \circ v$ in (2.3), by virtue of (2.1), we get

$$d(x \circ (u \circ v))(u \circ v) = 0 \text{ for all } x, u, v, \in N. \tag{2.4}$$

So that

$$\begin{aligned} 0 &= d\Big(x.(u \circ v) + (u \circ v).x\Big)(u \circ v) \\ &= \Big(d(x.(u \circ v)) + d((u \circ v).x)\Big)(u \circ v) \\ &= d(x)(u \circ v)^2 + xd(u \circ v)(u \circ v) + (u \circ v)d(x)(u \circ v) + d(u \circ v)x(u \circ v). \end{aligned}$$

Invoking (2.2), the last result reduces to $\Big(d(x)(u \circ v) + (u \circ v)d(x) + d(u \circ v)x\Big)(u \circ v) = 0$ and therefore

$$\Big(d(x) \circ (u \circ v) + d(u \circ v)x\Big)(u \circ v) = 0 \text{ for all } x, u, v \in N. \tag{2.5}$$

Let $z \in Z(N)$, replacing x and y by $(d(x) \circ (u \circ v))$ and z, respectively, in (2.1) and using 2-torsion freeness of N, we obtain

$$f\Big(z.\big(d(x) \circ (u \circ v)\big)\Big) = 0 \text{ for all } x, u, v \in N, z \in Z(N).$$

It follows that $d(z).\Big(d(x) \circ (u \circ v)\Big) + z.f\Big(d(x) \circ (u \circ v)\Big) = 0$ and thus

$$d(z)\Big(d(x) \circ (u \circ v)\Big) = 0 \text{ for all } x, u, v \in N, z \in Z(N). \tag{2.6}$$

In view of Lemma 2.1, we have $d(z) \in Z(N)$, then left multiplying the preceding relation by t, where $t \in N$, we get $d(z)t\Big(d(x) \circ (u \circ v)\Big) = 0$ for all $x, u, v, t \in N, z \in Z(N)$ and therefore

$$d(z)N\Big(d(x) \circ (u \circ v)\Big) = \{0\} \text{ for all } x, u, v \in N, z \in Z(N). \tag{2.7}$$

In the light of the 3-primeness of N , equation (2.7) yields

$$d(x) \circ (u \circ v) = 0 \text{ or } d(z) = 0 \text{ for all } x, u, v \in N, z \in Z(N). \tag{2.8}$$

Suppose that $d(Z(N)) \neq \{0\}$; then there exists an element $z \in Z(N)$ such that $d(z) \neq 0$. Thereby, equation (2.8) yields $d(x) \circ (u \circ v) = 0$ for all $x, u, v \in N$. Consequently, (2.5) shows that

$$d(u \circ v)x(u \circ v) = 0 \text{ for all } x, u, v \in N.$$

In such a way that, $d(u \circ v)N(u \circ v) = \{0\}$ for all $u, v \in N$. Once again using the 3-primeness, we conclude that $d(u \circ v) = 0$ or $u \circ v = 0$ for all $u, v \in N$. But the second case yields also $d(u \circ v) = 0$. Then,

$$d(u \circ v) = 0 \text{ for all } u, v \in N. \tag{2.9}$$

Substituting vu to v in (2.9), we obtain $(u \circ v)d(u) = 0$ for all $u, v \in N$ and hence $vud(u) = -uvd(u)$ for all $u, v \in N$. Writing vt instead of v in the preceding relation and using it, we get $vtud(u) = -uvtd(u) = (-u)vtd(u) = v(-u)td(u)$ for all $u, v, t \in N$. Accordingly,

$$[-u, v]td(u) = 0 \text{ for all } u, v, t \in N.$$

Replacing u by $-u$ in preceding expression, we obtain

$$[u, v]Nd(-u) = \{0\} \text{ for all } u, v \in N.$$

In view of the 3-primeness, we see that

$$d(u) = 0 \text{ or } u \in Z(N) \text{ for all } u \in N. \tag{2.10}$$

As, because of Lemma 2.1, $u \in Z(N)$ implies $d(u) \in Z(N)$, then (2.10) reduces to $d(u) \in Z(N)$ for all $u \in N$, which means that $d(N) \subseteq Z(N)$ and therefore N is a commutative ring by Lemma 2.2. Hence equation (2.1) together with 2-torsion freeness give $f(xy) = 0$, that is $d(x)y + xf(y) = 0$ for all $x, y \in N$. Replacing y by yx, we get $d(x)yx = 0$ and thus $d(x)Nx = \{0\}$ proving that $d(x) = 0$ for all $x \in N$, which contradicts our assumption that $d(Z(N)) \neq \{0\}$. Consequently, we conclude that $d(Z(N)) = \{0\}$. In this case, let z be an arbitrary element of $Z(N)$ and replacing y by z in (2.1) we have $f(x \circ z) = 0$ for all $x \in N$. Using 2-torsion freeness of N, the latter result shows that $f(zx) = 0$. Accordingly, $zf(x) = 0$ for all $x \in N$. Left multiplying by t, where $t \in N$, we get $ztf(x) = 0$ for all $x, t \in N$ which can rewritten as $zNf(x) = \{0\}$ for all $z \in Z(N), x \in N$ and thus, by 3-primeness of N, we obtain the required result. □

As application of theorem 2.3, we get the following results.

Corollary 2.4. *Let N be a 2-torsion free 3-prime near-ring and $\delta : N \to N$ is a nonzero map of N. If δ is a derivation, left generalized derivation or generalized derivation of N satisfying $\delta(x \circ y) = 0$ for all $x, y \in N$, then N cannot be a commutative ring.*

Theorem 2.5. *Let N be a 2-torsion free 3-prime near-ring. If N admits a left generalized derivation f satisfying $f(x \circ y) = x \circ y$ for all $x, y \in N$, then either f is the identity map on N or $Z(N) = \{0\}$.*

Proof. Assume that f is a left generalized derivation associated with a derivation d such that

$$f(x \circ y) = x \circ y \text{ for all } x, y \in N. \tag{2.11}$$

By properties of f, (2.11) gives

$$xf(y) + d(x)y + d(y)x + yf(x) = xy + yx \text{ for all } x, y \in N.$$

Replacing x and y by $u \circ v$ in the last equation and invoking (2.11), we obtain

$$2d(u \circ v)(u \circ v) = 0 \text{ for all } u, v \in N.$$

Which, because of the 2-torsion freeness, yields

$$d(u \circ v)(u \circ v) = 0 \text{ for all } u, v \in N. \tag{2.12}$$

Substituting xy for x in (2.11), we get

$$d(x \circ y)y + (x \circ y)f(y) = (x \circ y)y \text{ for all } x, y \in N.$$

Taking $y = u \circ v$ and using (2.11), one can see that

$$d(x \circ (u \circ v))(u \circ v) = 0 \text{ for all } x, u, v \in N. \tag{2.13}$$

Since equations (2.12) and (2.13) are the same as equations (2.2) and (2.4), we proceeding exactly as in the proof of Theorem 2.3, we get same result as equation (2.5), i.e

$$\Big(d(x) \circ (u \circ v) + d(u \circ v)x\Big)(u \circ v) = 0 \text{ for all } x, u, v \in N. \tag{2.14}$$

Now, replacing x and y by $d(x) \circ (u \circ v)$ and z , where $z \in Z(N)$, respectively in (2.11) and using (2.11), we obtain

$$\begin{aligned} 2(z.(d(x) \circ (u \circ v))) &= 2(f(z.(d(x) \circ (u \circ v))) \\ &= zf(d(x) \circ (u \circ v)) + d(z).(d(x) \circ (u \circ v)) + \\ &\quad d(z).(d(x) \circ (u \circ v)) + zf(d(x) \circ (u \circ v)) \\ &= z.(d(x) \circ (u \circ v)) + d(z).(d(x) \circ (u \circ v)) + \\ &\quad d(z).(d(x) \circ (u \circ v)) + z.(d(x) \circ (u \circ v)). \end{aligned}$$

So, after simplifying and in view of 2-torsion freeness of N, we find that

$$d(z).(d(x) \circ (u \circ v)) = 0 \text{ for all } x, u, v \in N, z \in Z(N). \tag{2.15}$$

Suppose that $d(Z(N)) \neq \{0\}$, since (2.14) and (2.15) are same as (2.5) and (2.6), we only need to consider the same arguments as used after (2.6) in the proof of Theorem 2.3, we arrive at a conclusion N is a commutative ring. Now, returning to the assumptions of theorem, we obtain $f(xy) = xy$ for all $x, y \in N$. Substituting yt for y in the last result and using it again, we get $d(x)yt = 0$ for all $x, y, t \in N$ and thus $d(x)Nt = \{0\}$ for all $x, t \in N$. By the use of 3-primeness of N, we get $d = 0$. Consequently, our assumption that $d(Z(N)) \neq \{0\}$ is not possible, so we conclude that $d(Z(N)) = \{0\}$. Replacing y by z, where $z \in Z(N)$, in (2.11), we get $f((x + x)z) = f(z(x + x)) = (x + x)z$ for all $x \in N, z \in Z(N)$. By defining of f, the latter relation shows that $(f(x + x) - (x + x))z = 0$. Right multiplying by t, where $t \in N$, and in view of 3-primeness of N, we obtain

$$f(2x) = 2x \text{ or } z = 0 \text{ for all } x \in N, z \in Z(N). \tag{2.16}$$

If $f(2x) = 2x$ for all $x \in N$, in this case taking $x = x(u \circ v)$ in the preceding relation and invoking (2.11), we obtain $2d(x)(u \circ v) = 0$ for all $x, u, v \in N$. Which, because of 2-torsion freeness of N, implies that

$$d(x)(u \circ v) = 0 \quad \text{for all } x, u, v \in N. \tag{2.17}$$

From (2.13), we have

$$\begin{aligned} 0 &= d(x \circ (u \circ v))(u \circ v) \\ &= d(x.(u \circ v))(u \circ v) + d((u \circ v).x)(u \circ v) \\ &= d(x)(u \circ v)^2 + xd(u \circ v)(u \circ v) + d(u \circ v)x(u \circ v) + (u \circ v)d(x)(u \circ v). \end{aligned} \tag{2.18}$$

In view of (2.12) and (2.17), equation (2.18) shows that $d(u \circ v)x(u \circ v) = 0$ for all $x, u, v \in N$. Accordingly, $d(u \circ v)N(u \circ v) = \{0\}$ for all $u, v \in N$. In the light of the 3-prime of N, the preceding result reduces to $d(u \circ v) = 0$ for all $u, v \in N$. Since this equation as the same as (2.9), then using the same arguments as used in the proof of theorem 2.3, we arrive at N is a commutative ring. Thus, (2.11) becomes $f(xy) = xy$ for all $x, y \in N$; it follows that $xf(y) + d(x)y = xy$. Taking $y = yx$, and after simplifying, we get $d(x)yx = 0$ for all $x, y \in N$ which can be rewritten as $d(x)Nx = \{0\}$ for all $x \in N$. Using again the 3-primeness of N, we arrive at $d = 0$. And therefore, by defining f we obtain $f(xy) = xf(y)$ for all $x, y \in N$. So, we conclude that $f(xy) = xy = xf(y)$ for all $x, y \in N$. In view of the 3-primeness of N, the latter result shows that $f(y) = y$ for all $y \in N$. Consequently, equation (2.16) reduces to $f = Id_N$ or $Z(N) = \{0\}$. This complete the proof of our theorem. □

Corollary 2.6. *Let N be a 2-torsion free 3-prime near-ring. If N admits a left generalized derivation f satisfying $f(x \circ y) = x \circ y$ for all $x, y \in N$ and $f \neq Id_N$, then N cannot be a commutative ring.*

The following example shows that the 3-primeness is necessary in the hypotheses of the above theorems.

Example 2.7. Let S be a 2-torsion free zero-symmetric right near-ring. Let us define N, d and $f : N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\},$$

$$d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \text{ and } f \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that N is not 3-prime near-ring and we have f is a nonzero left generalized derivation associated with the nonzero derivation d of N. Also, f satisfies the properties:
$f(A \circ B) = 0$, $f(A \circ B) = A \circ B$ for all $A, B \in N$, $f \neq Id_N$ and $Z(N) \neq \{0\}$. But, since the addition in N is not commutative, then N cannot be a commutative ring.

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