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IMPULSIVE DIFFERENTIAL EQUATIONS WITH ERDÉLYI-KOBER BOUNDARY CONDITIONS AND FRACTIONAL DERIVATIVES OF CAPUTO TYPE

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Abstract. This work takes up the study of the existence and uniqueness of solutions to a class of impulsive fractional boundary value problems of order $q \in (1, 2)$. The results are obtained by using multiple base points and by transforming the boundary value problem into an equivalent integral equation in a Banach space. Various properties of fractional calculus and a number of familiar fixed point theorems are used to obtain the results. A nonlinear operator is defined in a Banach space whose fixed point gives the solution. The obtained results can be seen as more general since Erdélyi-Kober integrals are known to be more general operators in fractional calculus and they reduce to Riemann-Liouville integrals with a power weight. An example is also provided which illustrates our abstract result.

1 Introduction

In the recent past, fractional differential equations have received reasonable importance because of their growing number of applications in many areas of science and engineering. The main advantage that fractional differential equations holds over their integer-order counterpart is that they provides an excellent tool in describing various processes with regard to their memory and hereditary properties. For detailed discussion of the theory and applications in this field, the readers are referred to the descriptions in the books [1, 2, 3] and in the articles [4, 5, 6].

Differential equations of an impulsive nature arise in real world problems while describing the dynamics of processes that exhibit sudden or discontinuous jumps. Such processes are naturally available in various areas of biology, physics, economics, engineering, etc. [7, 8].

The idea and understanding of the solution of an impulsive fractional differential equation have been the subject of discussion in many works, and for more details, the readers are referred to the works in [9, 10, 11, 12, 13, 14, 15]. An impulsive fractional differential equation is mainly of two types:
(i) the fractional derivative with a unique starting point: $D^q = D_0^q$,
(ii) the fractional derivative involving multiple starting points, i.e., $D^q = D_*^q$.

In [16], Wang considered the following problem:

$$\begin{aligned}
&{}^C D_*^q v(t) = F(t, v(t)), \;\; t \in [0,1] \setminus \{w_1, w_2, \dots, w_l\}, \;\; q \in (1,2],\\
&\Delta v(w_\kappa) = E_\kappa(v(w_\kappa^-)), \;\; \Delta v'(w_\kappa) = \bar{E}_\kappa(v(w_\kappa^-)), \;\; \kappa = 1, 2, \dots, l,\\
&av(0) - bv'(0) = v_0, \;\; cv(1) + dv'(1) = v_1.
\end{aligned}$$

In [17], Ahmad and Sivasundaram examined the existence of a solution of the impulsive problem governed by a fractional differential equation in Caputo derivative of the form

$$
\begin{aligned}
&{}^{C}D_{*}^{q}v(t) = F(t, v(t)), \;\; t \in [0,1] \setminus \{w_1, w_2, \ldots, w_l\}, \;\; q \in (1,2],\\
&\Delta v(w_\kappa) = E_\kappa(v(w_\kappa^-)), \;\; \Delta v'(w_\kappa) = \bar{E}_\kappa(v(w_\kappa^-)), \;\; \kappa = 1,2,\ldots,l,\\
&av(0) + bv'(0) = \int_0^1 q_1(v(s))ds, \;\; av(1) + bv'(1) = \int_0^1 q_2(v(s))ds.
\end{aligned}
$$

It may be noted that integral boundary conditions have found their place in several applications, mainly in computational fluid dynamics (CFD) and other fields of applied mathematics such as population dynamics, chemical engineering, etc. Hemodynamic conditions can be completely characterised by CFD techniques under appropriate boundary conditions. But the majority of the CFD-based hemodynamic investigations constitute in vitro conditions, which cannot fully represent the actual patient hemodynamic conditions [18, 20]. The difficulties caused are, in fact, related to the prescribed boundary conditions because it is not always justifiable to assume the geometry of the blood vessel to be circular. Therefore, it is advisable to utilise integral boundary conditions to model blood flow problems more accurately.

The following presents a practical example which can be converted into a problem with integral boundary condition [19]:

$$
\begin{aligned}
-v''(t) &= F(t)G(t, v(t)), \;\; t \in (0,1),\\
v(0) &= 0, \quad \mu v'(1) = v(\xi), \;\; \xi \in (0,1],
\end{aligned}
$$

where $\mu > 0$ is a constant. This represents the mathematical model of a thermostat. This problem can be solved to obtain the stationary solutions of the one-dimensional heat equation in a metallic bar with a controller placed at 1, which can add or remove heat as per the temperature detected by a sensor at ξ. This problem can be generalised by considering the heat equation with a nonlinear gradient source term that varies in time. In this case, the heated bar, with a controller at 1, adds or removes heat depending on the temperature detected by sensors located at any point of the bar depending on how the function H is defined in the following condition. Subsequently, this problem can be expressed in the following form which contains an integral boundary condition:

$$
\begin{aligned}
&v''(t) = G(t, v(t), v'(t)),\\
&v(0) = 0, \quad v'(1) = \int_0^1 v(s)dH(s).
\end{aligned}
$$

An important fractional integral operator, known as the Erdélyi-Kober fractional integral operator, was defined and introduced by Erdélyi and Kober in 1940 [21]. Its usual application is found in the theory of radiative transfer, the kinetic theory of gases, etc. For some recent developments in this direction, we refer the readers to the works carried out in [24, 23, 22]. To the best of the knowledge of the current authors, the existence of solutions to boundary value problems governed by impulsive fractional differential equations with an Erdélyi-Kober integral operator has not been studied to date. To fill this gap, the following integral boundary value problem is taken up:

$$
\left.
\begin{aligned}
&{}^{C}D_{*}^{q}v(t) = F(t, v(t)), \;\; t \in I' = I \setminus \{w_1, w_2, \ldots, w_l\}, \;\; q \in (1,2),\\
&\Delta v(w_\kappa) = \mathcal{I}_\kappa(v(w_\kappa^-)), \;\; \kappa = 1,2,\ldots,l,\\
&\Delta v'(w_\kappa) = \mathcal{J}_\kappa(v(w_\kappa^-)), \;\; \kappa = 1,2,\ldots,l,
\end{aligned}
\right\} \tag{1.1}
$$

with integral conditions on the boundary given by

$$
v(0) - \mu_1 v'(0) = \int_0^\Upsilon h(v(s))ds, \;\; v(\Upsilon) = \mu_2 I_\gamma^{\alpha,\beta} v(\xi), \;\; \xi \in (0, w_1), \tag{1.2}
$$

where $I = [0, \Upsilon]$, $\Upsilon > 0$. Here, ${}^{C}D_{*}^{q}$ represents the Caputo fractional derivative at the base points $t = w_\kappa, \kappa = 1,2,\ldots,l$, i.e., ${}^{C}D_{*}^{q}|_{(w_\kappa, w_{\kappa+1}]}v(t) = {}^{C}D_{w_\kappa^+}^{q}v(t)$ for all $t \in (w_\kappa, w_{\kappa+1}]$. The function $F\colon I \times \mathbb{R} \to \mathbb{R}$ is continuous, $\mathcal{I}_\kappa, \mathcal{J}_\kappa \in C(\mathbb{R}, \mathbb{R})$, $\kappa = 1,2,\ldots,l$, the function $h\colon \mathbb{R} \to \mathbb{R}$ is given to be continuous, $\mu_1 \geq 0$, $\mu_2 > 0$ are given constants. Further, $I_{\gamma}^{\alpha,\beta}$ is the Erdélyi-Kober fractional integral of order $\beta > 0$ where $\gamma > 0$, $\alpha \in \mathbb{R}$.

With this introduction section presenting relevant works, the problem statement, and motivation, the other sections in the paper are arranged as follows: Section 2 states some definitions, lemmas, and theorems required for establishing the results, Section 3 elaborates the results along with all hypotheses and proofs, an example is also provided. Section 4 presents expected results if any general sub-interval is considered, and Section 5 summarises the present study.

2 Preliminaries

Let $0 = w_0 < w_1 < \ldots < w_l < w_{l+1} = \Upsilon$. Then $PC(I,\mathbb{R}) = \{v : I \to \mathbb{R} | v \in C((w_\kappa, w_{\kappa+1}], \mathbb{R}), \kappa = 0, 1, \ldots, l$ and $v(w_\kappa^-), v(w_\kappa^+)$ exist, $\kappa = 1, \ldots, l,$ with $v(w_\kappa^-) = v(w_\kappa)\}$ is a Banach space with respect to the norm $\|v\|_{PC} = \sup_{t \in I} |v(t)|$. By Denoting $PC^1(I,\mathbb{R}) = \{v \in PC(I,\mathbb{R}) | \dot{v} \in PC(I,\mathbb{R})\}$ and setting $\|v\|_{PC^1} = \|v\|_{PC} + \|\dot{v}\|_{PC}$, we have $(PC^1(I,\mathbb{R}),$ $\|.\|_{PC^1})$ as a Banach space.

Let I_{a+}^q denote the Riemann-Liouville integral with base point a.

Definition 2.1. [22] The Erdélyi-Kober fractional integral of order of $\beta > 0$ with $\gamma > 0$, $\alpha \in \mathbb{R}$ of a continuous function $F : (0,\infty) \to \mathbb{R}$ is defined as follows:

$$I_{\gamma}^{\alpha,\beta} F(t) = \frac{\gamma t^{-\gamma(\alpha+\beta)}}{\Gamma(\beta)} \int_0^t \frac{s^{\gamma\alpha+\gamma-1} F(s)}{(t^\gamma - s^\gamma)^{1-\beta}} ds,$$

provided that the right-hand side is pointwise defined on $\mathbb{R}_+$.

Lemma 2.2. *[23] Let $\beta > 0$, $\gamma > 0$ and α, $q \in \mathbb{R}$. Then*

$$I_{\gamma}^{\alpha,\beta} t^q = \frac{t^q \Gamma(\alpha + \frac{q}{\gamma} + 1)}{\Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)}.$$

Definition 2.3. A function $v \in PC^1(I,\mathbb{R})$ with its Caputo derivative of order q existing on I is said to be a solution of (1.1) if it satisfies the problem (1.1).

Lemma 2.4. *[16] Letting $q > 0$, the differential equation ${}^C D^q F(t) = 0$ has the solution*

$$F(t) = a_0 + a_1 t + \ldots + a_{m-1} t^{m-1}$$

where $a_i \in \mathbb{R}$, for $i = 0, 1, \ldots, m-1$ with m denoting the least integer $\geq q$.

Lemma 2.5. *[16] Let $q > 0$. Then*

$$I^q \, {}^C D^q F(t) = F(t) + a_0 + a_1 t + \ldots + a_{m-1} t^{m-1}$$

for some $a_i \in \mathbb{R}$, for $i = 0, 1, \ldots, m-1$.

Theorem 2.6. *[16] Let S be an open bounded subset of a Banach space X with the zero element of X in S. Also, let $\Psi\colon \bar{S} \to X$ be a compact and continuous mapping such that $\|\Psi s\| \leq \|s\|$ for all $s \in \partial S$. Then Ψ has a fixed point in $\bar{S}$.*

Theorem 2.7. *(Schaefer's fixed point theorem)[12] Let $\Psi : X \to X$ be a completely continuous map on a Banach space X. If $E(\Psi) = \{x \in X : x = \Lambda \Psi x$ for some $\Lambda \in [0,1]\} \subset X$ is bounded, then Ψ has fixed points.*

Theorem 2.8. *(Leray-Schauder's nonlinear alternative)[23] Let S be a closed convex subset of a Banach space X, S_1 an open subset of S and the zero element of X belongs to S_1. Suppose that $\Psi\colon \bar{S_1} \to S$ is a completely continuous map. Then Ψ satisfies one of the following properties:*
(i) $\Psi(s) = s$ for $s \in \bar{S_1}$, or
(ii) there exist $s \in \partial S_1$ (represents boundary of S_1 in S) and a constant Λ satisfying $0 < \Lambda < 1$, with $s = \Lambda \Psi s$.

Throughout this work, for each $r > 0$, B_r represents the open ball of radius r in $PC(I,\mathbb{R})$, that is, $B_r = \{v \in PC(I,\mathbb{R})\colon \|v\|_{PC} < r\}$.

3 Existence Results

To investigate the existence of solution of (1.1) with boundary conditions given by (1.2), we require the following result:

Lemma 3.1. *For $f \in C([0,\Upsilon],\mathbb{R})$, the boundary value problem*

$$\left.\begin{aligned} {}^C D^q_* v(t) &= f(t), \quad t \in I' = I \setminus \{w_1, w_2, \ldots, w_l\}, \quad q \in (1,2), \\ \Delta v(w_\kappa) &= \mathcal{I}_\kappa(v(w_\kappa^-)), \quad \kappa = 1,2,\ldots,l, \\ \Delta v'(w_\kappa) &= \mathcal{J}_\kappa(v(w_\kappa^-)), \quad \kappa = 1,2,\ldots,l, \\ v(0) - \mu_1 v'(0) &= \int_0^\Upsilon h(v(s))ds, \quad v(\Upsilon) = \mu_2 I_\gamma^{\alpha,\beta} v(\xi), \quad \xi \in (0, w_1), \end{aligned}\right\} \tag{3.1}$$

is equivalent to the following integral equation

$$v(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + M_1^v + M_2^v t, \quad t \in [0, w_1], \\ \frac{1}{\Gamma(q)} \int_{w_\kappa}^t (t-s)^{q-1} f(s)ds + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} f(s)ds \\ + \sum_{j=1}^{\kappa} \frac{t - w_\kappa}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \\ + \sum_{j=1}^{\kappa-1} \frac{w_\kappa - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \\ + \sum_{j=1}^{\kappa} \mathcal{I}_j(v(w_j^-)) + \sum_{j=1}^{\kappa} (t - w_\kappa) \mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1} (w_\kappa - w_j) \mathcal{J}_j(v(w_j^-)) \\ + M_1^v + M_2^v t, \quad t \in (w_\kappa, w_{\kappa+1}], \quad \kappa = 1,2,\ldots,l, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} M_1^v = -\frac{1}{\omega}\Big[&- \mu_1\mu_2 I_\gamma^{\alpha,\beta} I_{0+}^q f(t)(\xi) - T_2 \int_0^\Upsilon h(v(s))ds + \mu_1 \sum_{j=1}^{l} \mathcal{I}_j(v(w_j^-)) \\ &+ \mu_1 \sum_{j=1}^{l} (\Upsilon - w_l) \mathcal{J}_j(v(w_j^-)) + \mu_1 \sum_{j=1}^{l-1} (w_l - w_j) \mathcal{J}_j(v(w_j^-) \\ &+ \frac{\mu_1}{\Gamma(q)} \sum_{j=1}^{l+1} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} f(s)ds + \mu_1 \sum_{j=1}^{l} \frac{\Upsilon - w_l}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \\ &+ \mu_1 \sum_{j=1}^{l-1} \frac{w_l - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \Big], \end{aligned}$$

$$\begin{aligned} M_2^v = -\frac{1}{\omega}\Big[&- \mu_2 I_\gamma^{\alpha,\beta} I_{0+}^q f(t)(\xi) + T_1 \int_0^\Upsilon h(v(s))ds + \sum_{j=1}^{l} \mathcal{I}_j(v(w_j^-)) \\ &+ \sum_{j=1}^{l} (\Upsilon - w_l) \mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{l-1} (w_l - w_j) \mathcal{J}_j(v(w_j^-) \\ &+ \sum_{j=1}^{l+1} \frac{1}{\Gamma(q)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} f(s)ds + \sum_{j=1}^{l} \frac{\Upsilon - w_l}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \\ &+ \sum_{j=1}^{l-1} \frac{w_l - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \Big], \end{aligned}$$

with

$$\omega = T_1\mu_1 + T_2 \neq 0, \ \ T_1 = 1 - \frac{\mu_2\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}, \ \ T_2 = \Upsilon - \frac{\mu_2\xi\Gamma(\alpha+\frac{1}{\gamma}+1)}{\Gamma(\alpha+\frac{1}{\gamma}+\beta+1)}.$$

Proof. Let v be a solution of (3.1). Then for $t \in [0,t_1]$, we have

$$v(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds - c_1 - c_2 t, \tag{3.3}$$

where c_1, $c_2 \in \mathbb{R}$ are constants, and

$$v'(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}f(s)ds - c_2. \tag{3.4}$$

Then, using the impulse conditions $\Delta v(w_\kappa) = \mathcal{I}_\kappa(v(w_\kappa^-))$ and $\Delta v'(w_\kappa) = \mathcal{J}_\kappa(v(w_\kappa^-))$, for $\kappa = 1,2,\ldots,l$ and $t \in (w_\kappa, w_{\kappa+1}]$, we have

$$v(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_{w_\kappa}^t (t-s)^{q-1}f(s)ds + \sum_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{w_{j-1}}^{w_j}(t-s)^{q-1}f(s)ds \\ + \sum_{j=1}^{\kappa}\frac{t-w_\kappa}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(t-s)^{q-2}f(s)ds \\ + \sum_{j=1}^{\kappa-1}\frac{w_\kappa-w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(t-s)^{q-2}f(s)ds + \sum_{j=1}^{\kappa}\mathcal{I}_j(v(w_j^-)) \\ + \sum_{j=1}^{\kappa}(t-w_\kappa)\mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1}(w_\kappa-w_j)\mathcal{J}_j(v(w_j^-)) - c_1 - c_2 t. \end{cases} \tag{3.5}$$

Using the condition $v(0) - \mu_1 v'(0) = \int_0^\Upsilon h(v(s))ds$, (3.3) and (3.4) imply $-c_1 = \int_0^\Upsilon h(v(s))ds - \mu_1 c_2$. Therefore, for $t \in [0,w_1]$,

$$v(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds + \int_0^\Upsilon h(v(s))ds - c_2(\mu_1+t).$$

For $\kappa = 1,2,\ldots,l$ and $t \in (w_\kappa, w_{\kappa+1}]$, (3.5) gives

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(q)}\int_{w_\kappa}^t (t-s)^{q-1}f(s)ds + \sum_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{w_{j-1}}^{w_j}(t-s)^{q-1}f(s)ds \\ &+ \sum_{j=1}^{\kappa}\frac{t-w_\kappa}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(t-s)^{q-2}f(s)ds \\ &+ \sum_{j=1}^{\kappa-1}\frac{w_\kappa-w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(t-s)^{q-2}f(s)ds + \sum_{j=1}^{\kappa}\mathcal{I}_j(v(w_j^-)) \\ &+ \sum_{j=1}^{\kappa}(t-w_\kappa)\mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1}(w_\kappa-w_j)\mathcal{J}_j(v(w_j^-)) + \int_0^\Upsilon h(v(s))ds \\ &- c_2(\mu_1+t). \end{aligned}$$

Next, we use the boundary condition $v(\Upsilon) = \mu_2 I_\gamma^{\alpha,\beta}v(\xi)$, where $\xi \in (0,w_1)$. For this, we have

$$\begin{aligned} v(\Upsilon) &= \sum_{j=1}^{l+1}\frac{1}{\Gamma(q)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-1}f(s)ds + \sum_{j=1}^{l}\frac{\Upsilon-w_l}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}f(s)ds \\ &+ \sum_{j=1}^{l-1}\frac{w_l-w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}f(s)ds + \sum_{j=1}^{l}\mathcal{I}_j(v(w_j^-)) + \int_0^\Upsilon h(v(s))ds \\ &+ \sum_{j=1}^{l}(\Upsilon-w_l)\mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{l-1}(w_l-w_j)\mathcal{J}_j(v(w_j^-)) - c_2\mu_1 + c_2\Upsilon, \end{aligned}$$

and for $t \in (0, w_1)$,

$$\mu_2 I_\gamma^{\alpha,\beta} v(\xi) = \mu_2 I_\gamma^{\alpha,\beta} I_{0+}^q f(t)(\xi) + \Big[\int_0^\Upsilon h(v(s))ds - c_2\mu_1 \Big] \frac{\mu_2 \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} - c_2 \frac{\mu_2 \xi \Gamma(\alpha + \frac{1}{\gamma} + 1)}{\Gamma(\alpha + \frac{1}{\gamma} + \beta + 1)},$$

which gives $c_1 = -M_1^v$ and $c_2 = -M_2^v$. Upon substitution of the values of c_1, c_2 in (3.3) and (3.5) gives (3.2).

For the converse part, it can be easily shown that the integral equation given by (3.2) satisfies the first three equations of (3.1).

Next, we have $v(0) = M_1^v$ and $v'(0) = M_2^v$, therefore $v(0) - \mu_1 v'(0) = M_1^v - \mu_1 M_2^v = \int_0^\Upsilon h(v(s))ds$.

It remains to verify that $v(\Upsilon) = \mu_2 I_\gamma^{\alpha,\beta} v(\xi)$, $\xi \in (0, w_1)$. From (3.2), we get

$$\begin{aligned} v(\Upsilon) = {} & \frac{1}{\Gamma(q)} \int_{w_l}^\Upsilon (\Upsilon - s)^{q-1} f(s)ds + \frac{1}{\Gamma(q)} \sum_{j=1}^{l} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} f(s)ds \\ & + \sum_{j=1}^{l} \frac{\Upsilon - w_l}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds \\ & + \sum_{j=1}^{l-1} \frac{w_l - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} f(s)ds + \sum_{j=1}^{l} \mathcal{I}_j(v(w_j^-)) \\ & + \sum_{j=1}^{l} (b - w_l)\mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{l-1} (w_l - w_j)\mathcal{J}_j(v(w_j^-)) + M_1^v + M_2^v \Upsilon, \end{aligned}$$

and for $\xi \in (0, w_1)$, we have

$$\mu_2 I_\gamma^{\alpha,\beta} v(\xi) = \mu_2 I_\gamma^{\alpha,\beta} J_0^q f(t)(\xi) + (1 - T_1)M_1^v + (\Upsilon - T_2)M_2^v.$$

Substituting the expressions for M_1^v, M_2^v and combining the terms, we get the desired equality. □

Theorem 3.2. *Assume that there exists a function $f \in L^\infty([0,\Upsilon],[0,\infty))$ and positive constants $\mathcal{I}$, $\mathcal{J}$ and H, such that for all $t \in I$, u, $v \in \mathbb{R}$ and $\kappa = 1, 2, \dots, l$, the following conditions hold*

$$\begin{aligned} |F(t,u) - F(t,v)| &\le f(t)|u - v|, \\ |\mathcal{I}_\kappa(u) - \mathcal{I}_\kappa(v)| &\le \mathcal{I}|u - v| \\ |\mathcal{J}_\kappa(u) - \mathcal{J}_\kappa(v)| &\le \mathcal{J}|u - v| \\ |h(u) - h(v)| &\le H|u - v|. \end{aligned}$$

Then, our considered problem has only one solution if

$$\frac{\|f\|_{L^\infty}}{|\omega|} \left\{ \frac{(1+\mu_1)\mu_2 \xi^q \Gamma(\alpha + \frac{q}{\gamma} + 1)}{\Gamma(q+1)\Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} + (\Upsilon + \mu_1 + |\omega|)\Upsilon^q \left(\frac{l+1}{\Gamma(q+1)} + \frac{2l-1}{\Gamma(q)} \right) \right\} + (l\mathcal{I} + (2l-1)\Upsilon\mathcal{J}) \left(1 + \frac{\mu_1 + \Upsilon}{|\omega|} \right) + (|T_1|\Upsilon + |T_2|) \frac{H\Upsilon}{|\omega|} < 1.$$

Proof. Define a map $\Psi\colon PC(I,\mathbb{R}) \to PC(I,\mathbb{R})$ by

$$(\Psi v)(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,v(s))ds + M_1^v + M_2^v t, \quad t\in[0,w_1], \\ \frac{1}{\Gamma(q)}\int_{w_\kappa}^t (t-s)^{q-1}F(s,v(s))ds \\ +\sum_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-1}F(s,v(s))ds \\ +\sum_{j=1}^{\kappa}\frac{t-w_\kappa}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}F(s,v(s))ds \\ +\sum_{j=1}^{\kappa-1}\frac{w_\kappa-w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}F(s,v(s))ds + \sum_{j=1}^{\kappa}\mathcal{I}_j(v(w_j^-)) \\ +\sum_{j=1}^{\kappa}(t-w_\kappa)\mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1}(w_\kappa-w_j)\mathcal{J}_j(v(w_j^-)) \\ +M_1^v + M_2^v t, \quad t\in(w_\kappa,w_{\kappa+1}], \quad \kappa=1,2,\ldots,l. \end{cases}$$

For $u, v \in PC(I,\mathbb{R})$ and $t \in [0,w_1] \cup \left(\bigcup_{\kappa=1}^{l}(w_\kappa,w_{\kappa+1}]\right)$, we have

$$\begin{aligned} &|(\Psi u)(t) - (\Psi v)(t)| \\ &\le \left[\|f\|_{L^\infty}\left(\frac{(l+1)\Upsilon^q}{\Gamma(q+1)} + \frac{(2l-1)\Upsilon^q}{\Gamma(q)}\right) + l\mathcal{I} + (2l-1)\mathcal{J}\Upsilon\right]\|u-v\|_{PC} \\ &+ |M_1^u - M_1^v| + |M_2^u - M_2^v|\Upsilon, \end{aligned}$$

where

$$\begin{aligned} |M_1^u - M_1^v| \le \frac{1}{|\omega|}\Bigg[\|f\|_{L^\infty}\Bigg(&\frac{\mu_1\mu_2\xi^q\Gamma(\alpha+\frac{q}{\gamma}+1)}{\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)} + \frac{(l+1)\mu_1\Upsilon^q}{\Gamma(q+1)} \\ &+ \frac{(2l-1)\mu_1\Upsilon^q}{\Gamma(q)}\Bigg) + |T_2|H\Upsilon + l\mu_1\mathcal{I} + (2l-1)\mu_1\Upsilon\mathcal{J}\Bigg]\|u-v\|_{PC} \end{aligned}$$

and

$$\begin{aligned} |M_2^u - M_2^v| \le \frac{1}{|\omega|}\Bigg[\|f\|_{L^\infty}\Bigg(&\frac{\mu_2\xi^q\Gamma(\alpha+\frac{q}{\gamma}+1)}{\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)} + \frac{(l+1)\Upsilon^q}{\Gamma(q+1)} \\ &+ \frac{(2l-1)\Upsilon^q}{\Gamma(q)}\Bigg) + |T_1|H\Upsilon + l\mathcal{I} + (2l-1)\Upsilon\mathcal{J}\Bigg]\|u-v\|_{PC}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\Psi u - \Psi v\|_{PC} \le \Bigg[&\frac{\|f\|_{L^\infty}}{|\omega|}\Bigg\{\frac{(1+\mu_1)\mu_2\xi^q\Gamma(\alpha+\frac{q}{\gamma}+1)}{\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)} + (\Upsilon+\mu_1+|\omega|)\Upsilon^q \\ &\times\left(\frac{l+1}{\Gamma(q+1)} + \frac{2l-1}{\Gamma(q)}\right)\Bigg\} + (l\mathcal{I} + (2l-1)\Upsilon\mathcal{J})\left(1 + \frac{\mu_1+\Upsilon}{|\omega|}\right) \\ &+ (|T_1|\Upsilon + |T_2|)\frac{H\Upsilon}{|\omega|}\Bigg]\|u-v\|_{PC}. \end{aligned}$$

Now, using Banach fixed point theorem we get a unique fixed point of Ψ on $PC(I,\mathbb{R})$. □

Theorem 3.3. *Assume that* $\lim_{v\to 0}\dfrac{F(t,v)}{v}=0$, $\lim_{v\to 0}\dfrac{\mathcal{I}_\kappa(v)}{v}=0$, $\lim_{v\to 0}\dfrac{\mathcal{J}_\kappa(v)}{v}=0$ *and* $\lim_{v\to 0}\dfrac{h(v)}{v}=0$, *then problem* (1.1) *with integral boundary condition possesses at least one solution.*

Proof. Let $r>0$ and define a map Ψ on B_r by

$$(\Psi v)(t)=\begin{cases}\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}F(s,v(s))ds+M_1^v+M_2^vt, \quad t\in[0,w_1],\\ \frac{1}{\Gamma(q)}\int_{w_\kappa}^t(t-s)^{q-1}F(s,v(s))ds\\ +\sum_{j=1}^{\kappa}\dfrac{1}{\Gamma(q)}\displaystyle\int_{w_{j-1}}^{w_j}(w_j-s)^{q-1}F(s,v(s))ds\\ +\sum_{j=1}^{\kappa}\dfrac{t-w_\kappa}{\Gamma(q-1)}\displaystyle\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}F(s,v(s))ds\\ +\sum_{j=1}^{\kappa-1}\dfrac{w_\kappa-w_j}{\Gamma(q-1)}\displaystyle\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}F(s,v(s))ds+\sum_{j=1}^{\kappa}\mathcal{I}_j(v(w_j^-))\\ +\sum_{j=1}^{\kappa}(t-w_\kappa)\mathcal{J}_j(v(w_j^-))+\sum_{j=1}^{\kappa-1}(w_\kappa-w_j)\mathcal{J}_j(v(w_j^-))\\ +M_1^v+M_2^vt, \quad t\in(w_\kappa,w_{\kappa+1}], \quad \kappa=1,2,\dots,l.\end{cases}$$

Then, we have

$$\begin{aligned}|M_1^v|\le{}&\frac{1}{|\omega|}\Big[\mu_1\mu_2|I_\gamma^{\alpha,\beta}I_{0+}^qF(t,v(t))(\xi)|+|T_2|\int_0^\Upsilon|h(v(s))|ds\\ &+\mu_1\sum_{j=1}^{l}|\mathcal{I}_j(v(w_j^-))|+\mu_1\sum_{j=1}^{l}(\Upsilon-w_l)|\mathcal{J}_j(v(w_j^-))|\\ &+\mu_1\sum_{j=1}^{l-1}(w_l-w_j)|\mathcal{J}_j(v(w_j^-))|+\frac{\mu_1}{\Gamma(q)}\sum_{j=1}^{l+1}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-1}|F(s,v(s))|ds\\ &+\mu_1\sum_{j=1}^{l}\frac{\Upsilon-w_l}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}|F(s,v(s))|ds\\ &+\mu_1\sum_{j=1}^{l-1}\frac{w_l-w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j-s)^{q-2}|F(s,v(s))|ds\Big]\\ \le{}&\frac{\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q+(l+1)\Upsilon^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}\mu_1L_1\\ &+\frac{(2l-1)\Upsilon^q}{|\omega|\Gamma(q)}\mu_1L_1+\frac{l}{|\omega|}\mu_1L_2+\frac{(2l-1)\Upsilon}{|\omega|}\mu_1L_3+\frac{|T_2|\Upsilon}{|\omega|}L_4,\end{aligned}$$

and

$$
\begin{aligned}
|M_2^v| \leq \frac{1}{|\omega|}\Big[&\mu_2|I_\gamma^{\alpha,\beta}I_{0+}^q F(t,v(t))(\xi)| + |T_1|\int_0^\Upsilon |h(v(s))|ds + \sum_{j=1}^{l}|\mathcal{I}_j(v(w_j^-))| \\
&+ \sum_{j=1}^{l}(\Upsilon - w_l)|\mathcal{J}_j(v(w_j^-))| + \sum_{j=1}^{l-1}(w_l - w_j)|\mathcal{J}_j(v(w_j^-)| \\
&+ \frac{1}{\Gamma(q)}\sum_{j=1}^{l+1}\int_{w_{j-1}}^{w_j}(w_j - s)^{q-1}|F(s,v(s))|ds \\
&+ \sum_{j=1}^{l}\frac{\Upsilon - w_l}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j - s)^{q-2}|F(s,v(s))|ds \\
&+ \sum_{j=1}^{l-1}\frac{w_l - w_j}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j - s)^{q-2}|F(s,v(s))|ds\Big] \\
\leq &\frac{\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q + (l+1)\Upsilon^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}L_1 + \frac{(2l-1)\Upsilon^q}{|\omega|\Gamma(q)}L_1 \\
&+ \frac{l}{|\omega|}L_2 + \frac{(2l-1)\Upsilon}{|\omega|}L_3 + \frac{|T_1|\Upsilon}{|\omega|}L_4,
\end{aligned}
$$

where the positive constants L_i, $(i = 1,2,3,4)$ satisfy $|F(t,v(t))| \leq L_1$, $|\mathcal{I}_\kappa(v(t))| \leq L_2$, $|\mathcal{J}_\kappa(v(t))| \leq L_3$, $|h(v(t))| \leq L_4$, for all $v \in \bar{B}_r$ and $t \in I$. Therefore, Ψ is well-defined on $\bar{B}_r$ and it can also be shown that $\Psi v \in PC(I,\mathbb{R})$ for $v \in \bar{B}_r$.

Next, to show that $\Psi\colon \bar{B}_r \to PC(I,\mathbb{R})$ is completely continuous, we split the proof into the following steps:

Step 1: To show that $\{\Psi v | v \in \bar{B}_r\}$ is equicontinuous in $(w_\kappa, w_{\kappa+1})$, $\kappa = 0,1,\ldots,l$.

Let $v \in \bar{B}_r$ and $0 \leq s_1 < s_2 \leq w_1$. Subsequently,

$$
\begin{aligned}
|(\Psi v)(s_2) - (\Psi v)(s_1)| &\leq \frac{1}{\Gamma(q)}\int_0^{s_1}[(s_2 - s)^{q-1} - (s_1 - s)^{q-1}]|F(s,v(s))|ds \\
&+ \frac{1}{\Gamma(q)}\int_{s_1}^{s_2}(s_2 - s)^{q-1}|F(s,v(s))|ds + |M_2^v|(s_2 - s_1) \\
&\leq \frac{L_1}{\Gamma(q+1)}[s_2^q - s_1^q] \longrightarrow 0 \quad \text{as } s_2 \to s_1.
\end{aligned}
$$

For $w_\kappa < s_1 < s_2 \leq w_{\kappa+1}$, $\kappa = 1,\ldots,l$,

$$
\begin{aligned}
|(\Psi v)(s_2) - (\Psi v)(s_1)| &\leq \frac{1}{\Gamma(q)}\int_{w_\kappa}^{s_1}[(s_2 - s)^{q-1} - (s_1 - s)^{q-1}]|F(s,v(s))|ds \\
&+ \frac{1}{\Gamma(q)}\int_{s_1}^{s_2}(s_2 - s)^{q-1}|F(s,v(s))|ds \\
&+ \Big|\sum_{j=1}^{\kappa}\frac{s_2 - s_1}{\Gamma(q-1)}\int_{w_{j-1}}^{w_j}(w_j - s)^{q-2}F(s,v(s))ds\Big| \\
&+ \Big|\sum_{j=1}^{\kappa}(s_2 - s_1)\mathcal{J}_j(v(w_j^-))\Big| + |M_2^v|(s_2 - s_1) \\
&\leq \frac{L_1}{\Gamma(q+1)}[q(s_2 - s_1) - (s_2 - s_1)^q] + \frac{L_1}{\Gamma(q+1)}(s_2 - s_1)^q \\
&+ (s_2 - s_1)L_1\sum_{j=1}^{l}\Upsilon^{q-1} + (s_2 - s_1)\sum_{j=1}^{l}L_3 + |M_2^v|(s_2 - s_1) \\
&\longrightarrow 0 \quad \text{as } s_2 \to s_1.
\end{aligned}
$$

Thus $\Psi\bar{B}_r$ is equicontinuous in $(w_\kappa, w_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$.
Step 2: To show that $\{\Psi v | v \in \bar{B}_r\}$ is a uniformly bounded subset of $PC(I, \mathbb{R})$.

For $v \in \bar{B}_r$ and $t \in (w_\kappa, w_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$, we have

$$
\begin{aligned}
|\Psi v(t)| \leq & \frac{1}{\Gamma(q)} \int_{w_\kappa}^{t} (t-s)^{q-1} |F(s, v(s))| ds \\
& + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} |F(s, v(s))| ds \\
& + \sum_{j=1}^{\kappa} \frac{t - w_\kappa}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} |F(s, v(s))| ds \\
& + \sum_{j=1}^{\kappa-1} \frac{w_\kappa - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} |F(s, v(s))| ds + \sum_{j=1}^{\kappa} |\mathcal{I}_j(v(w_j^-))| \\
& + \sum_{j=1}^{\kappa} (t - w_\kappa) |\mathcal{J}_j(v(w_j^-))| + \sum_{j=1}^{\kappa-1} (w_\kappa - w_j) |\mathcal{J}_j(v(w_j^-))| + |M_1^v| + |M_2^v| \Upsilon \\
& \leq \frac{L_1 \Upsilon^q}{\Gamma(q+1)} + \sum_{j=1}^{l} \frac{L_1 \Upsilon^q}{\Gamma(q+1)} + \sum_{j=1}^{l} \frac{L_1 \Upsilon^q}{\Gamma(q)} + \sum_{j=1}^{l-1} \frac{L_1 \Upsilon^q}{\Gamma(q)} + \sum_{j=1}^{l} L_2 \\
& + \sum_{j=1}^{l} \Upsilon L_3 + \sum_{j=1}^{l-1} \Upsilon L_3 + |M_1^v| + |M_2^v| \Upsilon \\
& \leq \frac{\Upsilon^q (l+1) |\omega| \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1) + \mu_2 \Gamma(\alpha + \frac{q}{\gamma} + 1) \xi^q}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} L_1 \\
& + \frac{(l+1) \Upsilon^q \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)(\mu_1 + \Upsilon)}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} L_1 + \frac{\Upsilon^q (2l-1)(|\omega| + \mu_1 + \Upsilon)}{|\omega| \Gamma(q)} L_1 \\
& + \frac{l(|\omega| + \mu_1 + \Upsilon)}{|\omega|} L_2 + \frac{\Upsilon(2l-1)(|\omega| + \mu_1 + \Upsilon)}{|\omega|} L_3 + \frac{|T_1| \Upsilon^2}{|\omega|} L_4 \\
& + \frac{\Upsilon |T_2|}{|\omega|} L_4 := L^*.
\end{aligned}
$$

It follows that $\|\Psi v\|_{PC} \leq L^*$ for all $v \in \bar{B}_r$.
Step 3: To show that Ψ is continuous on $\bar{B}_r$.

Using the continuity of the functions $F, \mathcal{I}_\kappa, \mathcal{J}_\kappa, h$, and Lemma 2.2, it can be shown that Ψ is continuous on $\bar{B}_r$.

Therefore, the Arzelá-Ascoli theorem ensures that $\Psi \colon \bar{B}_r \to PC(I, \mathbb{R})$ is completely continuous.

Further, since $\lim_{v \to 0} \frac{F(t,v)}{v} = 0$, $\lim_{v \to 0} \frac{\mathcal{I}_\kappa(v)}{v} = 0$, $\lim_{v \to 0} \frac{\mathcal{J}_\kappa(v)}{v} = 0$ and $\lim_{v \to 0} \frac{h(v)}{v} = 0$, for $\epsilon_i > 0$, $i = 1, 2, 3, 4$, there exists a $r_0 > 0$ such that $|F(t, v)| < \epsilon_1 |v|$, $|\mathcal{I}_\kappa(v)| < \epsilon_2 |v|$, $|\mathcal{J}_\kappa(v)| < \epsilon_3 |v|$ and $|h(v)| < \epsilon_4 |v|$ for $0 < |v| < r_0$ where ϵ_i, $i = 1, 2, 3, 4$ are chosen such that

$$
\begin{aligned}
& \frac{\Upsilon^q (l+1) |\omega| \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1) + \big(\mu_2 \Gamma(\alpha + \frac{q}{\gamma} + 1) \xi^q}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} \epsilon_1 \\
& \frac{(l+1) \Upsilon^q \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)\big)(\mu_1 + \Upsilon)}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} \epsilon_1 + \frac{\Upsilon^q (2l-1)(|\omega| + \mu_1 + \Upsilon)}{|\omega| \Gamma(q)} \epsilon_1 \\
& + \frac{l(|\omega| + \mu_1 + \Upsilon)}{|\omega|} \epsilon_2 + \frac{\Upsilon(2l-1)(|\omega| + \mu_1 + \Upsilon)}{|\omega|} \epsilon_3 + \frac{\Upsilon(|T_1| \Upsilon + |T_2|)}{|\omega|} \epsilon_4 \leq 1.
\end{aligned}
$$

Define $S = B_{r_0}$. Then we have

$$\begin{aligned}\|\Psi v\|_{PC} \leq \Bigg[&\frac{\Upsilon^q(l+1)|\omega|\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)+\big(\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}\epsilon_1\\ &+\frac{(l+1)\Upsilon^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)\big)(\mu_1+\Upsilon)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}\epsilon_1\\ &+\frac{\Upsilon^q(2l-1)(|\omega|+\mu_1+\Upsilon)}{|\omega|\Gamma(q)}\epsilon_1+\frac{l(|\omega|+\mu_1+\Upsilon)}{|\omega|}\epsilon_2\\ &+\frac{\Upsilon(2l-1)(|\omega|+\mu_1+\Upsilon)}{|\omega|}\epsilon_3+\frac{\Upsilon(|T_1|\Upsilon+|T_2|)}{|\omega|}\epsilon_4\Bigg]\|v\|_{PC},\end{aligned}$$

which gives $\|\Psi v\|_{PC} \leq \|v\|_{PC}$, for $v \in \partial B_{r_0}$. Therefore, by applying Theorem 2.6, it is established that the operator Ψ has at least one fixed point on $\bar{B}_{r_0}$. □

Theorem 3.4. *Assume that there exist $L_i > 0$, $i = 1, 2, 3, 4$ satisfying $|F(t, v)| \leq L_1$, $|\mathcal{I}_\kappa(v)| \leq L_2$, $|\mathcal{J}_\kappa(v)| \leq L_3$ and $|h(v)| \leq L_4$ for $t \in J$, $v \in \mathbb{R}$ and $\kappa = 1, 2, \ldots, l$. Then the problem defined by* (1.1) *and* (1.2) *has at least one solution.*

Proof. Under the above assumptions and using the continuity of the involved functions F, $\mathcal{I}_\kappa$, $\mathcal{J}_\kappa$, h, it can be shown that $\Psi\colon PC(I,\mathbb{R}) \to PC(I,\mathbb{R})$ is well-defined and completely continuous. Also, $\|v\|_{PC} \leq L^*$ for all $v \in E(\Psi)$ and for some $L^* \in \mathbb{R}$ (same as in theorem 3.3, Step 2). Therefore, Theorem 2.7 assures the existence of a fixed point of Ψ on $PC(I,\mathbb{R})$. □

Theorem 3.5. *Assume that*

(H1) *there exists a constant $q_1 \in (0,1)$ with $1+q_1 < q$ such that a function $f \in L^{\frac{1}{q_1}}([0,\Upsilon],[0,\infty))$ and a nondecreasing L^1 function $g\colon [0,\infty) \to (0,\infty)$ exist such that*

$$|F(t, v)| \leq f(t)g(|v|), \ \ \textit{for all } t \in I,\ v \in \mathbb{R},$$

(H2) *there exists a positive constant L such that*

$$\frac{L}{\frac{A}{|\omega|}+g(L)\|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}B} > 1,$$

where

$$a = \frac{q-q_1}{1-q_1},\ c = \frac{q-2}{1-q_1},$$
$$A = (\mu_1+\Upsilon)\big(lL_2+(2l-1)\Upsilon L_3\big)+(|T_2|+|T_1|\Upsilon)\Upsilon L_4,$$

$$\begin{aligned}B = &\frac{\Upsilon^{q-q_1}(|\omega|+\mu_1+\Upsilon)}{|\omega|}\left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}}+\frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}}\right)\\ &+\frac{\mu_2\xi^{q-q_1}(\mu_1+\Upsilon)\Gamma(\alpha+\frac{q-q_1}{\gamma}+1)}{|\omega|(a+1)^{1-q_1}\Gamma(\alpha+\frac{q-q_1}{\gamma}+\beta+1)}.\end{aligned}$$

Then problem (1.1) *with boundary conditions given by* (1.2) *has at least one solution.*

Proof. Here the proof is accomplished with the help of Leray-Schauder's nonlinear alternative.

Taking $S = PC(I, \mathbb{R})$, we define a nonlinear map Ψ on $\bar{B}_r$ by

$$(\Psi v)(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, v(s)) ds + M_1^v + M_2^v t, \quad t \in [0, w_1], \\ \frac{1}{\Gamma(q)} \int_{w_\kappa}^t (t-s)^{q-1} F(s, v(s)) ds \\ + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} F(s, v(s)) ds \\ + \sum_{j=1}^{\kappa} \frac{t - w_\kappa}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} F(s, v(s)) ds \\ + \sum_{j=1}^{\kappa-1} \frac{w_\kappa - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} F(s, v(s)) ds + \sum_{j=1}^{\kappa} \mathcal{I}_j(v(w_j^-)) \\ + \sum_{j=1}^{\kappa} (t - w_\kappa) \mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1} (w_\kappa - w_j) \mathcal{J}_j(v(w_j^-)) \\ + M_1^v + M_2^v t, \ t \in (w_\kappa, w_{\kappa+1}], \ \kappa = 1, 2, \ldots, l. \end{cases}$$

Under the given assumptions, it can be shown that Ψ is well-defined on $\bar{B}_r$ and $\Psi v \in PC(I, \mathbb{R})$ for $v \in \bar{B}_r$. Next, to show that $\Psi \colon \bar{B}_r \to PC(I, \mathbb{R})$ is completely continuous, the proof is split into the following parts:

Step 1: To show, $\{\Psi v | v \in \bar{B}_r\}$ is a uniformly bounded subset of $PC(I, \mathbb{R})$.

For $v \in \bar{B}_r$ and $t \in (w_\kappa, w_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$, we have

$$\begin{aligned} |\Psi v(t)| \leq{}& \frac{g(r)\Upsilon^{(a+1)(1-q_1)}}{\Gamma(q)(a+1)^{1-q_1}} \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} + \frac{g(r) l \Upsilon^{(a+1)(1-q_1)}}{\Gamma(q)(a+1)^{1-q_1}} \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} \\ &+ \frac{g(r) l \Upsilon^{(c+1)(1-q_1)+1}}{\Gamma(q-1)(c+1)^{1-q_1}} \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} + \frac{g(r)(l-1)\Upsilon^{(c+1)(1-q_1)+1}}{\Gamma(q-1)(c+1)^{1-q_1}} \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} \\ &+ l L_2 + (2l-1)\Upsilon L_3 + |M_1^v| + |M_2^v|\Upsilon, \end{aligned}$$

that is,

$$\begin{aligned} \|\Psi v\|_{PC} \leq{}& g(r)\Upsilon^{(q-q_1)} \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) \\ &+ l L_2 + (2l-1)\Upsilon L_3 + |M_1^v| + |M_2^v|\Upsilon, \end{aligned}$$

where

$$\begin{aligned} |M_1| \leq{}& \frac{\mu_1 g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{|\omega|} \Bigg[\frac{\mu_2 \xi^{q-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + 1)}{(a+1)^{1-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + \beta + 1)} \\ &+ \Upsilon^{q-q_1} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) \Bigg] \\ &+ \frac{\mu_1 l L_2 + \mu_1 (2l-1)\Upsilon L_3}{|\omega|} + \frac{|w_2|\Upsilon L_4}{|\omega|}, \end{aligned}$$

$$\begin{aligned} |M_2| \leq{}& \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{|\omega|} \Bigg[\frac{\mu_2 \xi^{q-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + 1)}{(a+1)^{1-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + \beta + 1)} \\ &+ \Upsilon^{q-q_1} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) \Bigg] \\ &+ \frac{l L_2 + (2l-1)\Upsilon L_3}{|\omega|} + \frac{|w_2|\Upsilon L_4}{|\omega|}, \end{aligned}$$

with the positive constants L_2, L_3 and L_4 satisfying $|\mathcal{I}_\kappa(v(t))| \leq L_2$, $|\mathcal{J}_\kappa(v(t))| \leq L_3$, $|h(v(t))| \leq L_4$, for all $t \in I$.

Step 2: To show that $\{\Psi v | v \in \bar{B}_r\}$ is equicontinuous in $(w_\kappa, w_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$.

Let $v \in \bar{B}_r$ and $0 \le s_1 < s_2 \le w_1$, then

$$\begin{aligned}
|(\Psi v)(s_2) - (\Psi v)(s_1)| &\le \frac{1}{\Gamma(q)} \int_0^{s_1} [(s_2 - s)^{q-1} - (s_1 - s)^{q-1}] |F(s, v(s))| ds \\
&+ \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} (s_2 - s)^{q-1} |F(s, v(s))| ds + |M_2^v| (s_2 - s_1) \\
&\le \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{\Gamma(q)(a+1)^{1-q_1}} \left[|s_2^{a+1} - s_1^{a+1}| + (s_2 - s_1)^{a+1}\right]^{1-q_1} \\
&+ \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{\Gamma(q)(a+1)^{1-q_1}} (s_2 - s_1)^{q-q_1} + |M_2^v| (s_2 - s_1) \\
&\longrightarrow 0 \quad \text{as } s_2 \to s_1.
\end{aligned}$$

For $w_\kappa < s_1 < s_2 \le w_{\kappa+1}$, $\kappa = 1, \ldots, l$,

$$\begin{aligned}
|(\Psi v)(s_2) - (\Psi v)(s_1)| &\le \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{\Gamma(q)(a+1)^{1-q_1}} \left[(a+1)(s_2 - s_1) - (s_2 - s_1)^{a+1}\right]^{1-q_1} \\
&+ \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{\Gamma(q)(a+1)^{1-q_1}} (s_2 - s_1)^{q-q_1} + (s_2 - s_1) l L_3 \\
&+ \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]}}{\Gamma(q-1)(c+1)^{1-q_1}} l \Upsilon^{q-q_1} (s_2 - s_1) + |M_2^v| (s_2 - s_1) \\
&\longrightarrow 0 \quad \text{as } s_2 \to s_1.
\end{aligned}$$

Thus, Arzelá-Ascoli theorem ensures that $\Psi \bar{B}_r$ is a relatively compact subset of $PC(I, \mathbb{R})$. Also, using (H1), it follows that Ψ is continuous on $\bar{B}_r$. Thus, $\Psi\colon \bar{B}_r \to PC(I, \mathbb{R})$ is completely continuous.

Now, for each $t \in [0, \Upsilon]$ and following the similar computation procedure as earlier, we find that

$$\frac{\|v\|_{PC}}{\frac{A}{|\omega|} + g(\|v\|_{PC}) \|f\|_{L^{\frac{1}{q_1}}[0,\Upsilon]} B} \le 1.$$

In view of (H2), we have $\|v\|_{PC} \ne L$. Set $S_1 = \{v \in PC(I, \mathbb{R}) | \|v\|_{PC} < L\}$. Then, the operator $\Psi\colon \bar{S}_1 \to PC(I, \mathbb{R})$ is completely continuous. Also, from the choice of S_1, there does not exist any $v \in \partial S_1$ such that $v = \Lambda \Psi v$ for some $\Lambda \in (0, 1)$. Thus, Theorem 2.8 gives the existence of a fixed point of the function Ψ on $\bar{S}_1$. $\square$

Example 3.6. Consider the following fractional impulsive differential equation:

$$\begin{aligned}
&{}^C D_*^q v(t) = \frac{2 \sin^7(v(t) + 1) + t^5 + |v(t)|^4}{1 + t^3 + |v(t)|^4}, \quad t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}, \quad q \in (1, 2), \\
&\Delta v\left(\frac{1}{2}\right) = \frac{\exp(-\frac{1}{2}) + \sin v(\frac{1}{2})}{10 + |v(\frac{1}{2})|}, \quad \Delta v'\left(\frac{1}{2}\right) = \frac{\exp\left(-v^2(\frac{1}{2})\right)}{5 + \sin v(\frac{1}{2})}, \\
&v(0) - \mu_1 v'(0) = \int_0^1 \frac{\cos v(s)}{11 + |v(s)|} ds, \\
&v(1) = \mu_2 I_\gamma^{\alpha,\beta} v(\xi), \quad \xi \in \left(0, \frac{1}{2}\right).
\end{aligned}$$

Here, $l = 1$, $w_1 = \frac{1}{2}$, $F(t, v(t)) = \frac{2 \sin^7(v(t)+1) + t^5 + |v(t)|^4}{1 + t^3 + |v(t)|^4}$, $\mathcal{I}_1(v(t)) = \frac{\exp(-t) + \sin v(t)}{10 + |v(t)|}$, $\mathcal{J}_1(v(t)) = \frac{\exp(-v^2(t))}{5 + \sin v(t)}$ and $h(v(t)) = \frac{\cos v(t)}{11 + |v(t)|}$. Thus all the assumptions of Theorem 3.4 are satisfied with $L_1 = 3$, $L_2 = \frac{1}{5}$, $L_3 = \frac{1}{4}$, $L_4 = \frac{1}{11}$ and therefore, we can conclude the existence of solution of our problem.

4 Further development with respect to general sub-intervals

In Section 3, the proofs were accomplished by using the right terminal condition where ξ belongs to the initial sub-interval $(0, w_1)$. In other words, in equation (1.2), at the end point $t = \Upsilon$, we consider the boundary condition $v(\Upsilon) = \mu_2 I_\gamma^{\alpha,\beta} v(\xi)$ for $\xi \in (0, w_1)$.

As a step towards more generalization, instead of taking $\xi \in (0, w_1)$, it may be assumed that $\xi \in (w_\kappa, w_{\kappa+1})$, $\kappa = 1, , \ldots, l$. Following the same procedure as in the earlier section, this extended problem may also be taken up by considering ξ in any arbitrary sub-interval. That is, for $\xi \in (w_\kappa, w_{\kappa+1})$, $\kappa = 1, , \ldots, l$, in view of Lemma 3.1, consider the following operator:

Ψ_i on $PC(I, \mathbb{R})$, $(i = 1, 2, 3)$ defined as

$$(\Psi_i v)(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, v(s)) ds + M_{i1}^v + M_{i2}^v t, \quad t \in [0, w_1], \\ \frac{1}{\Gamma(q)} \int_{w_\kappa}^t (t-s)^{q-1} F(s, v(s)) ds \\ + \displaystyle\sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-1} F(s, v(s)) ds \\ + \displaystyle\sum_{j=1}^{\kappa} \frac{t - w_\kappa}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} F(s, v(s)) ds \\ + \displaystyle\sum_{j=1}^{\kappa-1} \frac{w_\kappa - w_j}{\Gamma(q-1)} \int_{w_{j-1}}^{w_j} (w_j - s)^{q-2} F(s, x(s)) ds + \sum_{j=1}^{\kappa} \mathcal{I}_j(v(w_j^-)) \\ + \displaystyle\sum_{j=1}^{\kappa} (t - w_\kappa) \mathcal{J}_j(v(w_j^-)) + \sum_{j=1}^{\kappa-1} (w_\kappa - w_j) \mathcal{J}_j(v(w_j^-)) \\ + M_{i1}^v + M_{i2}^v t, \quad t \in (w_\kappa, w_{\kappa+1}], \quad \kappa = 1, 2, \ldots, l, \end{cases}$$

where, for $\xi \in (w_\kappa, w_{\kappa+1})$, $\kappa = 1, , \ldots, l-2$,

$$\begin{aligned} M_{11}^v = -\frac{1}{\omega} \Big[& -\mu_1 \mu_2 I_\gamma^{\alpha,\beta} I_{w_\kappa}^q F(t, v(t))(\xi) - T_2 \int_0^\Upsilon h(v(s)) ds \\ & + T_1 \mu_1 \sum_{j=1}^{\kappa} \mathcal{I}_j(v(w_j^-)) + \mu_1 \sum_{j=\kappa+1}^{l} \mathcal{I}_j(v(w_j^-)) + \mu_1 \sum_{j=1}^{\kappa} (T_2 - w_l) \mathcal{J}_j(v(w_j^-)) \\ & + \mu_1 \sum_{j=\kappa+1}^{l} (\Upsilon - w_l) \mathcal{J}_j(v(w_j^-)) + \mu_1 \sum_{j=1}^{\kappa} (w_l - T_1 w_j) \mathcal{J}_j(v(w_j^-) \\ & + \mu_1 \sum_{j=\kappa+1}^{l-1} (w_l - w_j) \mathcal{J}_j(v(w_j^-)) + w_1 \mu_1 \sum_{j=1}^{\kappa} I_{w_{j-1}}^q F(t, v(t))(w_j) \\ & + \mu_1 \sum_{j=\kappa+1}^{l+1} I_{w_{j-1}}^q F(t, v(t))(w_j) + \mu_1 \sum_{j=1}^{\kappa} (T_2 - w_l) I_{w_{j-1}}^{q-1} F(t, v(t))(w_j) \\ & + \mu_1 \sum_{j=\kappa+1}^{l} (\Upsilon - w_l) I_{w_{j-1}}^{q-1} F(t, v(t))(w_j) + \mu_1 \sum_{j=1}^{\kappa} (w_l - T_1 w_j) I_{w_{j-1}}^{q-1} F(t, v(t))(w_j) \\ & + \mu_1 \sum_{j=\kappa+1}^{l-1} (w_l - w_j) I_{w_{j-1}}^{q-1} F(t, v(t))(w_j) \Big], \end{aligned}$$

$$
\begin{aligned}
M_{12}^{v} = -\frac{1}{\omega}\Big[& -\mu_2 I_{\gamma}^{\alpha,\beta} I_{w_\kappa}^{q} F(t,v(t))(\xi) + T_1\int_0^{\Upsilon} h(v(s))ds + T_1\sum_{j=1}^{\kappa}\mathcal{I}_j(v(w_j^-)) \\
& + \sum_{j=\kappa+1}^{l}\mathcal{I}_j(v(w_j^-)) + \sum_{j=1}^{\kappa}(T_2-w_l)\mathcal{J}_j(v(w_j^-)) + \sum_{j=\kappa+1}^{l}(\Upsilon-w_l)\mathcal{J}_j(v(w_j^-)) \\
& + \sum_{j=1}^{\kappa}(w_l-T_1w_j)\mathcal{J}_j(v(w_j^-) + \sum_{j=\kappa+1}^{l-1}(w_l-w_j)\mathcal{J}_j(v(w_j^-) \\
& + T_1\sum_{j=1}^{\kappa} I_{w_{j-1}}^{q} F(t,v(t))(w_j) + \sum_{j=\kappa+1}^{l+1} I_{w_{j-1}}^{q} F(t,v(t))(w_j) \\
& + \sum_{j=1}^{\kappa}(T_2-w_l) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) + \mu_1\sum_{j=\kappa+1}^{l}(\Upsilon-w_l) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) \\
& + \sum_{j=1}^{\kappa}(w_l-T_1w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) + \sum_{j=\kappa+1}^{l-1}(w_l-w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)\Big].
\end{aligned}
$$

For $\xi \in (w_{l-2}, w_{l-1})$,

$$
\begin{aligned}
M_{21}^{v} = -\frac{1}{\omega}\Big[& -\mu_1\mu_2 I_{\gamma}^{\alpha,\beta} I_{w_{l-2}}^{q} F(t,v(t))(\xi) - T_2\int_0^{\Upsilon} h(v(s))ds \\
& + wT_1\mu_1\sum_{j=1}^{l-2}\mathcal{I}_j(v(w_j^-)) + \mu_1\sum_{j=l-1}^{l}\mathcal{I}_j(v(w_j^-)) + \mu_1\sum_{j=1}^{l-2}(T_2-T_1w_j)\mathcal{J}_j(v(w_j^-)) \\
& + \mu_1\sum_{j=l-1}^{l}(\Upsilon-w_j)\mathcal{J}_j(v(w_j^-)) + T_1\mu_1\sum_{j=1}^{l-2} I_{w_{j-1}}^{q} F(t,v(t))(w_j) \\
& + \mu_1\sum_{j=l-1}^{l+1} I_{w_{j-1}}^{q} F(t,v(t))(w_j) + \mu_1\sum_{j=1}^{l-2}(T_2-T_1w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) \\
& + \mu_1\sum_{j=l-1}^{l}(\Upsilon-w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)\Big],
\end{aligned}
$$

$$
\begin{aligned}
M_{22}^{v} = -\frac{1}{\omega}\Big[& -\mu_2 I_{\gamma}^{\alpha,\beta} I_{w_{l-2}}^{q} F(t,v(t))(\xi) + T_1\int_0^{\Upsilon} h(v(s))ds + T_1\sum_{j=1}^{l-2}\mathcal{I}_j(v(w_j^-)) \\
& + \sum_{j=l-1}^{l}\mathcal{I}_j(v(w_j^-)) + \sum_{j=1}^{l-2}(T_2-T_1w_j)\mathcal{J}_j(v(w_j^-)) + \sum_{j=l-1}^{l}(\Upsilon-w_j)\mathcal{J}_j(v(w_j^-)) \\
& + T_1\sum_{j=1}^{l-2} I_{w_{j-1}}^{q} F(t,v(t))(w_j) + \sum_{j=l-1}^{l+1} I_{w_{j-1}}^{q} F(t,v(t))(w_j) \\
& + \sum_{j=1}^{l-2}(T_2-T_1w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) + \mu_1\sum_{j=l-1}^{l}(\Upsilon-w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)\Big].
\end{aligned}
$$

For $\xi \in (w_l, \Upsilon)$,

$$M_{31}^v = -\frac{1}{\omega}\Big[-\mu_1\mu_2 I_\gamma^{\alpha,\beta} I_{w_l}^q F(t,v(t))(\xi) - T_2\int_0^\Upsilon h(v(s))ds$$

$$+ T_1\mu_1 \sum_{j=1}^{l} \mathcal{I}_j(v(w_j^-)) + \mu_1 \sum_{j=1}^{l}(T_2 - T_1 w_l)\mathcal{J}_j(v(w_j^-))$$

$$+ \mu_1 T_1 \sum_{j=1}^{l-1}(w_l - w_j)\mathcal{J}_j(v(w_j^-)) + \mu_1 I_{w_l}^q F(t,v(t))(\Upsilon)$$

$$+ \mu_1 T_1 \sum_{j=1}^{l} I_{w_{j-1}}^q F(t,v(t))(w_j) + \mu_1(T_2 - T_1 w_l)\sum_{j=1}^{l} I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)$$

$$+ \mu_1 T_1 \sum_{j=1}^{l-1}(w_l - w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)\Big],$$

$$M_{32}^v = -\frac{1}{\omega}\Big[-\mu_2 I_\gamma^{\alpha,\beta} I_{w_l}^q F(t,v(t))(\xi) + T_1\int_0^\Upsilon h(v(s))ds + T_1\sum_{j=1}^{l}\mathcal{I}_j(v(w_j^-))$$

$$+ \sum_{j=1}^{l}(T_2 - T_1 w_l)\mathcal{J}_j(v(w_j^-)) + T_1\sum_{j=1}^{l-1}(w_l - w_j)\mathcal{J}_j(v(w_j^-)) + I_{w_l}^q F(t,v(t))(\Upsilon)$$

$$+ T_1\sum_{j=1}^{l} I_{w_{j-1}}^q F(t,v(t))(w_j) + (T_2 - T_1 w_l)\sum_{j=1}^{l} I_{w_{j-1}}^{q-1} F(t,v(t))(w_j) + T_1$$

$$\times \sum_{j=1}^{l-1}(w_l - w_j) I_{w_{j-1}}^{q-1} F(t,v(t))(w_j)\Big].$$

Further, we have the following:

$$\omega = T_1\mu_1 + T_2 \neq 0,$$

$$\text{with } T_1 = 1 - \frac{\mu_2\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}, \quad T_2 = \Upsilon - \frac{\mu_2\xi\Gamma(\alpha+\frac{1}{\gamma}+1)}{\Gamma(\alpha+\frac{1}{\gamma}+\beta+1)}.$$

Subsequently, under similar kind of assumptions on the functions F, $\mathcal{I}_\kappa$, $\mathcal{J}_\kappa$ and h (as in Section 3), different results can be obtained for the existence and uniqueness of solutions for $\xi \in (w_\kappa, w_{\kappa+1})$, $\kappa = 1, \ldots, l$.

5 Conclusion

This work investigates the existence and uniqueness of the solution of a class of impulsive boundary value problems governed by a fractional differential equation with multiple base points. Here we define a nonlinear operator Ψ on the Banach space whose fixed point gives the solution of the boundary value problem. Our results are more general in the sense that Erdélyi-Kober integrals are known to be more general operators in fractional calculus, and they reduce to Riemann-Liouville integrals with a power weight for $\alpha = 0$, $\gamma = 1$. Furthermore, those integrals include Hadamard integrals as a special case. For further research in this direction, it is very likely possible to examine the existence and uniqueness of solutions of boundary value problems for impulsive evolution equations by considering Erdélyi-Kober conditions at both ends, and also two, three, or even multi-point boundary value problems involving Erdélyi-Kober conditions.

Remark 5.1. This work is an excerpt from the first author's thesis titled *On the existence, uniqueness and approximate controllability of some classes of differential equations with different types of fractional derivatives* which is available at **gyan.iitg.ernet.in**.

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