Palestine Journal of Mathematics

Vol. 12(3)(2023) , 151–162

COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE GEGENBAUER POLYNOMIAL

Gangadharan Murugusundaramoorthy and Hatun Özlem Güney

Communicated by Ayman Badawi

MSC 2010 Classifications: 30C45, 30C50.

Keywords and phrases: Analytic functions, bi-univalent functions, Gegenbauer polynomials, coefficient estimates, Fekete-Szegö inequalities.

The authors thanks the referees for their insightful suggestions to improve the paper in present form.

Abstract In this paper, we familiarize and explore new subclasses of bi-univalent functions defined in the open unit disk related with Gegenbauer polynomials. Furthermore, we find estimates for initial coefficients of functions in these classes. Also, we obtain the Fekete-Szegö inequalities for function in these classes. Several consequences of the results are also pointed out as corollaries.

1 Introduction

Let $\mathcal{A}$ represent the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n \tag{1.1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ comprising of functions of the form (1.1) which are also univalent in $\mathbb{D}$.

The Koebe one quarter theorem [3] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z,\ (z \in \mathbb{D}) \text{ and } f(f^{-1}(w)) = w\ (|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. Since $f \in \Sigma$ has the Maclaurian series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{1.2}$$

We notice that the class Σ is not empty. For instance, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the Koebe function is not a member of Σ. Lately, Srivastava et al. [21] have essentially revived the study of analytic and bi-univalent functions, it was followed by such works as those by(see [1, 2, 4, 13, 14, 18]). Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [1, 2, 6, 13, 21, 27, 28]), bi-close-to-convex functions[5, 11] .

An analytic function φ is subordinate to an analytic function ψ, written $\varphi(z) \prec \psi(z)$, provided there is an analytic function ω defined on $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $\varphi(z) = \psi(\omega(z))$.

We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

In Geometric Function Theory, there have been many interesting and fruitful usages of a wide variety of special functions, $q-$ calculus and special polynomials. The Fibonacci polynomials, Faber polynomials, the Lucas polynomials, the Pell polynomials, the Pell-Lucas polynomials, and the Chebyshev polynomials of the second kind and Horadam polynomials are potentially important in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences. These polynomials have been studied in several papers from a theoretical point of view and recently in case of bi-univalent functions (see[6, 15, 22, 23] also the references cited therein). In this article, we associate certain bi-univalent functions with Gegenbauer polynomials and then explores some properties of the class of bi-univalent functions based on earlier work of Jahangiri and Hamidi [8] .

We recall the Gegenbauer polynomials (for details see Kim et al., [9] and references cited therein) are given in terms of the Jacobi polynomials $P_n^{(\nu,\upsilon)}(x)$ with $\nu = \upsilon = \lambda - \frac{1}{2}; (\lambda > -\frac{1}{2}, \lambda \neq 0)$ by

$$\begin{aligned} \mathfrak{G}_n^{\lambda}(x) &= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) \\ &= \binom{n+2\lambda-1}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}(2\lambda+n)_k}{(\lambda+\frac{1}{2})_k} \left(\frac{x-1}{2}\right)^k \end{aligned} \tag{1.3}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. From (1.3), we note that $\mathfrak{G}_n^{\lambda}(x)$ is a polynomial of degree n with real coefficients and $\mathfrak{G}_n^{\lambda}(1) = \binom{n+2\lambda-1}{n}$. The leading coefficient of $\mathfrak{G}_n^{\lambda}(x)$ is $2^n\binom{n+\lambda-1}{n}$. By the theory of Jacobi polynomials with $\mu = \upsilon = \lambda - \frac{1}{2}, \lambda > -\frac{1}{2}$, and $\lambda \neq 0$, we get

$$\mathfrak{G}_n^{\lambda}(-x) = (-1)^n \mathfrak{G}_n^{\lambda}(x).$$

It is not difficult to show that $\mathfrak{G}_n^{\lambda}(x)$ is a solution of the following Gegenbauer differential equation:

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0.$$

The Rodrigues formula for the Gegenbauer polynomials is well known as the following:

$$(1-x^2)^{\lambda-\frac{1}{2}}\mathfrak{G}_n^{\lambda}(x) = \frac{(-2)^n(\lambda)_n}{n!(n+2\lambda)_n}\left(\frac{d}{dx}\right)^n (1-x^2)^{n+\lambda-\frac{1}{2}}.$$

The above equation can be easily derived from the properties of Jacobi polynomials.

As is well known, the generating function of Gegenbauer polynomials is given by(see[9, 25]

$$\frac{2^{\lambda-\frac{1}{2}}}{(1-2xt+t^2)^{\frac{1}{2}}(1-xt+\sqrt{1-2xt+t^2})^{\lambda-\frac{1}{2}}} = \sum_{n=0}^{\infty} \frac{(\lambda-\frac{1}{2})_n}{(2\lambda)_n}\mathfrak{G}_n^{\lambda}(x)t^n. \tag{1.4}$$

This equation can be derived from the generating function of Jacobi polynomials.

From above equation (1.4), we note that

$$\Phi(t,x) = \frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{\lambda}(x)t^n \qquad ; \quad (|t|<1, |x|\leq 1). \tag{1.5}$$

The proof of above is given in [25] and Kim et al., [9](also see [10]) extensively studied these results for different perspective. We note that , for $\lambda = 1$; we get the Chebyshev Polynomials and $\lambda = \frac{1}{2}$; we get the Legendre Polynomials. In 1935, Robertson [20] proved an integral representation for typically real valued T_R functions,has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are holomorphic in Δ, real for $z \in (-1;1)$ and satisfy the condition

$$Imz > 0 \quad z \in \Delta \notin (-1;1)$$

namely $f \in T_R$ if and only if it has the representation

$$f(z) = \int_{-1}^{1} \frac{z}{1 - 2xz + z^2} d(\mu) \qquad ; \qquad z \in \Delta$$

whereμ is a probability measure on $[-1, 1]$. The notion of the class $T_R(\lambda), \lambda > 0$, has been extended in [26] to the class which is defined by the integral formula

$$f(z) = \int_{-1}^{1} \frac{z}{(1 - 2xz + z^2)^{\lambda}} d(\mu) \qquad ; \qquad z \in \Delta \tag{1.6}$$

where μ is a probability measure on $[-1, 1]$. Of course, we have $T_R(1) \equiv T_R$ and if f given by(1.6) plays an important role in the geometric theory of holomorphic functions in the unit disk Δ then we have

$$a_n = \int_{-1}^{1} \mathfrak{G}_{n-1}^{\lambda}(x) d(x)$$

where $\mathfrak{G}_n^{\lambda}(x)$ is the Gegenbauer polynomial of degree n.

In this paper, motivated by recent works , we introduce a subclass of bi-univalent functions associated with Gegenbauer polynomials [16, 17] and obtain bounds on the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the functions $f \in \mathcal{HG}_{\Sigma}^{\kappa,\ \delta,\ \vartheta}\ (\Phi)$, by subordination and consider the celebrated Fekete-Szegö problem. We also provide relevant connections of our results with those considered in earlier investigations.

2 Class of Bi-univalent function related with Gegenbauer polynomials

To start with for our discussions unless otherwise stated we let

$$0 \leq \lambda \leq 1, \lambda > \frac{1}{2} \qquad \text{and} \qquad t \in (\frac{1}{2}, 1]$$

also

$$\Phi(t, z) = \frac{1}{(1 - 2tz + z^2)^{\lambda}} = \mathfrak{G}_0^{\lambda}(t) + \mathfrak{G}_1^{\lambda}(t) z^1 + \mathfrak{G}_2^{\lambda}(t) z^2 + \mathfrak{G}_3^{\lambda}(t) z^3 + ... \mathfrak{G}_n^{\lambda}(t) z^n + \cdots . \tag{2.1}$$

In particular,

$$\begin{aligned} \mathfrak{G}_0^{\lambda}(t) &= 1 \\ \mathfrak{G}_1^{\lambda}(t) &= 2\lambda t && (2.2) \\ \mathfrak{G}_2^{\lambda}(t) &= 2\lambda(\lambda+1)t^2 - \lambda = 2(\lambda)_2 t^2 - \lambda && (2.3) \\ 3\mathfrak{G}_3^{\lambda}(t) &= 4\lambda(\lambda+1)(\lambda+2)t^3 - 6\lambda(\lambda+1)t \\ &= 4(\lambda)_3 t^3 - 6(\lambda)_2 t && (2.4) \end{aligned}$$

where

$$(\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1).$$

Now we define first subclass of bi-univalent functions in the open unit disk, associated with Gegenbauer polynomials as below:

Definition 2.1. A function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{HG}_{\Sigma}^{\kappa,\ \delta,\ \vartheta}\ (\Phi)$, $\kappa \geq 0$, $\vartheta \geq 1$, $\delta \geq 0$, if the following conditions are satisfied:

$$(1 - \vartheta) \left(\frac{f(z)}{z} \right)^{\kappa} + \vartheta f'(z) \left(\frac{f(z)}{z} \right)^{\kappa - 1} + \delta z f''(z) \prec \Phi(t, z)$$

and

$$(1 - \vartheta) \left(\frac{g(w)}{w} \right)^{\kappa} + \vartheta g'(w) \left(\frac{g(w)}{w} \right)^{\kappa - 1} + \delta w g''(w) \prec \Phi(t, w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

By suitably specializing the parameters κ, ϑ and δ, the class $\mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$ reduces to various new subclasses, as illustrated in the following remark:

Remark 2.2. (i) For $\delta = 0$, we let $\mathcal{HG}_{\Sigma}^{\kappa,\,0,\,\vartheta}(\Phi) \equiv \mathcal{N}_{\Sigma}^{\kappa,\,\vartheta}(\Phi)$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{N}_{\Sigma}^{\kappa,\,\vartheta}(\Phi)$, if

$$(1-\vartheta)\left(\frac{f(z)}{z}\right)^{\kappa} + \vartheta f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} \prec \Phi(t,z)$$

and

$$(1-\vartheta)\left(\frac{g(w)}{w}\right)^{\kappa} + \vartheta g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} \prec \Phi(t,w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

(ii) For $\vartheta = 1$ and $\delta = 0$, we let $\mathcal{HG}_{\Sigma}^{\kappa,\,0,\,1}(\Phi) \equiv \mathcal{B}_{\Sigma}^{\kappa}(\Phi)$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{B}_{\Sigma}^{\kappa}(\Phi)$, if

$$f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} \prec \Phi(t,z) \quad \textit{and} \quad g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} \prec \Phi(t,w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

(iii) For $\vartheta = 1$ and $\delta = 0 = \kappa$, we let $\mathcal{HG}_{\Sigma}^{0,\,0,\,1}(\Phi) \equiv \mathcal{S}_{\Sigma}(\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{S}_{\Sigma}(\Phi)$, if

$$\frac{zf'(z)}{f(z)} \prec \Phi(t,z) \quad \textit{and} \quad \frac{wg'(w)}{g(w)} \prec \Phi(t,w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

(iv) For $\kappa = 1$, we let $\mathcal{HG}_{\Sigma}^{1,\,\delta,\,\vartheta}(\Phi) \equiv \mathcal{M}_{\Sigma}^{\delta,\,\vartheta}(\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{M}_{\Sigma}^{\delta,\,\vartheta}(\Phi)$, if

$$(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) + \delta z f''(z) \prec \Phi(t,z)$$

and

$$(1-\vartheta)\frac{g(w)}{w} + \vartheta g'(w) + \delta w g''(w) \prec \Phi(t,w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

(v) For $\vartheta = \kappa = 1$, we let $\mathcal{HG}_{\Sigma}^{1,\,\delta,\,1}(\Phi) \equiv \mathcal{Q}_{\Sigma}(\delta,\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{Q}_{\Sigma}(\delta,\Phi)$, if

$$f'(z) + \delta z f''(z) \prec \Phi(t,z)$$

and

$$g'(w) + \delta w g''(w) \prec \Phi(t,w)$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

(vi) For $\kappa = 1$ and $\delta = 0$, we let $\mathcal{HG}_{\Sigma}^{1,\,0,\,\vartheta}(\Phi), \equiv \mathcal{F}_{\Sigma}(\vartheta,\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{F}_{\Sigma}(\vartheta,\Phi(t,z))$, if

$$(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) \prec \Phi(t,z)$$

and

$$(1-\vartheta)\frac{g(w)}{w}+\vartheta g'(w)\prec \Phi(t,w)$$

where $g(w)=f^{-1}(w)$ assumed as in (1.2).

(vii) For $\vartheta=1$, $\kappa=1$ and $\delta=0$, we have the class $\mathcal{HG}_{\Sigma}^{1,\,0,\,1}(\Phi)\equiv \mathcal{H}_{\Sigma}(\Phi)$. A function $f\in\Sigma$ as assumed in (1.1) is said to be in $\mathcal{H}_{\Sigma}(\Phi)$, if

$$f'(z)\prec\Phi(t,z)\quad \textit{and}\quad g'(w)\prec\Phi(t,w)$$

where $g(w)=f^{-1}(w)$ assumed as in (1.2).

In the following section , we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$ and its special cases. Also, Fekete-Szegö inequality for functions in this subclass.

3 Coefficient estimates and Fekete-Szegö inequality

In order to discuss coefficient estimates and Fekete-Szegö inequality for $f\in\mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$ we define $u(z)$ and $v(w)$ in $\mathcal{P}$ as

$$u(z)=c_1z+c_2z^2+\cdots \tag{3.1}$$

and

$$v(w)=d_1w+d_2w^2+\cdots \tag{3.2}$$

are analytic in $\mathbb{D}$ with $u(0)=0=v(0)$ and $|u(z)|<1$, $|v(w)|<1$, for all $z,w\in\mathbb{D}$. It is well-known that if

$$|u(z)|=|c_1z+c_2z^2+\cdots|<1 \text{ and } |v(w)|=|d_1w+d_2w^2+\cdots|<1, z,w\in\mathbb{D}, \tag{3.3}$$

then

$$|c_j|\le 1\qquad\text{and}\qquad |d_j|\le 1\quad\text{for all}\quad j\in\mathbb{N}. \tag{3.4}$$

We now prove our first result asserted by Theorem 3.1 below.

Theorem 3.1. . *Let f be assumed as in (1.1) and $f\in\mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$. Then*

$$|a_2|\le min\{F_1,F_2,F_3\},$$

where

$$F_1=\frac{2\lambda|t|}{(\vartheta+\kappa+2\delta)},\qquad F_2=\sqrt{\frac{4\lambda t+2\lambda[2(\lambda+1)t^2-1]}{(2\vartheta+\kappa)(\kappa+1)+12\delta}}$$

$$\textit{and}\quad F_3=\frac{2\lambda t\sqrt{2\lambda t}}{\sqrt{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\,2\lambda^2t^2-(\vartheta+\kappa+2\delta)^2(2(\lambda)_2t^2-\lambda)}}.$$

Also

$$|a_3|\le min\{G_1,G_2,G_3\}$$

where

$$G_1=\frac{2\lambda t(\vartheta+\kappa+2\delta)^2+4\lambda^2t^2\,(2\vartheta+\kappa+6\delta)}{(2\vartheta+\kappa+6\delta)\,(\vartheta+\kappa+2\delta)^2},$$

$$G_2=\frac{(4\lambda t+2\lambda[2(\lambda+1)t^2-1])\,(2\vartheta+\kappa+6\delta)+2\lambda t\,[(2\vartheta+\kappa)(\kappa+1)+12\delta]}{(2\vartheta+\kappa+6\delta)\,[(2\vartheta+\kappa)(\kappa+1)+12\delta]}$$

$$G_3=\frac{8\lambda^3t^3\,(2\vartheta+\kappa+6\delta)+2\lambda t\left[[(2\vartheta+\kappa)(\kappa+1)+12\delta]\,2\lambda^2t^2-(\vartheta+\kappa+2\delta)^2(2(\lambda)_2t^2-\lambda)\right]}{(2\vartheta+\kappa+6\delta)\,[[(2\vartheta+\kappa)(\kappa+1)+12\delta]\,2\lambda^2t^2-(\vartheta+\kappa+2\delta)^2(2(\lambda)_2t^2-\lambda)]}.$$

Proof. Since $f \in \mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$, from the Definition 2.1, we have

$$\begin{aligned} & (1-\vartheta)\left(\frac{f(z)}{z}\right)^{\kappa} + \vartheta f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} + \delta z f''(z) \\ = \;& 1 + \mathfrak{G}_1^{\lambda}(x)u(z) + \mathfrak{G}_2(x)u^2(z) + \cdots, \\ = \;& 1 + 2\lambda t c_1 z + \{2\lambda t c_2 + \left[\frac{(\lambda)_2}{2!}(2t)^2 - \lambda\right] c_1^2\} z^2 + \cdots, \end{aligned} \tag{3.5}$$

and for $g = f^{-1}$

$$\begin{aligned} & (1-\vartheta)\left(\frac{g(w)}{w}\right)^{\kappa} + \vartheta g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} + \delta w g''(w) \\ = \;& 1 + \mathfrak{G}_1^{\lambda}(x)v(w) + \mathfrak{G}_2(x)v^2(w) + \cdots, \\ = \;& 1 - 2\lambda t d_1 z + \{2\lambda t d_2 + \left[\frac{(\lambda)_2}{2!}(2t)^2 - \lambda\right] d_1^2\} z^2 + \cdots, \end{aligned} \tag{3.6}$$

where $z,\ w \in \mathbb{D}$. Now by equating corresponding coefficients of z and z^2 in (3.5) and (3.6), we have

$$(\vartheta + \kappa + 2\delta)\, a_2 = 2\lambda t c_1, \tag{3.7}$$

$$(2\vartheta + \kappa)\left[\left(\frac{\kappa - 1}{2}\right) a_2^2 + \left(1 + \frac{6\delta}{2\vartheta + \kappa}\right) a_3\right] = 2\lambda t c_2 + \left[\frac{(\lambda)_2}{2!}(2t)^2 - \lambda\right] c_1^2, \tag{3.8}$$

$$-(\vartheta + \kappa + 2\delta)\, a_2 = 2\lambda t d_1, \tag{3.9}$$

and

$$(2\vartheta + \kappa)\left[\left(\frac{\kappa + 3}{2} + \frac{12\delta}{2\vartheta + \kappa}\right) a_2^2 - \left(1 + \frac{6\delta}{2\vartheta + \kappa}\right) a_3\right] = 2\lambda t d_2 + \left[\frac{(\lambda)_2}{2!}(2t)^2 - \lambda\right] d_1^2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$c_1 = -d_1,$$

and

$$2\left(\vartheta + \kappa + 2\delta\right)^2 a_2^2 = 4\lambda^2 t^2 (c_1^2 + d_1^2) \tag{3.11}$$

$$a_2^2 = \frac{2\lambda^2 t^2 (c_1^2 + d_1^2)}{\left(\vartheta + \kappa + 2\delta\right)^2}. \tag{3.12}$$

Now, by (3.4), we obtain

$$|a_2| \le \frac{2\lambda |t|}{(\vartheta + \kappa + 2\delta)} \tag{3.13}$$

By adding (3.8) and (3.10), we have

$$[(2\vartheta + \kappa)(\kappa + 1) + 12\delta] a_2^2 = 2\lambda t\,(c_2 + d_2) + \left[\frac{(\lambda)_2}{2!}(2t)^2 - \lambda\right](c_1^2 + d_1^2). \tag{3.14}$$

Again by (3.4), we obtain,

$$a_2 \le \sqrt{\frac{4\lambda t + 2\lambda[2(\lambda + 1)t^2 - 1]}{(2\vartheta + \kappa)(\kappa + 1) + 12\delta}}. \tag{3.15}$$

Now, by substituting (3.11) in (3.14), we reduce that

$$a_2^2 = \frac{4\lambda^3 t^3 (c_2 + d_2)}{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)}. \tag{3.16}$$

Now, by (3.4), we obtain

$$|a_2| \leq \frac{2\lambda t\sqrt{2\lambda t}}{\sqrt{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)}}. \tag{3.17}$$

By subtracting (3.10) from (3.8), we obtain

$$a_3 = \frac{\lambda t (c_2 - d_2)}{2\vartheta+\kappa+6\delta} + a_2^2. \tag{3.18}$$

Hence by(3.4), we have

$$|a_3| \leq \frac{\lambda t (|c_2| + |d_2|)}{2\vartheta+\kappa+6\delta} + |a_2|^2 = \frac{2\lambda t}{2\vartheta+\kappa+6\delta} + |a_2|^2 .$$

In view of (3.11), we get

$$|a_3| \leq \frac{2\lambda t (\vartheta+\kappa+2\delta)^2 + 4\lambda^2 t^2\, (2\vartheta+\kappa+6\delta)}{(2\vartheta+\kappa+6\delta)\, (\vartheta+\kappa+2\delta)^2}.$$

By using (3.15)

$$|a_3| \leq \frac{(4\lambda t + 2\lambda[2(\lambda+1)t^2 - 1])\, (2\vartheta+\kappa+6\delta) + 2\lambda t\, [(2\vartheta+\kappa)(\kappa+1)+12\delta]}{(2\vartheta+\kappa+6\delta)\, [(2\vartheta+\kappa)(\kappa+1)+12\delta]}.$$

Then in view of (3.16), we obtain

$$|a_3| \leq \frac{8\lambda^3 t^3\, (2\vartheta+\kappa+6\delta) + 2\lambda t\, [[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)]}{(2\vartheta+\kappa+6\delta)\, [[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)]}.$$

Similarly, we can prove the following theorem. □

Theorem 3.2. *For $\nu \in \mathbb{R}$, let f be given by (1.1) and $f \in \mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$,then*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{4\lambda t}{2\vartheta+\kappa+6\delta} & ; \quad 0 \leq |h(\nu)| \leq \dfrac{2\lambda t}{2\vartheta+\kappa+6\delta} \\ 2\,|h(\nu)| & ; \quad |h(\nu)| \geq \dfrac{2\lambda t}{2\vartheta+\kappa+6\delta} \end{cases}$$

where $h(\nu) = \dfrac{4\lambda^3 t^3 (1-\nu)}{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)}.$

Proof. From (3.18), we have

$$a_3 - \nu a_2^2 = \frac{\lambda t (c_2 - d_2)}{2\vartheta+\kappa+6\delta} + (1-\nu)\, a_2^2. \tag{3.19}$$

By substituting (3.16) in (3.19), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{2\lambda t\, (c_2 - d_2)}{2\vartheta+\kappa+6\delta} + \frac{4\lambda^3 t^3 (1-\nu)(c_2+d_2)}{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\, 2\lambda^2 t^2 - (\vartheta+\kappa+2\delta)^2 (2(\lambda)_2 t^2 - \lambda)} \\ &= \left(h(\nu) + \frac{2\lambda t}{2\vartheta+\kappa+6\delta}\right) c_2 + \left(h(\nu) - \frac{2\lambda t}{2\vartheta+\kappa+6\delta}\right) d_2, \end{aligned} \tag{3.20}$$

where

$$h(\nu)=\frac{4\lambda^3t^3(1-\nu)}{[(2\vartheta+\kappa)(\kappa+1)+12\delta]\,2\lambda^2t^2-(\vartheta+\kappa+2\delta)^2(2(\lambda)_2t^2-\lambda)}. \tag{3.21}$$

Thus by taking modulus of (3.20),

$$|a_3-\nu a_2^2| \quad\leq\quad \left|h(\nu)+\frac{2\lambda t}{2\vartheta+\kappa+6\delta}\right|+\left|h(\nu)-\frac{2\lambda t}{2\vartheta+\kappa+6\delta}\right|,$$

and $h(\nu)$ is given by (3.21).

Thus, we conclude that

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\dfrac{4\lambda t}{2\vartheta+\kappa+6\delta} & ; 0\leq|h(\nu)|\leq\dfrac{2\lambda t}{2\vartheta+\kappa+6\delta}\\ 2\,|h(\nu)| & ; |h(\nu)|\geq\dfrac{2\lambda t}{2\vartheta+\kappa+6\delta}\end{cases} \tag{3.22}$$

where $h(\nu)$ is given by (3.21). □

By taking $\nu=1$ in above Theorem one can easily state the following:

Remark 3.3. Let f be given by (1.1) and $f\in\mathcal{HG}_{\Sigma}^{\kappa,\,\delta,\,\vartheta}(\Phi)$. Then

$$\left|a_3-a_2^2\right|\leq\frac{4\lambda t}{2\vartheta+\kappa+6\delta}.$$

4 Subclass of Bi-univalent function $\mathcal{M}_{\Sigma}^{\tau}(\Phi)$

In [19], Obradovic et.al gave some criteria for univalence expressing by $\Re(f'(z))>0$, for the linear combinations

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)}>0,\qquad(\tau\geq1,z\in\mathbb{U}).$$

Based on the above definitions recently, in [12], Lashin introduced and studied the new subclasses of bi-univalent function.

Definition 4.1. A function $f(z)\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_{\Sigma}^{\tau}(\Phi)$ if it satisfies the following conditions :

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)}\prec\Phi(t,z) \tag{4.1}$$

and

$$\tau\left(1+\frac{zg''(w)}{g'(w)}\right)+(1-\tau)\frac{1}{g'(w)}\prec\Phi(t,w) \tag{4.2}$$

where $\tau\geq1$, $z,w\in\mathbb{D}$ and the function g is given by (1.2).

Remark 4.2. For a function $f(z)\in\Sigma$ given by (1.1),is said to be in the class $\mathcal{M}_{\Sigma}^{1}(\Phi)\equiv\mathcal{K}_{\Sigma}(\Phi)$ if it satisfies the following conditions :

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\Phi(t,z)\ \text{ and }\ \left(1+\frac{wg''(w)}{g'(w)}\right)\prec\Phi(t,w)$$

where $z,w\in\mathbb{D}$ and the function g is given by (1.2).

Theorem 4.3. *Let $f(z)$ be given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{\tau}(\Phi)$, and $\tau\geq1$. Then*

$$|a_2|\leq\min\begin{cases}\frac{\lambda t}{(2\tau-1)},\\ \frac{\lambda t\sqrt{2}}{(2\tau-1)}\\ \frac{2\lambda t\sqrt{\lambda t}}{\sqrt{2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]}}\end{cases} \tag{4.3}$$

and

$$|a_3| \leq \min \begin{cases} \frac{2\lambda t}{3(3\tau-1)} + \frac{\lambda^2 t^2}{(2\tau-1)^2}, \\ \frac{2\lambda t}{3(3\tau-1)} + \frac{2\lambda^2 t^2}{(2\tau-1)^2} \\ \frac{2\lambda t\left(2(1+\tau)\lambda^2 t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]\right)+12\lambda^3 t^3(3\tau-1)}{3(3\tau-1)(2(1+\tau)\lambda^2 t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda])}. \end{cases} \tag{4.4}$$

Proof. It follows from (4.1) and (4.2) that

$$\tau\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\tau)\frac{1}{f'(z)}=\Phi(t,u(z)) \tag{4.5}$$

and

$$\tau\left(1+\frac{zg''(w)}{g'(w)}\right)+(1-\tau)\frac{1}{g'(w)}=\Phi(t,v(w)). \tag{4.6}$$

From (4.5) and (4.6), we have

$$\begin{aligned} &1+2(2\tau-1)a_2 z+\left[3(3\tau-1)a_3+4(1-2\tau)a_2^2\right]z^2+\cdots \\ &\quad=1+2\lambda t c_1 z+\{2\lambda t c_2+\left[\frac{(\lambda)_2}{2!}(2t)^2-\lambda\right]c_1^2\}z^2+\cdots, \end{aligned}$$

and

$$\begin{aligned} &1-2(2\tau-1)a_2 w+\left(2(5\tau-1)a_2^2-3(3\tau-1)a_3\right)w^2-\cdots \\ &\quad=1-2\lambda t d_1 z+\{2\lambda t d_2+\left[\frac{(\lambda)_2}{2!}(2t)^2-\lambda\right]d_1^2\}z^2+\cdots,. \end{aligned}$$

Now, equating the coefficients, we get

$$2(2\tau-1)a_2=2\lambda t c_1, \tag{4.7}$$

$$\left[3(3\tau-1)a_3+4(1-2\tau)a_2^2\right]=2\lambda t c_2+\left[\frac{(\lambda)_2}{2!}(2t)^2-\lambda\right]c_1^2, \tag{4.8}$$

$$-2(2\tau-1)a_2=2\lambda t d_1, \tag{4.9}$$

and

$$\left(2(5\tau-1)a_2^2-3(3\tau-1)a_3\right)=2\lambda t d_2+\left[\frac{(\lambda)_2}{2!}(2t)^2-\lambda\right]d_1^2. \tag{4.10}$$

From (4.7) and (4.9), we get

$$c_1=-d_1 \tag{4.11}$$

and by(3.4) in (4.7) ,

$$|a_2|\leq\frac{\lambda t}{(2\tau-1)}. \tag{4.12}$$

Also

$$\begin{aligned} (2\tau-1)^2 a_2^2 &= \lambda^2 t^2(c_1^2+d_1^2) \\ a_2^2 &= \frac{\lambda^2 t^2(c_1^2+d_1^2)}{(2\tau-1)^2} \end{aligned} \tag{4.13}$$

Thus by(3.4), we get

$$|a_2|\leq\frac{\lambda t\sqrt{2}}{(2\tau-1)}. \tag{4.14}$$

Now from (4.8), (4.10) and using (4.13), we obtain

$$\left(2(1+\tau)\lambda^2 t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]\right)a_2^2=2\lambda^3 t^3(c_2+d_2). \tag{4.15}$$

Thus,by we obtain

$$|a_2| \leq \frac{2\lambda t\sqrt{\lambda t}}{\sqrt{2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]}}.$$

From (4.8) from (4.10) and using(4.11), we get

$$a_3 = \frac{2\lambda t(c_2-d_2)}{6(3\tau-1)} + a_2^2. \tag{4.16}$$

Then taking modulus and using (3.4), we get

$$|a_3| \leq \frac{2\lambda t}{3(3\tau-1)} + |a_2^2|. \tag{4.17}$$

Using (4.12) and (4.14), we get

$$|a_3| \leq \frac{2\lambda t}{3(3\tau-1)} + \frac{\lambda^2t^2}{(2\tau-1)^2}$$

and

$$|a_3| \leq \frac{2\lambda t}{3(3\tau-1)} + \frac{2\lambda^2t^2}{(2\tau-1)^2}.$$

Now by using(4.15) in(4.17) ,

$$\begin{aligned}|a_3| &\leq \frac{4\lambda t}{6(3\tau-1)} + |a_2^2| \\ &= \frac{2\lambda t}{3(3\tau-1)} + \frac{4\lambda^3t^3}{(2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda])} \\ &= \frac{2\lambda t\left(2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]\right)+12\lambda^3t^3(3\tau-1)}{3(3\tau-1)\left(2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]\right)}.\end{aligned}$$

□

Theorem 4.4. *For $\nu \in \mathbb{R}$, let f be given by (1.1) and $f \in \mathcal{M}_\Sigma^\tau(\Phi)$,then*

$$\left|a_3-\nu a_2^2\right| \leq \begin{cases} \dfrac{2\lambda t}{3(3\tau-1)} & ; 0 \leq |h(\nu)| \leq \dfrac{\lambda t}{3(3\tau-1)} \\ 2\,|h(\nu)| & ; |h(\nu)| \geq \dfrac{\lambda t}{3(3\tau-1)} \end{cases}$$

where $h(\nu) = \dfrac{2\lambda^3t^3(1-\nu)}{2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]}$.

Proof. From (4.16), we have

$$a_3-\nu a_2^2 = \frac{\lambda t(c_2-d_2)}{3(3\tau-1)} + (1-\nu)\,a_2^2. \tag{4.18}$$

By substituting (4.15) in (4.18), we have

$$\begin{aligned} a_3-\nu a_2^2 &= \frac{\lambda t(c_2-d_2)}{3(3\tau-1)} + \frac{2\lambda^3t^3(1-\nu)(c_2+d_2)}{(2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda])} \\ &= \left(h(\nu)+\frac{\lambda t}{3(3\tau-1)}\right)c_2 + \left(h(\nu)-\frac{\lambda t}{3(3\tau-1)}\right)d_2, \end{aligned} \tag{4.19}$$

where

$$h(\nu) = \frac{2\lambda^3t^3(1-\nu)}{2(1+\tau)\lambda^2t^2-(2\tau-1)^2[2t^2(\lambda)_2-\lambda]}. \tag{4.20}$$

Thus by taking modulus of (4.19),

$$|a_3 - \nu a_2^2| \leq \left|h(\nu) + \frac{\lambda t}{3(3\tau-1)}\right| + \left|h(\nu) - \frac{\lambda t}{3(3\tau-1)}\right|,$$

where $h(\nu)$ is given by (4.20). Thus, we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{2\lambda t}{3(3\tau-1)} & ; 0 \leq |h(\nu)| \leq \dfrac{\lambda t}{3(3\tau-1)} \\ 4\,|h(\nu)| & ; |h(\nu)| \geq \dfrac{\lambda t}{3(3\tau-1)} \end{cases} \tag{4.21}$$

where $h(\nu)$ is given by (4.20). □

By taking $\nu = 1$ in above Theorem one can easily state the following:

Remark 4.5. Let f be given by (1.1) and $f \in \mathcal{M}_{\Sigma}^{\tau}(\Phi)$. Then

$$\left|a_3 - a_2^2\right| \leq \frac{2\lambda t}{3(3\tau-1)}.$$

Concluding Remark: By fixing $\lambda = \frac{1}{2}$ one can get the new analogues results for the subclasses discussed in this article based on Legendre polynomials further by taking $\lambda = 1$ we get the results related with Chebyshev polynomials for the function classes given in Definitions 2.1and 4.1. Further, suitably specificating the parameters as mentioned in Remark 2.2 and 4.2,one can easily obtain upper bounds for the coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $|a_3 - \nu a_2^2|$ for function classes illustrated in Remark 2.2 and 4.1.Therefore, we believe that this research will encourage many researchers to expand the concept of meromorphic bi-univalent functions and also to define a new class of Σ based on quantum calculus operator[24] (see documentation on this) and certain special functions

References

[1] D.A. Brannan, J. Clunie and W.E. Kirwan, *Coefficient estimates for a class of star-like functions,* Canad. J. Math., vol.22, pp.476-485, 1970.

[2] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions,* Studia Univ. Babes-Bolyai Math., vol. 31, no.2, pp.70-77, 1986.

[3] P.L. Duren, *Univalent Functions. In: Grundlehren der Mathematischen Wissenschaften, Band 259,* New York, Berlin, Heidelberg and Tokyo, Springer-Verlag, 1983.

[4] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions,* Appl. Math. Lett.,vol.24, pp.1569-1573, 2011.

[5] H.Ö.Güney, G. Murugusundaramoorthy, and H.M. Srivastava, *The second Hankel determinant for a certain class of bi-close-to-convex functions,* Results Math 2019, vol.74,Article ID 93, 2019.

[6] H.Ö.Güney, G. Murugusundaramoorthy, K.Vijaya and K.Thilagavathi, *Coefficient bounds for certain subclasses of bi-prestarlike functions associated with the Chebyshev polynomials* Mathematica Moravica ,Vol. 24,No. 2 pp.71–82, 2020.

[7] D.J. Hallenbeck, *Convex hulls and extreme points of families of starlike and close-to-convex mappings,* Pacific J. Math. 57, pp.167–176, 1975.

[8] Jay M. Jahangiri and S. G. Hamidi, *Advances on the coefficients of bi-prestarlike functions,*C. R. Acad. Sci. Paris, Ser. I 354 ,pp. 980-985, 2016.

[9] D. S. Kim,, T. Kim and S.H. Rim, *Some identities involving Gegenbauer polynomials.* Adv. Differ. Equ., 2012, 219 (2012). https://doi.org/10.1186/1687-1847-2012-219

[10] K.Kiepiela,I.Naraniecka and J.Szynal, *The Gegenbauer polynomials and typically real functions,* J.Comp.& Appl.Math,vol.153, no.(1-2),pp 273-282, 2003.

[11] B. Kowalczyk, A. Lecko, and H. M. Srivastava, *A note on the Fekete-Szegö problem for close-to-convex functions with respect to convex functions,* Publ. Inst. Math. (Beograd) (Nouvelle Ser.),vol. 101,no.115,pp.143-149,2017.

[12] A.Y.Lashin, *Coefficients estimates for two subclasses of analytic and bi-univalent functions*, Ukranian Mathematical Journal Vol. 70, no 9, pp.1484-1492, 2019.

[13] M. Lewin,*On a coefficient problem for bi-univalent functions,* Proc. Amer. Math. Soc., vol.18, pp.63-68, 1967.

[14] X-F. Li and A-P Wang, *Two new subclasses of bi-univalent functions,* Inter.Math. Forum, vol.7, no.30, pp.1495-1504, 2012.

[15] G. Murugusundaramoorthy and S.Yalçın, *On λ Pseudo bi-starlike functions related (p; q)-Lucas polynomial,* Libertas Mathematica (new series) vol 39, no 2 pp.59-77,2019.

[16] G. Murugusundaramoorthy, H.Ö.Güney and K.Vijaya, *Coefficient bounds for certain suclasses of bi-prestarlike functions associated with the Gegenbauer polynomial* Adv.Stud.Contemp.Math.**32** (2022)5–15. http://dx.doi.org/10.17777/ascm2022.32.1.5

[17] G. Murugusundaramoorthy, T. Bulboaca, *Subclasses of Yamakawa-type Bi-starlike functions associated with Gegenbauer polynomials*, Axioms **11**.3 (2022), 92.

[18] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$. Arch. Ration. Mech. Anal., vol.32, pp. 100-112, 1969.

[19] M.Obradovic, T.Yaguchi and H.Saitoh, *On some conditions for univalence and starlikeness in the unit disc,* Rend.Math.Ser.VII.12, pp.869–877, 1992.

[20] M.S. Robertson *On the coefficients of typically-real functions* Bull. Amer. Math. Soc.,vol 41 , pp. 565-572,1935.

[21] H.M. Srivastava, A.K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions,* Appl. Math. Lett., vol.23, no.10, pp.1188-1192, 2010.

[22] H.M.Srivastava, S.S.Eker,S.G.Hamidi, J.M.Jahangiri,*Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator,* Bull. Iran. Math. Soc., vol 44, pp.149-157, 2018.

[23] H.M.Srivastava, A.K.Wanas and G.Murugusundaramoorthy, *Certain family of bi-univalent functions associated with Pascal distribution series based on Horadam polynomials*. Surveys Math. Appl., **16** (2021), 193–205.

[24] H.M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. A Sci., **44**(2020), 327-–344.

[25] E. M. Stein G. Weiss,*Introduction to Fourier Analysis in Euclidean Space,* Princeton University Press,1971

[26] J. Szynal, $\widehat{\text{A}}$n extension of typically-real functions, Annales UMCS vol48, pp.193–201, 1994.

[27] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava,*Coefficinet estimates for a certain subclass of analytic and bi-univalent functions,* Appl. Math. Lett., vol.25, pp.990-994, 2012.

[28] P. Zaprawa, *On the Fekete-Szegö problem for classes of bi-univalent functions,*Bull. Belg. Math. Soc. Simon Stevin,vol.21, no.1, pp. 169-178, 2014.

Author information

Gangadharan Murugusundaramoorthy, Department of Mathematics , School of Advanced Sciences, Vellore Institute of Technology(VIT), Vellore - 632014, INDIA.
E-mail: `gmsmoorthy@yahoo.com; gms@vit.ac.in`

Hatun Özlem Güney, Faculty of Science, Department of Mathematics,
Dicle University, TR-21280 Diyarbakır., TURKEY.
E-mail: `ozlemg@dicle.edu.tr`

Received: 2021-12-06
Accepted: 2023-06-22