ON STRESS OF A VERTEX IN A GRAPH

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Abstract The stress of a vertex in a graph is the number of geodesics passing through it (A. Shimbel, 1953). A graph is k-stress regular if stress of each of its vertices is k. In this paper, we investigate some results and compute stress of vertices in some standard graphs and give a characterization of graphs with all vertices of zero stress except for one. Also we compute stress of vertices in graphs of diameter 2 and in the corona product $K_m \circ G$. Further we prove that any strongly regular graph is stress regular and characterize k-stress regular graphs for $k = 0, 1, 2$.

1 Introduction

For standard terminology and notion in graph theory, we follow the text-book of Harary [3].

Let $G = (V, E)$ be a graph (finite, undirected, simple). The number of edges in a path P is its length $l(P)$. A shortest path between two vertices u and w in G is called a $u - w$ geodesic. We say that a geodesic P is passing through a vertex v in G if v is an internal vertex of P. The length of a longest geodesic in G is called the diameter of G, denoted by $d(G)$. Eccentricity $e(v)$ of a vertex v denotes the distance between v and a vertex farthest from v. For any vertex v, its open neighborhood $N_G(v)$ (or simply $N(v)$) is the set of all vertices which are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v is called a simplicial vertex if $N(v)$ induces a complete subgraph. A maximal connected subgraph of G is a component of G. A cut-vertex of G is the one whose removal increases the number of components of G. A nonseparable graph is connected, nontrivial and has no cut-vertex. A block of G is a maximal nonseparable subgraph. A graph is said to be vertex transitive if the automorphism group of G acts transitively on $V(G)$. We denote the minimum of the degrees of vertices of G by $\delta(G)$. The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 with disjoint vertex sets is defined as the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ number of copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2.

The concept of stress of a vertex in a graph was defined by Alfonso Shimbel [8] in 1953. The concept has many applications in the study of biological networks, social networks etc. Some related works can be found in [9], [7], [4], [6] and [5]. In this paper, we study some of the properties of this concept.

In section 2, followed by definitions, we make some observations, characterize zero stress vertices and give a formula for the total stress of a graph. In section 3, we compute stress of vertices in some standard graphs. In section 4, we give a characterization of graphs with all vertices of zero stress except for one. In section 5, we study stress imposing and stress neutral vertices. We obtain formulae for computing stress of vertices in graphs of diameter 2 and in the corona product $K_m \circ G$. Also we prove that strongly regular graphs are stress regular. In section 6, we characterize k-stress regular graphs for $k = 1, 2$.

2 Stress of a vertex

Definition 2.1 (Alfonso Shimbel[8]). Let G be a graph and v be a vertex in G. The stress of v, denoted by $\text{str}_G(v)$ or simply $\text{str}(v)$, is defined as the number of geodesics in G passing through v.

Definition 2.2 (S. Arumugam). A graph is said to be k-stress regular if all of its vertices have stress k.

Definition 2.3. Let $G = (V, E)$ be a graph. The total stress of G, denoted by $N_{str}(G)$, is defined as,

$$N_{str}(G) = \sum_{v \in V} str(v).$$

Few Observations:

 (i) For any vertex v in a graph G, $0 \leq str(v) \leq N$, where N is the number of geodesics of length at least 2 in G.

 (ii) If there is no geodesic of length at least 2 passing through a vertex v in a graph G, then $str(v) = 0$. Hence for any vertex v in a complete graph K_n, we have $str(v) = 0$.

(iii) If η is an automorphism of a graph G and v is any vertex in G, then $str(v) = str(\eta(v))$. Hence it follows that any vertex transitive graph is stress regular. However the converse is not true. The graph in the Figure 2 is a 2-stress regular graph, but it is not vertex transitive since it is not even regular. The same example shows that a stress regular graph may not be regular.

(iv) A regular graph may not be stress regular. The graph in the Figure 1 is a regular graph of degree 3, but it is not stress regular. (The numbers written near the vertices indicate the stress of the corresponding vertex.)

Figure 1. A regular graph which is not stress regular

Theorem 2.4. *Let G be any graph and let v be any vertex in G. Then $str(v) = 0$ if and only if v is a simplicial vertex.*

Proof. Let $str(v) = 0$. If two neighbors u and w of v are not adjacent, then uvw is a geodesic passing through v, which will give a stress on v, a contradiction. If v is a pendant vertex, clearly the result holds.

Conversely, assume the given condition. Let P be a $u - w$ path in G through v. Then P contains a $u - w$ sub-path of length one less than that of P, since any two neighbors of v are adjacent. Thus P cannot be a geodesic through v. Hence there are no geodesics passing through v and thus $str(v) = 0$. □

The following corollary directly follows from the above theorem.

Corollary 2.5. *A connected graph is 0-stress regular if and only if it is a complete graph.*

Proposition 2.6. *For any graph G with diameter d, the total stress of G, is given by*

$$N_{str}(G) = \sum_{i=1}^{d} (i - 1) f_i,$$

where f_i is the number of geodesics of length i in G.

Proof. We know that, in a graph

 (i) every geodesic of length k has $k - 1$ internal vertices;

 (ii) every geodesic contributes one count to the stress of each of its internal vertices.

Since the length of the longest geodesic in G is d, the result follows from (i) and (ii). □

3 Stress of vertices in some standard graphs

The results in the following proposition follow by direct computations.

Proposition 3.1. *(i) In a complete bipartite graph K_{mn}, if A and B are the partite sets with $|A| = m$ and $|B| = n$, then*

$$str(v) = \begin{cases} \dfrac{n(n-1)}{2}, & if\ v \in A; \\ \dfrac{m(m-1)}{2}, & if\ v \in B. \end{cases}$$

Consequently, $N_{str}(K_{m,n}) = \dfrac{mn}{2}(m+n-2)$.

(ii) For every vertex v in a cycle C_n on n vertices,

$$str(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8}, & if\ n\ is\ odd; \\ \dfrac{n(n-2)}{8}, & if\ n\ is\ even. \end{cases}$$

Thus,

$$N_{str}(C_n) = \begin{cases} \dfrac{n(n-1)(n-3)}{8}, & if\ n\ is\ odd; \\ \dfrac{n^2(n-2)}{8}, & if\ n\ is\ even. \end{cases}$$

(iii) Let $Wd(n,m)$ denote the windmill graph [2] constructed for $n \geq 2$ and $m \geq 2$ by joining m copies of the complete graph K_n at a shared universal vertex v. Then

$$str(v) = m(m-1)(n-1)^2/2$$

and the rest of the vertices have zero stress. Hence

$$N_{str}(Wd(n,m)) = m(m-1)(n-1)^2/2.$$

Proposition 3.2. *Let v be an internal vertex of a tree T and let $C_1, \ldots, C_m$ be the components of $T - v$. Then $str(v) = \sum_{i<j} |C_i||C_j|$.*

Proof. Since T is a tree, if v_i and v_j are any two vertices in two different components C_i and C_j of $T - v$, then there is a unique path P in T joining them. Clearly v must be an internal vertex of P. Hence the pair (v_i, v_j), $i < j$ contributes exactly 1 to the stress count of v. Thus the result follows. □

4 Some characterizations

Theorem 4.1. *Let $G = (V, E)$ be a connected graph with at least 3 vertices. The graph G has all vertices of zero stress except for one if and only if G is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of G.*

Proof. Suppose that the graph G has all vertices of zero stress except for one vertex, say v. Define a relation $\sim$ on $V - \{v\}$ as follows: for $u, w \in V - \{v\}$, $u \sim w \iff u \in N[w]$. We show that $\sim$ is an equivalence relation on $V - \{v\}$. Clearly $\sim$ is reflexive and symmetric.

We prove the transitivity of $\sim$: Suppose $u \sim w$ and $w \sim x$ in $V - \{v\}$. we need to show that $u \sim x$. If $u = x$, then clearly $u \sim x$. If $u \neq x$, we need to show that u and x are adjacent. If either u or x is equal to w, then u and x are adjacent. Assume that $u \neq w$ and $x \neq w$. If u and x are not adjacent, then uwx is a geodesic of length 2 passing through w, so that $str(w) > 0$. But then $w \neq v$ contradicts the hypothesis. Thus u and x are adjacent in G. So $u \sim x$.

Thus $\sim$ is an equivalence relation on $V - \{v\}$. Hence $\sim$ partitions the set $V - \{v\}$ into disjoint equivalence classes, say $V_1, \ldots, V_k$. Note that $k > 1$. If $k = 1$, then $N(v) \subset V_1$, and so $N(v)$ induces a complete subgraph of G; hence $str(v) = 0$, which is a contradiction. Clearly, by the

definition of $\sim$, each V_i induces a complete subgraph of G and the induced subgraphs $\langle V_i \rangle$ and $\langle V_j \rangle$ for $i \neq j$ are mutually disjoint (vertex disjoint) subgraphs of G.

We claim that v is adjacent to all the vertices in G. Let $u \in V - \{v\}$. Then $u \in V_i$ for exactly one i. If v is not adjacent to u, then there is a geodesic of length 2 from v to u passing through a vertex w in V_i (because G is connected and V_i induces a complete subgraph of G). This implies $\mathrm{str}(w) > 0$, which is not possible. Hence v is adjacent to u. This proves the claim.

Now its clear that the removal of v from G disconnects the graph into components $\langle V_1 \rangle, \ldots, \langle V_k \rangle$, where $k > 1$. So v is a cut-vertex in G. There is no other cut-vertex in G, because each vertex u different from v belongs to the complete subgraph induced by $V_i \cup \{v\}$ for exactly one i. Since v is adjacent to all other vertices in G, $V_i \cup \{v\}$ induces a block that is a complete subgraph of G, for each i.

Conversely, suppose that G is a graph with a unique cut-vertex v such that all its blocks are complete. Clearly, there are at least 2 blocks. Let u be a vertex in G, $u \neq v$. Then u belong to a block in G. Since each block in G is a complete subgraph, it follows that the neighbors of u induce a complete subgraph of G and so $\mathrm{str}(u) = 0$. Let B_1 and B_2 be two distinct blocks in G. Let x be a vertex in B_1 and y be a vertex in B_2. Clearly, x and y are not adjacent. Since v is the unique cut-vertex in the connected graph G, every path between x and y must pass through v and so there is a geodesic between x and y passing through v. Hence $\mathrm{str}(v) > 0$.   □

Corollary 4.2. *Let G be a connected graph with $n + 1$ vertices. Then $G \cong K_{1n}$ if and only if G has exactly one vertex of stress $n(n-1)/2$, with the stress of remaining vertices being zero.*

Proof. The direct part follows by computation. Conversely, let $v, w_1, \ldots, w_n$ be the vertices of G, where we assume $\mathrm{str}(v) = n(n-1)/2$ and $\mathrm{str}(w_i) = 0$ for all i. From the proof of the direct part of the Theorem 4.1, it follows that $\deg(v) = n$. Now if $\deg(w_i) > 1$ for some i, then there exists a $j \neq i$ such that w_i and w_j are adjacent. Thus there is no stress on v due to the pair (w_i, w_j) and hence, the maximum possible stress on v is $(n-2) + (n-2) + (n-3) + (n-4) + \cdots + 2 + 1$ which is less than $n(n-1)/2$, a contradiction. Hence $\deg(w_i) = 1$ for all i. Thus $G \cong K_{1n}$.   □

5   Stress imposing and stress neutral vertices

Definition 5.1. A vertex u in a graph G is called a stress imposing vertex with respect to a vertex v if there exists a vertex w and a $u - w$ geodesic P passing through v. A vertex u which is not stress imposing with respect to v is called a stress neutral vertex with respect to v. A pair (x, y) of vertices is said to impose a stress on v if there exists a $x - y$ geodesic passing through v.

Lemma 5.2. *Let u and v be any two vertices in a connected graph G. Then u is a stress imposing vertex with respect to v if and only if there exists $w \in N(v)$ such that $d(u, w) = d(u, v) + 1$. Consequently u is stress neutral with respect to v if and only if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$.*

Proof. Suppose u imposes a stress on v. Then there exists a vertex w and a $u - w$ geodesic P passing through v. Let x be the vertex in P which lies immediately next to v. Then $x \in N(v)$ and $d(u, x) = d(u, v) + 1$.

Conversely suppose there exists $w \in N(v)$ such that $d(u, w) = d(u, v) + 1$. Let P be a $u - v$ geodesic. Then the $u - w$ path P' formed using P followed by w is a $u - w$ geodesic, which passes through v and hence u imposes a stress on v.   □

Corollary 5.3. *For any vertex v in a graph G of diameter 2, $\mathrm{str}(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$.*

Proof. Since $d(G) = 2$, if u imposes a stress on v, it follows from the Lemma 5.2 that, $u \in N(v)$. Also if $u, w \in N(v)$, then the pair (u, w) imposes exactly one stress on v if and only if u and w are non-adjacent. Hence the result follows.   □

Let G be a regular graph with v vertices and degree k. G is said to be strongly regular [1] if there exist integers λ and μ such that any two adjacent vertices have λ common neighbors and any two non-adjacent vertices have μ common neighbors. We write $G = \mathrm{srg}(v, k, \lambda, \mu)$ in this case.

Corollary 5.4. *Any strongly regular graph $G = srg(v, k, \lambda, \mu)$ is stress regular.*

Proof. If $\mu = 0$, then it a well-known result that G is a disjoint union of one or more equal-sized complete graphs and hence G is stress regular by Corollary 2.5.

If $\mu \neq 0$, then by definition, it follows that $d(G) = 2$. Hence using the Corollary 5.3, for any vertex v in G, we have $str(v) = k(k - 1 - \lambda)/2$. □

Remark 5.5. (i) The converse of the Corollary 5.4 is not true. The graph in the Figure 2 is a 2-stress regular graph, but it is not strongly regular.

(ii) For each natural number k, there is k-stress regular graph. For example the graph $C_{2k+2}^{k} = srg(2k + 2, 2k, 2k - 2, 2k)$ is k-stress regular, by Corollary 5.4.

Proposition 5.6. *Let G be any graph with n vertices and let G' be the corona product $K_m \circ G$ of the complete graph K_m, $m \geq 2$ and G. Let $v \in V(G')$. If v lies in the copy of K_m, then*

$$str(v) = \frac{mn(m - 1)(n + 1)}{2}$$

and if v lies in a copy of G, then $str(v)$ equals the number of unordered pairs of non-adjacent vertices in $N_G(v)$.

Proof. The first part follows by computation. For the second, let $H_1, \ldots, H_m$ be the components of $G' - V(K_m)$ (the graph obtained from G' by deleting the vertices in the copy of K_m). Note that $H_i \cong G$, for all i. Let $w \in V(H_i)$ for some i. Note that if $u \in V(G') - V(H_i)$, then $d_{G'}(u, x) = d_{G'}(u, w)$ for all $x \in N_{G'}(w)$, so all such vertices u are stress neutral with respect to w, by Lemma 5.2. Now, note that if $x, y \in V(H_i)$, then $d_{G'}(x, y) \leq 2$ and hence any vertex $x \in V(H_i) - N_G(v)$, must be stress neutral with respect to w, by Lemma 5.2. Hence it follows that $str(w)$ equals the number of unordered pairs of non-adjacent vertices in $N_G(w)$. □

6 Stress regular graphs

Lemma 6.1. *Let G be a k-stress regular connected graph with $k \geq 1$ and $d(G) = 2$. Let n be the smallest positive integer satisfying $k \leq \binom{n}{2}$. Then $\delta(G) \geq n$.*

Proof. Suppose $d(v) = m < n$ for some vertex v. Since $d(G) = 2$, the $str(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$ and hence $str(v) \leq \binom{m}{2} < k$ since $m < n$ and by the choice of n. The contradiction proves the result. □

Lemma 6.2. *Let G be any graph. Then for any vertex v of a geodesic P in G, we have $str_P(v) \leq str_G(v)$. Let $k = max\{str(v) : v \in V(G)\}$. If n is a positive integer such that $\lfloor n/2 \rfloor \lceil n/2 \rceil > k$, then all geodesics in G are of length less than n.*

Proof. The first part follows by noting that any sub-path of a geodesic in G is also a geodesic in G. For the second part, suppose P is a geodesic in G with $m = l(P) \geq n$. If w is a vertex in the center of P, then $str_P(w) = \lfloor m/2 \rfloor \lceil m/2 \rceil \geq \lfloor n/2 \rfloor \lceil n/2 \rceil > k$. By the first part, we have $str_P(w) \leq str_G(w)$. Hence $str_G(w) > k$, a contradiction. Hence the result follows. □

Lemma 6.3. *Let G be a connected graph which is not complete and let v, w be any two vertices in G, with $e(v) = 1$. Then $str(w) \leq str(v)$, where equality holds if and only if $e(w) = 1$. Further if $str(v) \geq 1$, then there exist a vertex u in G for which $str(u) < str(v)$.*

Proof. Since $e(v) = 1$ and G is connected, it follows that v must be adjacent to all the vertices in G. Now since G is not complete, we must have $d(G) = 2$. Hence whenever a pair (x, y) of vertices impose a stress on w, they impose a stress on v too. Hence $str(w) \leq str(v)$. Now suppose $str(w) = str(v)$. If w is not adjacent to a vertex u in G, then the pair (u, w) imposes a stress on v, but not on w. Consequently $str(w) < str(v)$, a contradiction. Thus $e(w) = 1$. Conversely if $e(w) = 1 = e(v)$, then $str(v) \leq str(w) \leq str(v)$ from the first part and so the equality.

For the second part, since $str(v) \geq 1$, there exist a pair (u, x) of non adjacent vertices in G. Then $e(u) > 1$. So using the first part, we have $str(u) < str(v)$. □

Corollary 6.4. *Lt G be a k-stress regular connected graph with $k \geq 1$. Then $e(v) > 1$ for any vertex v in G.*

Proof. Since $k \geq 1$, by Corollary 2.5, it follows that G is not complete. Further G is connected implies $e(v) \geq 1$ for any vertex v. Suppose $e(v) = 1$ for some vertex, then by the second part of the Lemma 6.3, there exists a vertex w such that $\text{str}(w) < \text{str}(v)$, contradicting the assumption. Hence the result. □

Theorem 6.5. *A connected graph G is 1-stress regular if and only if it is isomorphic to the cycle C_4 or C_5.*

Proof. The converse follows from Proposition 3.1(ii). We need only prove the direct part.

Let G be a connected graph which is 1-stress regular. By Lemma 6.2, we observe that all geodesics in G are of length less than 3. Hence $e(v) \leq 2$ for all $v \in V(G)$. But by Corollary 6.4, we have $e(v) > 1$ for all $v \in V(G)$. Hence we must have $e(v) = 2$ for all $v \in V(G)$. Also by Lemma 6.1, we must have $d(v) \geq 2$ for all $v \in V(G)$. We claim that $d(v) = 2$ for all $v \in V(G)$.

Suppose $d(v) > 2$ for some vertex v of G. Since $d(G) = 2$, only the vertices in the neighborhood of v can impose stress on v, by Lemma 5.2. Now since $\text{str}(v) = 1$, there exist two vertices u and w in the neighborhood of v which are not adjacent to each other. Since $d(v) > 2$, we can find another vertex a which is adjacent to v. But then a must be stress neutral, so we must have that a is adjacent to every vertex in the neighborhood of v. Now since $e(v) = 2$, we can find a vertex x which is at a distance 2 from v. If x is adjacent to a, then there will be at least two geodesics passing through a, namely uaw and xav, which is not possible since G is 1-stress regular. Similarly x cannot be adjacent to any other vertex in the neighborhood of v other than u and w. Hence x must be adjacent to either u or to w, say to u. But then there will be at least two different geodesics passing through u, namely xuv and xua. Again this is not possible. Hence $d(v) = 2$ for all $v \in V(G)$.

Thus G is a connected 2-regular graph and hence G must be a cycle. Now by Proposition 3.1(ii), it follows that G must be isomorphic to either C_4 or C_5. □

Theorem 6.6. *A connected graph G is 2-stress regular if and only if it is isomorphic to one of the graphs given by the Figures 2 or 3 or 4.*
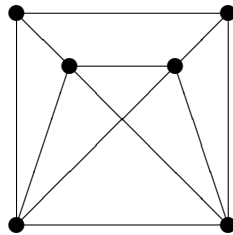

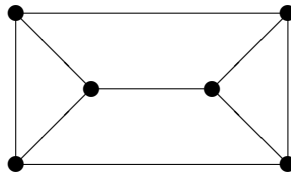**Figure 2.** A 2-stress regular graph which is not regular

Figure 3. A 2-stress regular graph which is 3 regular

Proof. The converse follows from direct computations. We prove the direct part.

Let G be a connected graph which is 2-stress regular. By Lemma 6.2, we observe that all geodesics in G are of length less than 4. We claim that G does not contain a geodesic of length 3.
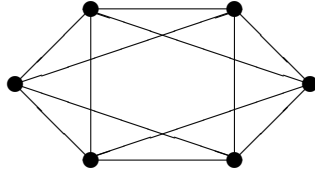
Figure 4. A 2-stress regular graph which is 4 regular

Suppose $F : uabv$ is a geodesic of length 3 in G. Note that $\text{str}_F(a) = \text{str}_F(b) = 2$. We claim that $d(a) = d(b) = 2$. For suppose a vertex x is adjacent to a, where $x \neq u, b$. Then, since G is 2-stress regular and there are two geodesics passing through a already, we must have that x must be stress neutral with respect to a. Then by Lemma 5.2, it follows that x must be adjacent to both u and b. But again, by similar argument, we must have that x must be adjacent to v. But then uxv is a path of length 2 joining u and v, contradicting the fact that F is a $u - v$ geodesic. Thus we must have $d(a) = 2$. Similarly $d(b) = 2$. Now u cannot be adjacent to v and u cannot be a pendant vertex in G too. So we can find a vertex $x \neq a$ such that x is adjacent to u. Note that since F is a $u - v$ geodesic, x cannot be adjacent to v. Now since $d(b) = 2$, x cannot be adjacent to b. So it follows that $xuab$ is a $x - b$ geodesic. But then there will be at least 3 different geodesics passing through a, a contradiction. Thus G does not contain a geodesic of length 3.

Hence all geodesics in G are of length less than 3. Hence $e(v) \leq 2$ for all $v \in V(G)$. But by Corollary 6.4, we have $e(v) > 1$ for all $v \in V(G)$. Hence we must have $e(v) = 2$ for all $v \in V(G)$. Also by Lemma 6.1, we must have $d(v) \geq 3$ for all $v \in V(G)$. We claim that $d(v) \leq 4$ for all $v \in V(G)$.

Suppose $d(v) > 4$ for some vertex v of G, say $d(v) = 5$. Since $e(u) = 2$ for all vertices u in G, only the vertices in the neighborhood of v can impose stress on v, by Lemma 5.2.

Suppose there is no vertex in the neighborhood of v imposing more than one stress on v, then the arrangement of vertices in the neighborhood of v is as shown in Figure 5, where a is not adjacent to b and c is not adjacent to d. Consider a vertex x of G which is at a distance 2 from v. If x is adjacent to u, then aub, cud and xuv will be 3 distinct geodesics passing through u, a contradiction. Hence x cannot be adjacent to u. So x must be adjacent to one of the other 4 vertices, say to a. Then cad, xav and xau will be 3 distinct geodesics passing through a, a contradiction. Similar contradictions are obtained if x is adjacent to b or c or d.
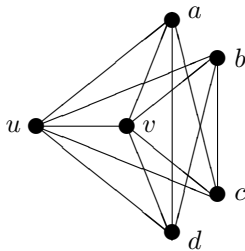

Figure 5. $d(v) = 5$ and no vertex imposes more than one stress on v

Suppose there is a vertex in the neighborhood of v imposing more than one stress on v, then the arrangement of vertices in the neighborhood of v is as shown in Figure 6, where x is not adjacent to y and z. Consider a vertex u of G which is at a distance 2 from v. If u is adjacent to a, then xay, xaz and uav will be 3 distinct geodesics passing through a, a contradiction. Hence u cannot be adjacent to a. Similarly, u cannot be adjacent to b. So u must be adjacent to one of the other 3 vertices, say to x. Then uxa, uxb and uxv will be 3 distinct geodesics passing through x, a contradiction. Similar contradictions are obtained if u is adjacent to y or to z.

Thus we cannot have $d(v) = 5$. Similarly $d(v) \geq 6$ is not possible. Hence it follows that $d(v) \leq 4$ for all $v \in V(G)$. Now we complete the proof by considering Case 1 and Case 2 below:
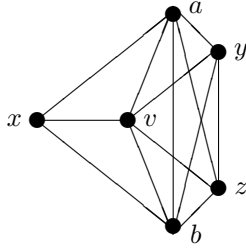
Figure 6. $d(v) = 5$ and there is a vertex imposing more than one stress on v

Case 1. Suppose $d(v) = 4$ for some vertex v in G. Let the vertices adjacent to v be a, b, c, d. Suppose none of these impose more than one stress on v. Then the arrangement of vertices in the neighborhood of v is as shown in Figure 7, where a is not adjacent to c and b is not adjacent to d.
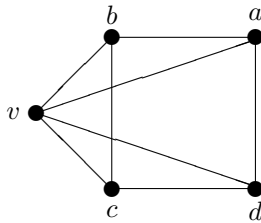

Figure 7. $d(v) = 4$ and no vertex imposes more than one stress on v

Let u be a vertex in G which is at a distance 2 from v. Then u must be adjacent to either a or b or c or d, say to a. Now since uav and bad are 2 distinct geodesics passing through a and since $\text{str}_G(a) = 2$, we must have that u must be adjacent to both b and d, by Lemma 5.2. But then ubv and abc will be 2 distinct geodesics through b and since $\text{str}_G(b) = 2$, we must have that u must be adjacent to c, by Lemma 5.2. Hence we obtain a 2-stress regular subgraph H of G which is isomorphic to the graph given by the Figure 4. Note that, since G is connected and $d(v) \leq 4$ for all $v \in V(G)$, it follows that $V(G) = V(H)$. Clearly H is the maximal subgraph with the vertex set $V(H)$. Hence we must have $G = H$.

Now suppose one of the vertices in the neighborhood of v, say a, imposes more than one stress on v. Then a must impose a stress count of 2 on v and after some relabeling, we may assume that the arrangement of vertices in the neighborhood of v is as shown in the Figure 8, where a is not adjacent to b and c.
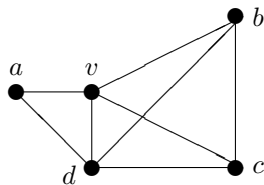

Figure 8. $d(v) = 4$ and there is a vertex imposing more than one stress on v

Since $e(v) = 2$, we can find a vertex u which is at a distance 2 from v. Note that u cannot be adjacent to d, since G does not contain a vertex of degree 5. We prove that u must be adjacent to a, b and c. Note that if u is adjacent to either b or c, then by Lemma 5.2, it must be adjacent to both.

Suppose u is adjacent to a and that it is not adjacent to both b and c. Since all geodesics in G are of length less than 3, we can find a $u-b$ geodesic of length 2, say uxb. Note that $x \neq a, v, d, c$. Further, since G does not contain any vertex of degree 5, x cannot be adjacent to v and d. By

Lemma 5.2, it follows that x must be adjacent to c. Also note that, since there are 2 distinct geodesics passing through a namely uav and uad, it follows that x cannot be adjacent to a. The situation is shown in the Figure 9.
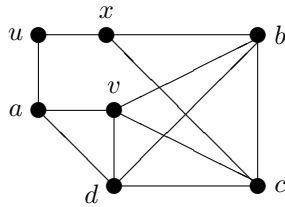

Figure 9. A figure to illustrate a situation in Case 1

Since $3 \le d(p) \le 4$ for every vertex p in G, and since $\text{str}_G(u) = 2$, we can find a vertex y in the neighborhood of u different from a and x, having the property that y is adjacent to either a or x, but not both. Since u is not adjacent to v, b, c, d, we have that $y \ne v, b, c, d$. If y is adjacent to a, then by Lemma 5.2, it follows that y must be adjacent to v and to d, which is not possible, since $d(v) = 4$. If y is adjacent to x, then by Lemma 5.2, it follows that y must be adjacent to b and to c, which is not possible, since $d(b) = 4$. This contradiction proves that if u is adjacent to a, then it must be adjacent to both b and c.

Suppose, on the other hand, u is adjacent to b and hence to c. We prove that u must be adjacent to a. For otherwise, since all geodesics in G are of length less than 3, we can find a $u - a$ geodesic of length 2, say uxa. Clearly $x \ne v, b, c, d$ and x cannot be adjacent to v, b, c, d and the situation is similar to the one in Figure 9, with the roles of x and a interchanged and hence will lead to a contradiction. Hence u must be adjacent to a.

Hence, in any case, u must be adjacent to a, b and c. Thus we obtain a 2-stress regular subgraph K of G which is isomorphic to the graph given by the Figure 2. Note that if G contains any other vertex x, then since G is connected, there must exist a vertex y which is adjacent to at least one vertex in K. But then, by Lemma 5.2, it follows that y must be adjacent to each vertex in K and hence G will contain a vertex of degree 5, a contradiction. Hence we have $V(G) = V(K)$. Clearly K is the maximal subgraph with the vertex set $V(K)$. Hence it follows that $G = K$.

Case 2. Suppose $d(v) = 3$ for all $v \in V(G)$. Let p be any vertex in G. Let the points adjacent to p be a, b and c. Since $\text{str}(p) = 2$, we must have that exactly one pair of vertices among a, b, c are adjacent to each other. Let b and c be adjacent to each other and a be non-adjacent to both b and c. Let v and w be the other two neighbors of a. Since $d(p) = 3$, it follows that p must be non-adjacent to both v and w and hence, v must be adjacent to w, since $\text{str}(a) = 2$. The arrangement of vertices is as shown in the Figure 10. Note that all the vertices shown are distinct.
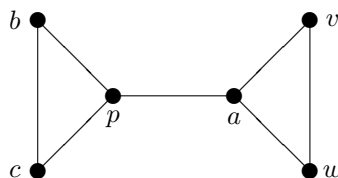

Figure 10. A figure to illustrate a situation in Case 2

Now we claim that if b is adjacent to v, then c must be adjacent to w. On the contrary, suppose b is adjacent to v and c is not adjacent to w. Now since all geodesics in G are of length less than 3, there must be a $c - w$ geodesic of length 2, say cxw. The present situation is shown in the Figure 11.

Note that $x \ne p$ since p is not adjacent to w and $x \ne a$ since c cannot be adjacent to a. Suppose $x = v$, then c would be adjacent to v, but then v will have degree 4, a contradiction.
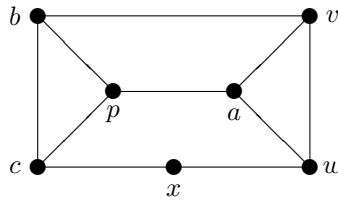
Figure 11. A figure to illustrate a situation in Case 2

Hence $x \neq v$. Similarly $x \neq b$. Further, since $d(u) = 3$ for all $u \in V(G)$, we have that x cannot be adjacent to p, a, b, v. Let $y \neq c, w$ be the other neighbor of x. Since x is not adjacent to p, a, b, v, we have $y \neq p, a, b, v$. Now since str$(x) = 2$, by Lemma 5.2 it follows that y must be adjacent to either c or to w, both of which are not possible, since $d(c) = d(w) = 3$. Hence if b is adjacent to v, then c must be adjacent to w. Similarly, if c is adjacent to w, then b must be adjacent to v. Thus we get that b is adjacent to v if and only if c is adjacent to w. Similarly b is adjacent to w if and only if c is adjacent to v.

Now we prove that b is adjacent to either v or to w. Suppose not. Then from the above part, it follows that c cannot be adjacent to v and w. Now since, all geodesics in G are of length less than 3, there must be a $b - v$ geodesic of length 2, say bxv. The present situation is as shown in the Figure 12.
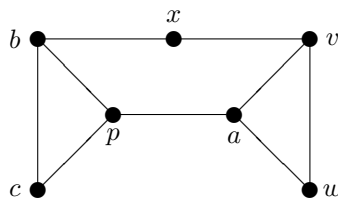

Figure 12. A figure to illustrate a situation in Case 2

As earlier we have $x \neq p$ and $x \neq a$. Further, since c is not adjacent to v, we have $x \neq c$ and since b is not adjacent to w, we have $x \neq w$. Again, since b and w are not adjacent, it follows that there must be a $b - w$ geodesic, say P of length 2. Now since $d(b) = 3$ and since w is not adjacent to c and p, it follows that $P = bxw$ so that x and w are adjacent. But then, since c is not adjacent to any of the neighbors of w, it follows that there is no path of length less than 3 between c and w, which is a contradiction.

Hence b is adjacent to either v or w. But since $d(b) = 3$, it cannot be adjacent to both. We may assume b is adjacent to v. But then, from the earlier part, it follows that c must be adjacent to w and hence we obtain a 2-stress regular subgraph H of G which is isomorphic to the graph given by the Figure 3. Since $d(u) = 3$ for all $u \in V(G)$ and G is connected, it follows that $V(G) = V(H)$. Clearly H is the maximal subgraph with the vertex set $V(H)$. Hence we must have $G = H$. $\square$

Acknowledgments

References

[1] R.C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* **13**, 389–419 (1963).

[2] J.A. Gallian, A Dynamic Survey of Graph Labeling, *The Electron. J. Combin.* **DS6**, 1–535 (2019).

[3] F. Harary, *Graph theory*, Narosa Publishing house, New Delhi (1969).

[4] M. Indhumathy, S. Arumugam, V. Baths and T. Singh, Graph Theoretic Concepts in the Study of Biological Networks, *Proceedings in Mathematics & Statistics : Applied Analysis in Biological and Physical Sciences*, **186**, Springer, 187–200 (2016).

[5] C. Kang, C. Molinaro, S. Kraus, Y. Shavitt, V. Subrahmanian, Diffusion centrality in social networks, *Proceedings of the 2012 international conference on advances in social networks analysis and mining (ASONAM 2012)*, IEEE Computer Society, 558–564 (2012).

[6] D. Koschützki, K.A. Lehmann, L. Peeters, S. Richter, D. Tenfelde-Podehl, O. Zlotowsk, *Centrality Indices, in Network Analysis*, Lecture Notes in Computer Science, **3418**, Springer, Berlin, Heidelberg, 16–61 2005).

[7] G. Scardoni, M. Petterlini, C. Laudanna, Analyzing Biological Network Parameters with CentiScaPe., *Bioinformatics*. **25**, 2857–2859 (2009) .

[8] A. Shimbel, Structural Parameters of Communication Networks, *Bulletin of Mathematical Biophysics*. **15**, 501–507 (1953) .

[9] K. Shinozaki, Y.S. Kazuko, S. Motoaki, Regulatory Network of Gene Expression in the Drought and Cold Stress responses, *Curr. Opin. Plant Biol.* **6**(5), 410–417 (2003).

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