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ONE MODULO N-DIFFERENCE MEAN GRAPHS

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Abstract In this paper, we introduce a new labeling namely one modulo N-difference mean labeling. We establish that the graphs such as $B_{m,n}$, $S_{m,n}$, $P_n@P_m$, $B(l,m,n)$, $T(n,m)$, shrub, caterpillar and $K_{1,n}$ are one modulo N-difference mean graph. In addition we show that the graph C_3 is not a one modulo N-difference mean graph.

1 Introduction

Here we consider only finite and simple graphs.The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey on graph labeling is available in [2].The concept of mean labeling was introduced by Somasundaram et.al[5]. A graph $G = (V, E)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps V to $\{0, 1, 2, ..., q\}$ such that each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u) + f(v)$ is odd and the resulting labels of the edges are distinct and are $1, 2, 3, ..., q$. Several papers have been published on mean labeling and its variations.
The concept of one modulo N graceful labeling was introduced by Ramachandran et.al[4]. A function f is called a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function φ from the vertex set of G to $\{0, 1, N, (N + 1), 2N, (2N + 1), ..., N(q - 1), N(q - 1) + 1\}$ in such a way that (i) φ is 1-1 (ii)φ induces a bijection φ^* from the edge set of G to $\{1, N + 1, 2N + 1, ..., N(q - 1) + 1\}$ where $\varphi^*(uv) = |\varphi(u) - \varphi(v)|$.

The concept of one modulo three geometric mean labeling was introduced by Maheswari et.al[3]. A graph G is said to be one modulo three geometric mean graph if there is an injective function φ from the vertex set of G to the set $\{a \mid 1 \leq a \leq 3q - 2$ and either $a \equiv 0(mod3)$ or $a \equiv 1(mod3)\}$, where q is the number of edges of G and φ induces a bijection φ^* from the edge set of G to $\{a \mid 1 \leq a \leq 3q - 2$ and $a \equiv 1(mod3)\}$ given by $\varphi^*(uv) = \left\lceil \sqrt{\varphi(u)\varphi(v)} \right\rceil$ or $\left\lfloor \sqrt{\varphi(u)\varphi(v)} \right\rfloor$ and the function φ is called one modulo three geometric mean labeling of G.

Motivated by the concept and the results in[3,4], we introduce a new labeling called one modulo N-difference mean labeling. A graph $G = (p, q)$ is said to be one modulo N-difference mean graph if there is an injective function f from the vertex set of G to the set $\{a/0 \leq a \leq 2(q - 1)N + 1$ and either $a \equiv 0(modN)$ or $a \equiv 1(modN)\}$, where N is a positive integer and f induces a bijection f^* from the edge set of G to $\{a/1 \leq a \leq (q - 1)N + 1$ and $a \equiv 1(modN)\}$ given by $f^*(uv) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the function f is called a one modulo N-difference mean labeling of G. A graph that admits a one modulo N-difference mean labeling is called a one modulo N-difference mean graph. The bistar $B_{m,n}$ is obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ by an edge. The graph $S_{m,n}$ has n spokes in which each spoke is a path of length m. The graph $P_n@P_m$ is obtained by identifying one pendant vertex of the path P_m to each a vertex of the path P_n. The graph $B(l,m,n)$ is obtained by joining the centers of the stars $K_{1,l}$ and $K_{1,m}$ by a path of length n. The coconut tree $T(n,m)$ is obtained by identifying the apex vertex of the star $K_{1,m}$ with a pendant vertex of the path P_n. In this paper, we show that the graphs such as $B_{m,n}$, $S_{m,n}$, $P_n@P_m$, $B(l,m,n)$, $T(n,m)$, shrub, caterpillar and $K_{1,n}$ are one modulo N-difference mean graph. In addition we show that the graph C_3 is not a one modulo N-difference mean graph.

2 One Modulo N-difference Mean Graphs

Theorem 2.1. *Any path $P_n(n \geq 1)$ is a one modulo N-difference mean graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the path P_n.
Define $f : V(P_n) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(n-2)+1\}$ as follows:
$$f(v_i) = \begin{cases} N(i-1) & \text{if } 1 \leq i \leq n, \ i \text{ is odd} \\ N(2n-i-2)+1 & \text{if } 2 \leq i \leq n, \ i \text{ is even.} \end{cases}$$
Let $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$.
The corresponding edge label f^* is
$f^*(e_i) = N(n-i-1)+1$ for $1 \leq i \leq n-1$.
Therefore P_n admits a one modulo N-difference mean labeling. □

A one modulo 5- difference mean labeling of P_6 is given in Figure 1.

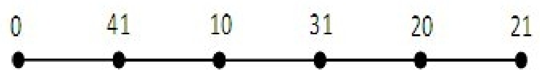

Figure 1.

Theorem 2.2. *The star graph $K_{1,n}$ admits a one modulo N-difference mean labeling.*

Proof. Let v_0, v_i $(1 \leq i \leq n)$ be the vertices of $K_{1,n}$ and $E(K_{1,n}) = \{v_0 v_i : 1 \leq i \leq n\}$. Then $K_{1,n}$ has $n+1$ vertices and n edges.
Define $f : V(K_{1,n}) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(n-1)+1\}$ by
$f(v_0) = 0$, $f(v_i) = 2N(i-1)+1$ for $1 \leq i \leq n$.
Let $e_i = v_0 v_i$ for $1 \leq i \leq n$.
Then the label is $f^*(e_i) = N(i-1)+1$ for $1 \leq i \leq n$.
Therefore $K_{1,n}$ is a one modulo N-difference mean graph. □

A one modulo 9- difference mean labeling of $K_{1,8}$ is given in Figure 2.

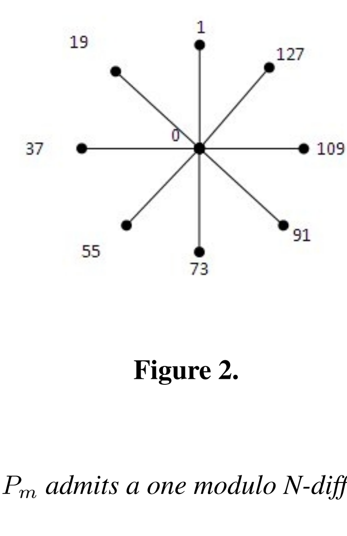

Figure 2.

Theorem 2.3. *The graph $P_n@P_m$ admits a one modulo N-difference mean labeling.*

Proof. Let v_i $(1 \leq i \leq n)$ be the vertices of P_n and u_i^j $(1 \leq i \leq n, 1 \leq j \leq m)$ be the vertices of P_m. $P_n@P_m$ is obtained by identifying the vertices v_i with u_i^1. Then $E(P_n@P_m) = \{v_iv_{i+1}(1 \leq i \leq n-1), u_i^ju_i^{j+1}(1 \leq i \leq n, 1 \leq j \leq n-1\}$.
Define $f : V(P_n@P_m) \to \{0, 1, N, N+1, 2N, 2N+1, 4N, 4N+1, ..., 2N(mn-2)+1\}$ as follows:

$$f(v_i) = \begin{cases} Nm(i-1) & \text{if } 1 \leq i \leq n, \ i \text{ is odd} \\ N[2(mn-1) - mi] + 1 & \text{if } 1 \leq i \leq n, \ i \text{ is even.} \end{cases}$$

For odd i and odd j,
$f(u_i^j) = Nm(i-1) + N(j-1)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.
For odd i and even j,
$f(u_i^j) = N[m(2n-i+1) - 4] - N(j-2) + 1$ for $1 \leq i \leq n$ and $2 \leq j \leq m$.
For even i and odd j,
$f(u_i^j) = N[m(2n-i) + j - 3] + 1$ for $2 \leq i \leq n$ and $1 \leq j \leq m$.
For even i and even j,
$f(u_i^j) = N(mi - j)$ for $2 \leq i \leq n$ and $2 \leq j \leq m$.
The corresponding edge labels are given below:
Let $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$ and $e_i^j = u_i^ju_i^{j+1}$ for $1 \leq i \leq n$ and $1 \leq j \leq m-1$.
$f^*(e_i) = N[m(n-i) - 1] + 1$ for $1 \leq i \leq n-1$.
For odd i, $f^*(e_i^j) = N[(n-i+1)m - j - 1] + 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m-1$,
For even i, $f^*(e_i^j) = N[(n-i)m + j - 1] + 1$ for $2 \leq i \leq n$ and $1 \leq j \leq m-1$.
It establishes that $P_n@P_m$ admits a one modulo N-difference mean labeling. □

A one modulo 3- difference mean labeling of $P_5@P_4$ is given in Figure 3.

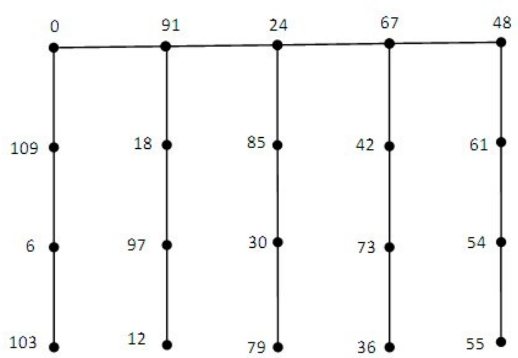

Figure 3.

Theorem 2.4. *The bistar $B_{m,n}$ is a one modulo N-difference mean graph.*

Proof. Let $V(B_{m,n}) = \{u_0, v_0, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and
$E(B_{m,n}) = \{u_0v_0, u_0u_i, v_0v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.
Define $f : V(B_{m,n}) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(m+n)+1\}$ by
$f(u_0) = 0$,
$f(v_0) = 2Nn + 1$,
$f(u_i) = 2N(m+n+1-i) + 1$ for $1 \leq i \leq m$,
$f(v_j) = 2Nj$ for $1 \leq j \leq n$.
The induced edge label f^* is given below:
$f^*(u_0v_0) = Nn + 1$
$f^*(u_0u_i) = (m+n-i+1)N + 1$ for $1 \leq i \leq m$,
$f^*(v_0v_i) = N(n-j) + 1$ for $1 \leq j \leq n$.
Hence $B_{m,n}$ admits a one modulo N-difference mean labeling. □

A one modulo 2- difference mean labeling of $B_{4,5}$ is given in Figure 4.

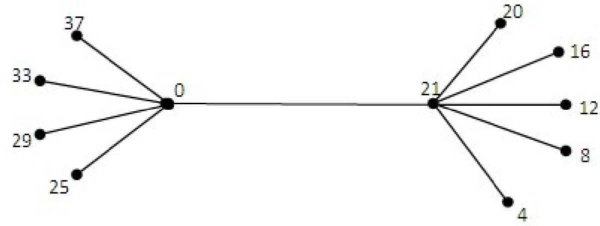

Figure 4.

Theorem 2.5. *The graph $B(l,m,n)$ is a one modulo N-difference mean graph.*

Proof. Let $V(B(l,m,n)) = \{v_1, v_2, ..., v_n, v_1^j, v_n^k : 1 \leq j \leq l, 1 \leq k \leq m\}$
and $E(B(l,m,n)) = \{v_1 v_1^j, v_n v_n^k, v_i v_{i+1} : 1 \leq i \leq n-1, 1 \leq j \leq l, 1 \leq k \leq m\}$.
Define $f : V(B(l,m,n)) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(m+l+n-2)+1\}$ by
$$f(v_i) = \begin{cases} N(i-1) & \text{if } 1 \leq i \leq n, \ \ i \text{ is odd} \\ N[2(n+m-1)-i]+1 & \text{if } 2 \leq i \leq n, \ \ i \text{ is even.} \end{cases}$$
$f(v_1^j) = 2N(l+m+n-2) - 2N(j-1) + 1$ for $1 \leq j \leq l$,
For n odd, $f(v_n^k) = N(n+2k-3)+1$ for $1 \leq k \leq m$.
For n even, $f(v_n^k) = N(n+2k-2)$ for $1 \leq k \leq m$.
The induced edge label f^* is given below:
Let $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$, $e_j = v_1 v_1^j$ $(1 \leq j \leq l)$
and $e_k = v_n v_n^k$ $(1 \leq k \leq m)$.
$f^*(e_i) = N(n+m-i-1)+1$ for $1 \leq i \leq n-1$,
$f^*(e_j) = N(n+m+l-j-1)+1$ for $1 \leq j \leq l$,
For odd n, $f^*(e_k) = N(k-1)+1$ for $1 \leq k \leq m$,
For even n, $f^*(e_k) = N(m-k)+1$ for $1 \leq k \leq m$.
Hence $B(l,m,n)$ is a one modulo N-difference mean graph. □

A one modulo 2- difference mean labeling of $B(4,5,4)$ is given in Figure 5.

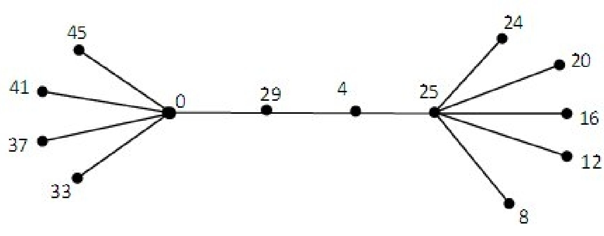

Figure 5.

Theorem 2.6. *The coconut tree $T(n,m)$ is a one modulo N-difference mean graph.*

Proof. Let $v_0, v_1, v_2, ..., v_n$ be the vertices of P_n having path length n $(n \geq 1)$. Let u be the central vertex and $u_1, u_2, ..., u_m$ be the pendant vertices of the star $K_{1,m}$. Let $T(n,m)$ be a tree obtained by identifying v_0 with u.
Define $f : V(T(n,m)) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(m+n-1)+1\}$ as
$f(v_0) = 0$,
$f(u_j) = 2N[m+n-j]+1$ for $1 \leq j \leq m$,

$$f(v_i) = \begin{cases} N(2n-i-1)+1 & \text{if } 1 \le i \le n,\ \ i \text{ is odd} \\ Ni & \text{if } 2 \le i \le n,\ \ i \text{ is even.} \end{cases}$$

Let $e_i = v_i v_{i+1}$ $(1 \le i \le n-1)$, $e_j = v_0 u_j$ $(1 \le j \le m)$.
$f^*(v_0 v_1) = N(n-1)+1$,
$f^*(e_i) = N(n-i-1)+1$ for $1 \le i \le n-1$,
$f^*(e_j) = N(m+n-j)+1$ for $1 \le j \le m$.
Therefore, the coconut tree $T(n,m)$ admits a one modulo N-difference mean labeling. □

A one modulo 3-difference mean labeling of $T(3,7)$ is given in Figure 6.

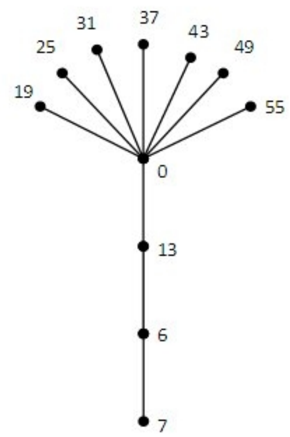

Figure 6.

Theorem 2.7. *The shrub graph constructed by joining the centers of different stars to a new vertex is a one modulo N-difference mean graph.*

Proof. Let $K_{1,m_1}, K_{1,m_2}, ..., K_{1,m_n}$ be n stars and let G be a graph obtained by joining the apex vertices of k stars to a new vertex v_0. We label the vertices as follows:
Let $m = m_1 + m_2 + ... + m_n$.
$f(v_0) = 0$,
$f(v_i) = 2N(m_1 + m_2 + ... + m_n + i - 1) + 1$ for $1 \le i \le n$,
$f(u_i^j) = 2N(m_1 + m_2 + ... + m_{i-1} + i + j - 1)$ for $1 \le i \le n$ and $1 \le j \le m_i$.
The edge label f^* is given below:
$f^*(v_0 v_i) = N(m_1 + m_2 + ... + m_n + i - 1) + 1$ for $1 \le i \le n$,
$f^*(v_1 u_1^j) = N(m-j) + 1$ for $1 \le j \le m_1$,
$f^*(v_i u_i^j) = N[m - j - (m_1 + m_2 + ... + m_{i-1})] + 1$ for $2 \le i \le n$ and $1 \le j \le m_i$.
Hence G is a one modulo N-difference mean graph. □

An example for the 3 modulo N- difference mean labeling of a shrub is given below.

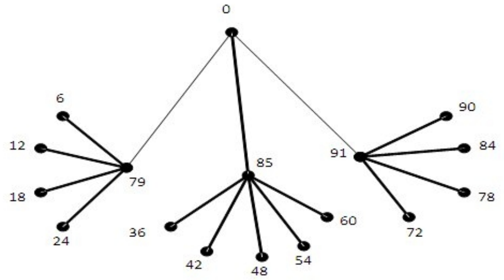

Figure 7.

Theorem 2.8. *The caterpillar $S(n_1, n_2, ..., n_m)$ is a one modulo N-difference mean graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the path P_n and u_i^j $(1 \leq i \leq n, 1 \leq j \leq m_i)$ be the pendant vertices joined with $v_i(1 \leq i \leq n)$ by an edge.
Let $m = m_1 + m_2 + ... + m_n$.
Then $V(S(n_1, n_2, ..., n_m)) = \{v_i, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m_i\}$
and $E(S(n_1, n_2, ...n_m)) = \{v_i v_{i+1}, v_i u_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m_i\}$.
Define $f : V(S(n_1, n_2, ..., n_m)) \to \{0, 1, N, N+1, 2N, 2N+1, ..., 2N(2m+n-2)+1\}$ as follows:
$f(v_1) = 0$,

$$f(v_i) = \begin{cases} 2N[m_2 + m_4 + ... + m_{i-1} + (\frac{i-1}{2})] & \text{if } 3 \leq i \leq n, \ i \text{ is odd} \\ 2N(m+n-2) + 1 - 2N[m_1 + m_3 + ... + m_{i-1} + (\frac{i-2}{2})] & \text{if } 2 \leq i \leq n, \ i \text{ is even.} \end{cases}$$

$f(u_1^j) = 2N(m+n-2) - 2N(j-1) + 1$ for $1 \leq j \leq m_1$,
$f(u_i^j) = 2N[(m+n-j-1) - [(m_1+1) + (m_3+1) + ... + (m_{i-2}+1)]] + 1$ for i is odd and $3 \leq i \leq n, 1 \leq j \leq m_i$,
$f(u_i^j) = 2N[j + (m_2+1) + (m_4+1) + ... + m_{i-2}+1)]$ for i is even and $2 \leq i \leq n, 1 \leq j \leq m_i$.
The corresponding edge label f^* is given below:
$f^*(v_i v_{i+1}) = N[m+n-i-1-(m_1+m_2+...+m_i)] + 1$ for $1 \leq i \leq n-1$,
$f^*(v_i u_i^j) = N[m+n-2-j+i-(m_1+m_2+...+m_i)] + 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$.
Hence $S(n_1, n_2, ..., n_m)$ is a one modulo N-difference mean graph. □

A one modulo 2-difference mean labeling of $S(4, 4, 4, 4)$ is shown in Figure 8.

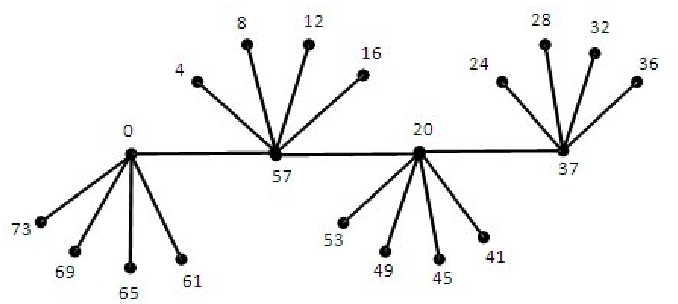

Figure 8.

Theorem 2.9. *The graph $S_{m,n}$ is a one modulo N-difference mean graph.*

Proof. Let v_0 be the root vertex. Denote the vertices of level 1 by $v_1^1, v_2^1, ..., v_m^1$ and the vertices of level 2 by $v_1^2, v_2^2, ..., v_m^2$. Now we can denote the vertices in level j by $v_1^j, v_2^j, ..., v_m^j$ for $1 \leq j \leq n$.
Define a one modulo N-difference mean labeling as follows:
$f(v_0) = 0$,
for $1 \leq i \leq m$,

$$f(v_i^j) = \begin{cases} 2N[mn-i] - Nm(j-1) + 1 & \text{if } 1 \leq j \leq n, \ j \text{ is odd} \\ N[4(m-j) + 3j - 4] & \text{if } 2 \leq j \leq n, \ j \text{ is even.} \end{cases}$$

Let $e_i = v_0 v_i^1$ $(1 \leq i \leq m)$ and $e_i^j = v_i^j v_i^{j+1}$ for $1 \leq i \leq m, 1 \leq j \leq n-1$.
The corresponding edge label f^* is given below:
$f^*(e_i) = Nmn - Ni + 1$ for $1 \leq i \leq m$,
$f^*(e_i^j) = Nmn - N[m(j+1) - i] - N + 1$ for $1 \leq i \leq m, 1 \leq j \leq n-1$.
Then $S_{m,n}$ admits a one modulo N-difference mean labeling. □

A one modulo 2-difference mean labeling of $S_{3,4}$ is shown in Figure 9.

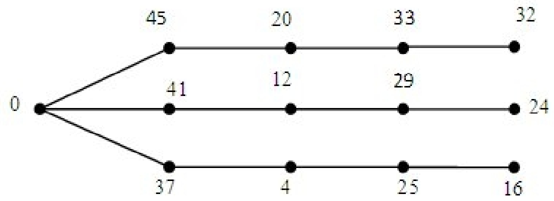

Figure 9.

Theorem 2.10. *The graph C_3 is not a one modulo N-difference mean graph.*

Proof. Suppose C_3 is a one modulo N-difference mean graph with one modulo N-difference mean labeling f. Let u, v, w be the vertices of C_3. Assume $f(u) = 0$ and $f(v) = 4N + 1$, so that we get the edge label $2N + 1$. To get 1 as an edge label, $f(w) \in \{1, 2, 4N, 4N - 1\}$. In the above cases, N+1 cannot occur as an edge label of C_3. Hence proved. □

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