Palestine Journal of Mathematics

Vol. 12(3)(2023) , 170–179

ON WEAVING $PG-$FRAMES

S.K. Sharma, Shashank Goel and S.K. Kaushik

Communicated by Ayman Badawi

MSC 2010 Classifications: 42C30, 42C15, 42C40, 92C55.

Keywords and phrases: Frames, generalized frames, weaving frames, weaving $p-$frames, weaving $g-$frames, distributed signal processing.

The authors are thankful to the referee(s) for their helpful comments, suggestions and constructive remarks towards the improvement of the paper.

Abstract Recently, a new application of frame theory in Hilbert spaces has been emerged related to distributed signal processing. Bemrose et al. [4] have developed a theory of weaving frames to handle such types of problems. In this article, we introduce and study woven $pg-$frames for Banach spaces. It has been shown that a family of $pg-$Bessel sequences is always woven Bessel family. The image of woven $pg-$frame under a bounded invertible operator is a woven $pg-$frame has been proved. Also, a necessary and sufficient condition for the image of woven $pg-$frame under a bounded operator to be a woven $pg-$frame has been given. Further, some characterizations of woven $pg-$ frames and characterization of woven $pg-$Bessel sequences are given. Furthermore, woven $qg-$Riesz bases are defined and prove that these are particular cases of woven $pg-$frames. Finally, some equivalent conditions for woven $qg-$Riesz basis are given.

1 Introduction

Duffin and Schaeffer [12] introduced the concept of frames in Hilbert spaces while studying the problems of non-harmonic Fourier series. They gave the following definition of frames in Hilbert spaces:

Definition 1.1. A family of vectors $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$, if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2,\ \forall\, x \in \mathcal{H}.$$

Later, in 1986, Daubechies et al. [10] reintroduced frames and observed that frames can be used to approximate functions in $L^2(\mathbb{R})$. These days theory of frames become an integral and important tool to study the problems of applied mathematics and engineering. For nice introduction of frames, one may refer [8].

The concept of frames was extended to Banach spaces by Feichtinger and Gröchenig [13]. They introduced the notion of atomic decomposition for Banach spaces. Later, Gröchenig [14] introduced a more general concept for Banach spaces called Banach frames. He gave the following definition:

Definition 1.2. Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_d \to \mathcal{X}$ be given. Then, the pair $(\{f_n\}, S)$ is called a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$, if

(i) $\{f_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}},\ \ x \in \mathcal{X}.$$

(iii) S is a bounded linear operator such that $S(\{f_n(x)\}) = x,\ \ x \in \mathcal{X}$.

In the last decade, various generalization of frames for Hilbert spaces have been introduced and studied. Sun [15] generalized the concept of frames and introduced $g-$frames for Hilbert spaces which includes ordinary frames as well as many recent generalization of frames. Recently, Bhandari and Mukherjee [5] gave the notion of atomic subspaces with respect to a bounded linear operator on a separable Hilbert space which leads to the concept of K-fusion frames, a generalization of fusion frames.

2 Preliminaries

Throughout this paper, $\mathbb{N}_M = \{1, 2, \dots M\}$, $M \in \mathbb{N}$, $\mathcal{X}$ is a Banach space with dual $\mathcal{X}^*$ and also $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ is a sequence of Banach spaces. For each $n \in \mathbb{N}$, $B(\mathcal{X}, \mathcal{Y}_n)$ is the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}_n$.

In order to make this paper complete, we review some concepts related to $pg-$frames and weaving frames.

2.1 $pg-$frames and $qg-$ Riesz basis in Banach spaces

In 2001, Aldroubi et al. [2] introduced $p-$frames in Banach spaces and observed that $p-$frames can be used to obtain series expansions in shift invariant subspaces of $L^p(\mathbb{R})$. In [9], O. Christensen and D. Stoeva studied $p-$frames in separable Banach spaces and proved that a p-frame allows every $g \in \mathcal{X}^*$ to be represented as an unconditionally convergent series $g = \sum d_i g_i$ for coefficients $\{d_i\} \in \ell^q$, where $1/p + 1/q = 1$. Abdollahpour et al. [1] generalized the concepts of $p-$frames for Banach Spaces and defined $pg-$Banach frames in Banach spaces. They also observed that a $pg-$frame allows every $g \in \mathcal{X}^*$ to be represented as an unconditionally convergent series $g = \sum_{n\in\mathbb{N}} g_n \mathfrak{T}_n(x)$, where $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$ is a $pg-$frame and $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. They gave the following definition:

Definition 2.1. Let $\{\mathfrak{T}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y}_n)$ be a sequence of operators. Then $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$ is said to be $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, if there exist two constants $0 < c_1 \le c_2 < \infty$ such that

$$c_1\|x\|_{\mathcal{X}} \le \left(\sum_{n\in\mathbb{N}} \|\mathfrak{T}_n(x)\|^p\right)^{1/p} \le c_2\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{2.1}$$

A sequence $\{\mathfrak{T}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y}_n)$ is said to be $pg-$Bessel sequence for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, if it satisfies the right hand side upper inequality in (2.1).

Towards the existence of $pg-$frames, we give the following example:

Example 2.2. Let $\mathcal{X} = \ell^p$ and $\mathcal{Y}_n = \ell^p$, for all $n \in \mathbb{N}$ and $1 < p < \infty$.
For each $n \in \mathbb{N}$, we define $\mathfrak{T}_n : \mathcal{X} \to \mathcal{Y}_n$ as

$$\mathfrak{T}_n(x) = \delta_n^{x_n},\ x = \{x_n\} \in \mathcal{X},$$

where $\delta_n^x = \{0, \dots, 0 \underbrace{x}_{n^{th}\text{place}}, 0, \dots\}$ for all $n \in \mathbb{N}$ and $x \in \mathcal{X}$.
Then $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$ is a $pg-$frame for $\mathcal{X}$ with bound 1.

Definition 2.3. Let $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then

$$\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p} = \left\{\{x_n\}_{n\in\mathbb{N}} : x_n \in \mathcal{Y}_n, n \in \mathbb{N} \text{ and } \left(\sum_{n\in\mathbb{N}} \|x_n\|^p\right)^{1/p} < \infty\right\}$$

is a Banach space with the norm given by

$$\|\{x_n\}_{n\in\mathbb{N}}\|_{\ell_p} = \left(\sum_{n\in\mathbb{N}} \|x_n\|^p\right)^{1/p}.$$

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $g^* = \{g_n^*\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ then

$$\langle g, g^* \rangle = \sum_{n\in\mathbb{N}} \langle g_n, g_n^* \rangle, \; g = \{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$$

defines a continuous functional on $\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$, whose norm is equal to $\|g^*\|_q$ and its dual can be characterized with the following lemma:

Lemma 2.4. *[3] Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}^* = \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q},$$

where the equality holds under the duality

$$\langle g, g^* \rangle = \sum_{n\in\mathbb{N}} \langle g_n, g_n^* \rangle.$$

Further, if $\{\mathfrak{T}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y}_n)$ be a $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. Then, the operators

$$U : \mathcal{X} \to \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p} \text{ as } U(x) = \{\mathfrak{T}_n(x)\}_{n\in\mathbb{N}}, x \in \mathcal{X}$$

$$\text{and } T : \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q} \to \mathcal{X}^* \text{ as } T(\{g_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} g_n \mathfrak{T}_n, \{g_n\} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$$

are called analysis operator and synthesis operator of $pg-$frames $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$, respectively.

Abdollahpour et al. [1] also generalized the concept of $q-$Riesz bases and defined $qg-$Riesz basis for dual Banach space $\mathcal{X}^*$. They observed that every $qg-$Riesz basis for $\mathcal{X}^*$ is a $pg-$frame for $\mathcal{X}$ and that $qg-$Riesz basis bounds coincides with bounds of $pg-$frames. They gave the following definition of $qg-$Riesz basis for $\mathcal{X}^*$:

Definition 2.5. Let $1 < q < \infty$. A sequence $\{\mathfrak{T}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y}_n)$ is said to be $qg-$Riesz basis for $\mathcal{X}^*$, if

(i) $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$ is $g-$complete in $\mathcal{X}$.

(ii) there are positive constants $0 < c_1 \leq c_2 < \infty$ such that, for any finite subset $J \subseteq \mathbb{N}$ and $g_n \in \mathcal{Y}_n^*, n \in J$,

$$c_1 \left(\sum_{n\in J} \|g_n\|^q\right)^{1/q} \leq \left\|\sum_{n\in J} g_n \mathfrak{T}_n\right\| \leq c_2 \left(\sum_{n\in J} \|g_n\|^q\right)^{1/q}.$$

In view of Definition 2.5, one may observe that $\sum_{n\in\mathbb{N}} g_n \mathfrak{T}_n$ converges unconditionally, for all $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ and

$$c_1 \left(\sum_{n\in\mathbb{N}} \|g_n\|^q\right)^{1/q} \leq \left\|\sum_{n\in\mathbb{N}} g_n \mathfrak{T}_n\right\| \leq c_2 \left(\sum_{n\in\mathbb{N}} \|g_n\|^q\right)^{1/q}.$$

Therefore $\{\mathfrak{T}_n\}_{n\in\mathbb{N}}$ is $qg-$Riesz basis for $\mathcal{X}^*$ if and only if the synthesis operator T is an invertible operator from $\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ onto $\mathcal{X}^*$. For the existence of $qg-$Riesz basis, one may refer [15].

2.2 Weaving frames in Hilbert spaces

Recently, a new problem in frame theory related to distributed signal processing has been emerged. Two frames $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$, where each set of frames is labelled by a sensor or a node $i \in I$. Then, at each sensor $i \in I$, a signal x is measured with either ϕ_i or ψ_i, so that the collected information is the set of numbers $\{\langle x, \phi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \psi_i\rangle\}_{i\in\sigma^c}$ for some subset $\sigma \subseteq I$. Now, a natural question arises: "Can x still be recovered robustly from these measurements, regardless of the kind of measurement considered at each node?" In other words: "Is the set $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ a frame for all subsets $\sigma \subseteq I$?" In this direction, a theory of weaving frames was developed by Bemrose et al. [4]. They gave the following definition:

Definition 2.6. A family of frames $\{\psi_{ij}\}_{i\in I}$ for $j \in \mathbb{N}_M$ for a Hilbert space $\mathcal{H}$ is said to be *woven* if there are universal constants c_1 and c_2 so that for every partition $\{\sigma_j\}_{j\in\mathbb{N}_M}$ of I, the family $\{\psi_{ij}\}_{i\in\sigma_j,\, j\in\mathbb{N}_M}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds c_1 and c_2, respectively. Each family $\{\psi_{ij}\}_{i\in\sigma_j,\, j\in\mathbb{N}_M}$ is called a *weaving*, where $\mathbb{N}_M = \{1, 2, \dots M\}$ and $\mathbb{N}_M^c = \mathbb{N}\backslash\mathbb{N}_M$ for every $M \in \mathbb{N}$.

Casazza et al. [7] generalized the notion of weaving fames in Hilbert spaces to Banach spaces and defined weaving approximate scahuder frames for Banach spaces. They gave the following definition:

Definition 2.7. [7] Let $(x_j^0, f_j^0)_{j=1}^\infty$ and $(x_j^1, f_j^1)_{j=1}^\infty$ be two approximate Schauder frames for a Banach space $\mathcal{X}$. A sequence $(x_j^{\sigma(j)}, f_j^{\sigma(j)})_{j=1}^\infty$ with $\sigma \in \{0,1\}^{\mathbb{N}}$ is called a *weaving* of $(x_j^0, f_j^0)_{j=1}^\infty$ and $(x_j^1, f_j^1)_{j=1}^\infty$. For given $C \geq 1$, then $(x_j^0, f_j^0)_{j=1}^\infty$ and $(x_j^1, f_j^1)_{j=1}^\infty$ are $C-$*woven* if every weaving is a C-approximate Schauder frame.

They also showed that it is a true generalization of woven frames for Hilbert spaces as two frames $\{x_j^0\}_{j=1}^\infty$ and $\{x_j^1\}_{j=1}^\infty$ for a Hilbert space $\mathcal{H}$ are woven if and only if the approximate Schauder frames $(x_j^0, f_j^0)_{j=1}^\infty$ and $(x_j^1, f_j^1)_{j=1}^\infty$ are woven.

Weaving frames have many useful applications in sensor networks. Likewise, weaving K-frames [11] have been proved to be useful during signal reconstructions from the range of a bounded linear operator K. On this development, Bhandari, Borah and Mukherjee [6] studied weaving K-frames and gave characterization of weaving K-frames in different spaces.

3 Woven $pg-$Frames

We begin this section with the following definition of woven $pg-$frames:

Definition 3.1. Let $1 < p < \infty$ and $\mathfrak{F} = \{\{\mathfrak{T}_{ni} : \mathcal{X} \to \mathcal{Y}_n\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a family of $pg-$frames for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. Then $\mathfrak{F}$ is said to be a *woven $pg-$frame*, if there exist universal constants $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$ such that for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$, the family $\{\mathfrak{T}_{ni}\}_{n\in\sigma_i, i\in\mathbb{N}_M}$ is a $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\sigma_i, i\in\mathbb{N}_M}$ with lower and upper $pg-$frame bounds $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$, respectively.

In this case, the family $\mathfrak{F}$ is called a *weaving $pg-$frame* for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and the bounds $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$ are called the *woven $pg-$frame bounds*. If every weaving is a $pg-$Bessel sequence, then it is called a *woven $pg-$Bessel sequence* for $\mathcal{X}$.

For the existence of woven $pg-$frames, we give the following example:

Example 3.2. Let $\mathcal{X} = \ell^p$ and $\mathcal{Y}_n = \ell^p$, for all $n \in \mathbb{N}$ and $1 < p < \infty$.
For each $n \in \mathbb{N}$, we define $\mathfrak{T}_{n1} : \mathcal{X} \to \mathcal{Y}_n$ as

$$\mathfrak{T}_{n1}(x) = \delta_n^{x_n} - \delta_{n+1}^{x_n},\ x = \{x_n\} \in \mathcal{X},$$

where $\delta_n^x = \{0, \dots, 0\ \underbrace{x}_{n^{th}\text{place}}\ , 0, \dots\}$ for all $n \in \mathbb{N}$ and $x \in \mathcal{X}$.
Then $\{\mathfrak{T}_{n1}\}_{n\in\mathbb{N}}$ is a $pg-$frame for $\mathcal{X}$.
Again, for each $n \in \mathbb{N}$, we define $\mathfrak{T}_{n2} : \mathcal{X} \to \mathcal{Y}_n$ as

$$\mathfrak{T}_{n2}(x) = \begin{cases} \delta_2^{x_1} - \delta_1^{x_1},\ n = 1 \\ \delta_n^{x_n} - \delta_{n+1}^{x_n},\ n \geq 2,\ n \in \mathbb{N},\ x = \{x_n\} \in \mathcal{X}, \end{cases}$$

where $\delta_n^x = \{0, \ldots, 0 \underbrace{x}_{n^{th}\text{place}}, 0, \ldots\}$ for all $n \in \mathbb{N}$ and $x \in \mathcal{X}$.
Then $\{\mathfrak{T}_{n2}\}_{n\in\mathbb{N}}$ is also a $pg-$frame for $\mathcal{X}$.
Now, we let $\sigma = \{1, 2\} \subseteq \mathbb{N}$ be any partition, then for every $x \in \mathcal{X}$, we have

$$\sum_{n\in\sigma} \|\mathfrak{T}_{n1}(x)\|^p + \sum_{n\in\sigma^c} \|\mathfrak{T}_{n2}(x)\|^p = 2\|x\|^p.$$

Therefore $\{\mathfrak{T}_{n1}\}_{n\in\sigma} \cup \{\mathfrak{T}_{n2}\}_{n\in\sigma^c}$ is a weaving $pg-$frame with bounds $2^{1/p}$.

The following proposition shows that the family of $pg-$Bessel sequences is always woven.

Proposition 3.3. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a family of $pg-$Bessel sequences for X with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ with Bessel bounds B_i, $i \in \mathbb{N}_M$. Then $\mathfrak{F}$ is a woven $pg-$Bessel sequence with $pg-$ Bessel bound $\left(\sum_{i\in\mathbb{N}_M} B_i^p\right)^{1/p}$.*

Proof. Let $\{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$. Then, for each $x \in \mathcal{X}$, we have

$$\sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} \|\mathfrak{T}_{ni}(x)\|^p \leq \sum_{i\in\mathbb{N}_M} \sum_{n\in\mathbb{N}} \|\mathfrak{T}_{ni}(x)\|^p \leq \sum_{i\in\mathbb{N}_M} B_i^p \|x\|^p.$$

Thus $\mathfrak{F}$ is a woven $pg-$Bessel sequence with $pg-$ Bessel bound $\left(\sum_{i\in\mathbb{N}_M} B_i^p\right)^{1/p}$. □

Next, we show that the image of woven pg-frame under a bounded and invertible operator is also a woven pg-frame.

Theorem 3.4. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ with universal frame bounds $A_{\mathfrak{F}}$, $B_{\mathfrak{F}}$ and $\mathfrak{U}$ be a bounded invertible operator on $\mathcal{X}$. Then, the family $\mathfrak{U}\mathfrak{F} = \{\{\mathfrak{T}_{ni}\mathfrak{U}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$frame for $\mathcal{X}$ with universal frame bounds $A_{\mathfrak{F}}\|\mathfrak{U}^{-1}\|^{-1}$ and $B_{\mathfrak{F}}\|\mathfrak{U}\|$.*

Proof. Let $\{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$. Then, we have

$$A_{\mathfrak{F}}^p \|\mathfrak{U}x\|^p \leq \sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} \|\mathfrak{T}_{ni}(\mathfrak{U}x)\|^p \leq B_{\mathfrak{F}}^p \|\mathfrak{U}x\|^p, \ x \in \mathcal{X}.$$

Therefore,

$$A_{\mathfrak{F}}^p \|\mathfrak{U}^{-1}\|^{-p} \|x\|^p \leq \sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} \|(\mathfrak{T}_{ni}\mathfrak{U})(x)\|^p \leq B_{\mathfrak{F}}^p \|\mathfrak{U}\|^p \|x\|^p, \ x \in \mathcal{X}.$$

Thus $\mathfrak{U}\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$ with universal $pg-$frame bounds $A_{\mathfrak{F}}\|\mathfrak{U}^{-1}\|^{-1}$ and $B_{\mathfrak{F}}\|\mathfrak{U}\|$.
□

Corollary 3.5. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and $\mathfrak{U}$ be an isometry on $\mathcal{X}$. Then, the family $\mathfrak{U}\mathfrak{F} = \{\{\mathfrak{T}_{ni}\mathfrak{U}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$frame for $\mathcal{X}$.*

Theorem 3.6. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ with universal frame bounds $A_{\mathfrak{F}}$, $B_{\mathfrak{F}}$ and $\mathfrak{U}$ be a bounded operator on $\mathcal{X}$. Then, the family $\mathfrak{U}\mathfrak{F} = \{\{\mathfrak{T}_{ni}\mathfrak{U}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$frame for $\mathcal{X}$ if and only if $\mathfrak{U}$ is bounded below.*

Proof. Let $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$. Let $A_{\mathfrak{U}\mathfrak{F}}$ and $B_{\mathfrak{U}\mathfrak{F}}$ be an universal $pg-$frame bounds for $\mathfrak{U}\mathfrak{F}$. Then, we have

$$A_{\mathfrak{U}\mathfrak{F}}^p \|x\|^p \leq \sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} \|(\mathfrak{T}_{ni}\mathfrak{U})(x)\|^p \leq B_{\mathfrak{U}\mathfrak{F}}^p \|x\|^p, \ x \in \mathcal{X}. \tag{3.1}$$

Also,

$$A_{\mathfrak{F}}^p \|\mathfrak{U}x\|^p \leq \sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} \|\mathfrak{T}_{ni}(\mathfrak{U}x)\|^p \leq B_{\mathfrak{F}}^p \|\mathfrak{U}x\|^p, \ x \in \mathcal{X}. \tag{3.2}$$

Therefore, by using (3.1) and (3.2), we get

$$A^p_{\mathfrak{U}\mathfrak{F}}\|x\|^p \leq B^p_{\mathfrak{F}}\|\mathfrak{U}x\|^p,\ x \in \mathcal{X}.$$

Thus $\|\mathfrak{U}x\| \geq M\|x\|, x \in \mathcal{X}$, where $M = \dfrac{A_{\mathfrak{U}\mathfrak{F}}}{B_{\mathfrak{F}}} > 0$. Conversely, let $\mathfrak{U}$ be bounded below. Then there exists a $\delta > 0$ such that $\|\mathfrak{U}x\| \geq \delta\|x\|,\ \forall\, x \in \mathcal{X}$. So,

$$A^p_{\mathfrak{F}}\delta^p\|x\|^p \leq A^p_{\mathfrak{F}}\|\mathfrak{U}x\|^p \leq \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i}\|\mathfrak{T}_{ni}(\mathfrak{U}x)\|^p \leq B^p_{\mathfrak{F}}\|\mathfrak{U}x\|^p \leq B^p_{\mathfrak{F}}\|\mathfrak{U}\|^p\|x\|^p,\ x \in \mathcal{X}.$$

Since $\{\sigma_i\}_{i\in\mathbb{N}_M}$ is any arbitrary partition of $\mathbb{N}$, therefore $\mathfrak{U}\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$ with universal $pg-$frame bounds $A_{\mathfrak{F}}\delta$ and $B_{\mathfrak{F}}\|\mathfrak{U}\|$. □

Next, we provide the example which validate the characterization of Theorem 3.6.

Example 3.7. Let $\mathcal{X} = \ell^p,\ 1 < p < \infty$ and $\{e_n\}_{n\in\mathbb{N}}$ be the standard basis of the canonical unit vectors in ℓ_p. Let $\mathcal{Y}_n = [e_n]$ and define $\mathfrak{T}_{ni} : X \to \mathcal{Y}_n$ as

$$\mathfrak{T}_{ni}(x) = \alpha_n e_n,\ x = \sum_{n\in\mathbb{N}}\alpha_n e_n \in \mathcal{X}, \tag{3.3}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of scalars.
Let $\{\sigma_i\}$ be any partition of $\mathbb{N}$, where

$$\begin{cases}\sigma_i = i,\ i = 1, 2, \ldots, M-1,\\ \sigma_M = \{M, M+1, M+2, \ldots\},\ M \in \mathbb{N}.\end{cases}$$

Then

$$\sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i}\|\mathfrak{T}_{ni}(x)\|^p = \|x\|^p.$$

Therefore $\{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. Now, we define an operator $\mathfrak{U} : \mathcal{X} \to \mathcal{X}$ as

$$\mathfrak{U}(x) = (0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots),$$

where, the vector x is as defined in equation (3.3).
Then $\mathfrak{U}$ is bounded operator on $\mathcal{X}$ such that $\{\{\mathfrak{T}_{ni}\mathfrak{U}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$frame for $\mathcal{X}$ if and only if $\mathfrak{U}$ is bounded below.

Now, we give the definition of the analysis and synthesis operators of woven $pg-$frame.

Definition 3.8. Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}_M$. Then, the operators

$$U_\sigma : X \to \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}_n\right)_{\ell_p} \text{ as } U_\sigma(x) = \{\mathfrak{T}_{ni}(x)\}_{n\in\sigma_i},\ x \in \mathcal{X},\ i \in \mathbb{N}_M$$

and

$$T_\sigma : \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}^*_n\right)_{\ell_q} \to X^* \text{ as } T_\sigma(\{g_n\}_{n\in\mathbb{N}}) = \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni},\ \{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}^*_n\right)_{\ell_q}$$

are called the analysis and synthesis operators, respectively, of woven $pg-$frame $\mathfrak{F}$ with respect to σ.

Next, we give the characterization of woven $pg-$Bessel sequences.

Theorem 3.9. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni} : \mathcal{X} \to \mathcal{Y}_n\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a family of operators and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$. Then $\mathfrak{F}$ is a woven $pg-$Bessel sequences for $\mathcal{X}$ with universal bound $B_{\mathfrak{F}}$ if and only if for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$, the synthesis operator T_σ is well-defined and bounded.*

Proof. Let $\mathfrak{F}$ be a family of woven $pg-$Bessel sequences for $\mathcal{X}$ with universal bound $B_{\mathfrak{F}}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$.
Then, for $J_2 \subsetneqq J_1 \subset \mathbb{N}_M$, we have

$$\begin{aligned}\left\|\sum_{i\in J_1}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni} - \sum_{i\in J_2}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}\right\| &= \left\|\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}\right\| \\ &= \sup_{\|x\|=1}\left\|\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x)\right\| \\ &\le \sup_{\|x\|=1}\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i}\|g_n\|\|\mathfrak{T}_{ni}(x)\| \\ &\le \left(\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i}\|g_n\|^q\right)^{1/q}\sup_{\|x\|=1}\left(\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i}\|\mathfrak{T}_{ni}(x)\|^p\right)^{1/p} \\ &\le B_{\mathfrak{F}}\left(\sum_{i\in J_1\setminus J_2}\sum_{n\in\sigma_i}\|g_n\|^q\right)^{1/q}, \ \{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}_n^*\right)_{\ell_q}.\end{aligned}$$

Therefore $\sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}$ is unconditionally convergent and

$$\|T_\sigma(\{g_n\}_{n\in\mathbb{N}})\| \le B_{\mathfrak{F}}\|\{g_n\}_{n\in\mathbb{N}}\|_q.$$

Thus T_σ is well-defined and bounded with $\|T_\sigma\| \le B_{\mathfrak{F}}$.
Conversely, let T_σ be well-defined and bounded. Then, for each partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$, define $\tilde{T}_x : \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}_n^*\right)_{\ell_q} \to \mathbb{C}$ as

$$\tilde{T}_x(\{g_n\}_{n\in\mathbb{N}}) = \left(T_\sigma(\{g_n\}_{n\in\mathbb{N}})\right)(x) = \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x), \ x \in \mathcal{X}.$$

Then $\tilde{T}_x \in \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}_n^*\right)_{\ell_q}^*$ and $\|\tilde{T}_x(\{g_n\}_{n\in\mathbb{N}})\| \le \|T_\sigma\|\|\{g_n\}_{n\in\mathbb{N}}\|\|x\|$. Therefore, there exists $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}}\oplus\mathcal{Y}_n^*\right)_{\ell_q}$ with $\|\{g_n\}_{n\in\mathbb{N}}\|_q \le 1$ such that

$$\|\{\mathfrak{T}_{ni}(x)\}\|_p = \left|\sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x)\right|, \ x \in \mathcal{X}.$$

So, for each $x \in \mathcal{X}$, we have

$$\begin{aligned}\left(\sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i}\|\mathfrak{T}_{ni}(x)\|^p\right)^{1/p} &= \|\{\mathfrak{T}_{ni}(x)\}\|_p \\ &\le \sup_{\|\{g_n\}_{n\in\mathbb{N}}\|_q=1}\left|\sum_{i\in J}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x)\right| = \|\tilde{T}_x\| \le \|T_\sigma\|\|x\|.\end{aligned}$$

Therefore $\{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ is a woven $pg-$Bessel sequence for $\mathcal{X}$ with Bessel bound $\|T_\sigma\|$. □

Theorem 3.10. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$. Then, the analysis operator U_σ of $\mathfrak{F}$ has a closed range.*

Proof. Let $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$ be universal $pg-$frame bounds for woven $pg-$frame $\mathfrak{F}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$. Then,

$$A_{\mathfrak{F}}\|x\| \leq \|U_\sigma(x)\|_p \leq B_{\mathfrak{F}}\|x\|,\ x \in \mathcal{X} \tag{3.4}$$

Let $U_\sigma(x) = 0$, then by (3.4), $x = 0$. Therefore U_σ is injective and so $\mathcal{X} \simeq \mathcal{R}_{U_\sigma}$. □

Corollary 3.11. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven pg−frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, where $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ is a sequence of reflexive Banach spaces. Then $\mathcal{X}$ is reflexive.*

Theorem 3.12. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven pg−Bessel sequence for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$. Then*

(i) $U_\sigma^* = T_\sigma$

(ii) $U_\sigma = T_\sigma^*$, if $\mathfrak{F}$ has universal lower pg−frame bound and $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ is a sequence of reflexive Banach spaces.

Proof. (i). Let $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$. Then, for any $x \in \mathcal{X}$ and $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$, we have

$$\begin{aligned}\langle U_\sigma(x), \{g_n\}_{n\in\sigma_i}\rangle &= \langle \{\mathfrak{T}_{ni}(x)\}_{n\in\sigma_i}, \{g_n\}_{n\in\sigma_i}\rangle \\ &= \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} \langle \mathfrak{T}_{ni}(x), g_n\rangle \\ &= \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x),\ x \in \mathcal{X}\end{aligned} \tag{3.5}$$

and

$$\langle x, T_\sigma(\{g_n\}_{n\in\mathbb{N}})\rangle = \left\langle x, \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}\right\rangle = \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}(x). \tag{3.6}$$

Therefore from (3.5) and (3.6), we have $U_\sigma^* = T_\sigma$.
(ii). By Theorem 3.10, $\mathcal{R}_{U_\sigma}$ is a closed subspace of $\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$ and so is reflexive. Then $U_\sigma = T_\sigma^*$. □

The next theorem provides the characterization of woven $pg-$frame in terms of synthesis operator.

Theorem 3.13. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni} : \mathcal{X} \to \mathcal{Y}_n\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a family of operators and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$. Then $\mathfrak{F}$ is a woven pg−frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ if and only if T_σ is bounded and onto.*

Proof. Let $\mathfrak{F}$ be a woven $pg-$frame for $\mathcal{X}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$. Then, by Theorem 3.9, T_σ is well-defined and bounded. Also, by Theorem 3.10, U_σ is injective. Then, $U_\sigma^* = T_\sigma$ is onto. Conversely, let $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be any partition of $\mathbb{N}$ and T_σ is bounded and onto. Then, by Theorem 3.9, $\mathfrak{F}$ is a $pg-$Bessel sequence for $\mathcal{X}$. Since $T_\sigma = U_\sigma^*$ is onto. So U_σ has bounded inverse. Therefore, there exists a constant $A > 0$ such that

$$\|U_\sigma(x)\|_p \geq A\|x\|,\ x \in \mathcal{X}.$$

This gives

$$\left(\sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} \|\mathfrak{T}_{ni}(x)\|^p\right)^{1/p} \geq A\|x\|,\ x \in \mathcal{X}.$$

Hence $\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$. □

Corollary 3.14. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven pg−frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ and $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$. Then, for any $g \in \mathcal{X}^*$, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ such that*

$$g = \sum_{i\in\mathbb{N}_M}\sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}.$$

4 Woven $qg-$Riesz Bases

We begin this section with the following definition of woven $qg-$Riesz basis for $\mathcal{X}^*$.

Definition 4.1. Let $1 < q < \infty$ and $\mathfrak{F} = \{\{\mathfrak{T}_{ni} : \mathcal{X} \to \mathcal{Y}_n\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a family of $qg-$Riesz basis for $\mathcal{X}^*$ with respect to $\{\mathcal{Y}_n^*\}_{n\in\mathbb{N}}$. Then $\mathfrak{F}$ is said to be a *woven qg-Riesz basis*, if there exist universal constants $0 < A_{\mathfrak{F}} \leq B_{\mathfrak{F}} < \infty$ such that for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$, the family $\mathfrak{F}_{\sigma_i}$ is a $qg-$Riesz basis for $\mathcal{X}^*$ with respect to $\{\mathcal{Y}_n^*\}_{n\in\mathbb{N}}$.

In this case, the family $\mathfrak{F}$ is called a *weaving* $qg-$*Riesz basis* for $\mathcal{X}^*$ and the bounds $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$ are called the *woven* $qg-$*Riesz basis bounds*.

Next, we prove that the woven $qg-$Riesz basis for $\mathcal{X}^*$ is a particular case of woven $pg-$frame for $\mathcal{X}$.

Theorem 4.2. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $qg-$Riesz basis for $\mathcal{X}^*$ with respect to $\{\mathcal{Y}_n^*\}_{n\in\mathbb{N}}$ and with universal $qg-$Riesz bounds $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$. Then $\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$.*

Proof. Let $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ be a partition of $\mathbb{N}$ and $\{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $qg-$Riesz basis for $\mathcal{X}^*$. Then, for all $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$, the series $\sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni}$ is unconditionally convergent and satisfies

$$A_{\mathfrak{F}} \left(\sum_{n\in\mathbb{N}} \|g_n\|^q\right)^{1/q} \leq \left\| \sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni} \right\| \leq B_{\mathfrak{F}} \left(\sum_{n\in\mathbb{N}} \|g_n\|^q\right)^{1/q}.$$

Therefore T_σ is bounded and invertible from $\left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ to $\mathcal{X}^*$. Thus $\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$. □

Theorem 4.3. *Let $\mathfrak{F} = \{\{\mathfrak{T}_{ni}\}_{n\in\mathbb{N}} : i \in \mathbb{N}_M\}$ be a woven $pg-$frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, where $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ is a sequence of reflexive Banach spaces. Then, for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$, the following conditions are equivalent:*

(i) *$\mathfrak{F}$ is a $qg-$Riesz basis for $\mathcal{X}^*$.*

(ii) *If $\{g_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q}$ and $\sum_{i\in\mathbb{N}_M} \sum_{n\in\sigma_i} g_n\mathfrak{T}_{ni} = 0$, then $g_n = 0,\ n \in \mathbb{N}$.*

(iii) $\mathcal{R}_{U_\sigma} = \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$.

Proof. Clearly, (1) $\implies$ (2).
(2) $\implies$ (1). Since $\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$, then by the Theorem 3.13, T_σ is bounded and onto, for each partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$.
Also, by the hypothesis, T_σ is injective. Therefore, T_σ is invertible for each partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$. Then $\mathfrak{F}$ is a $qg-$Riesz basis for $\mathcal{X}^*$.
(1) $\implies$ (3). Let $\mathfrak{F}$ is a $qg-$Riesz basis for $\mathcal{X}^*$. Then T_σ has a bounded inverse on $\mathcal{R}_{T_\sigma}$. So, the adjoint $T_\sigma^* : \mathcal{X}^{**} \to \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$ of T_σ is surjective on $\mathcal{R}_{T_\sigma}$. Since $\mathfrak{F}$ is a woven $pg-$frame for $\mathcal{X}$, then by Corollary 3.11, $\mathcal{X}$ is reflexive. Thus $\mathcal{R}_{U_\sigma} = \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n\right)_{\ell_p}$, for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$.
(3) $\implies$ (1). As $T_\sigma = U_\sigma^* : \left(\sum_{n\in\mathbb{N}} \oplus \mathcal{Y}_n^*\right)_{\ell_q} \to \mathcal{X}^*$ is invertible, for every partition $\sigma = \{\sigma_i\}_{i\in\mathbb{N}_M}$ of $\mathbb{N}$. Therefore $\mathfrak{F}$ is a $qg-$Riesz basis for $\mathcal{X}^*$. □

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Author information

S.K. Sharma, Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India.
E-mail: `sumitkumarsharma@gmail.com`

Shashank Goel, Department of Mathematics, Jaypee Institue of Information Technology, Noida, Uttar Pradesh-201309, India.
E-mail: `goel.shashank25@gmail.com; shashank.goel@jiit.ac.in`

S.K. Kaushik, Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India.
E-mail: `shikk2003@yahoo.co.in`

Received: 2022-01-05
Accepted: 2023-08-02