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SOME ORDER PROPERTIES OF C-SPACES

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Abstract In this paper, we discuss the properties of functions and t-closed subsets of an order induced c-space. Here we prove that the order induced c-spaces corresponding to the order isomorphic linearly ordered sets are c-isomorphic. We also give a characterization of dense linearly ordered set in terms of t-closed subsets.

1 Introduction

In 1983, R. Börger [1] started a systematic study of connectivity class or c-structure and further carried out by J. Serra, H. J. A. M. Heijmans, C. Ronse, S. Dugowson etc.[10, 3, 8, 2, 5, 12, 7, 9, 11]

Precisely, a c-structure [5, page 2] in X is a collection $\mathfrak{C}$ of subsets of X such that the following conditions hold:

(i) $\emptyset \in \mathfrak{C}$ and $\{x\} \in \mathfrak{C}$ for every $x \in X$,

(ii) If $\{C_i : i \in I\}$ be a nonempty collection of subsets in $\mathfrak{C}$ with $\bigcap_{i\in I} C_i \neq \emptyset$, then $\bigcup_{i\in I} C_i \in \mathfrak{C}$.

The set X together with a c-structure $\mathfrak{C}$, that is $(X, \mathfrak{C})$ is called a c-space and elements of $\mathfrak{C}$ are called connected sets in X with respect to $\mathfrak{C}$. For any non empty set X, the collection $\mathfrak{D} = \{\emptyset\} \cup \{\{x\} : x \in X\}$ is a c-structure on X, called the discrete c-structure and $(X, \mathfrak{D})$ is called discrete c-space.

Let $(X, \leq)$ be a linearly ordered set, then the collection of all intervals of X is a c-structure on X and is denoted by $\mathfrak{C}_{\leq}$. Then the c-space $(X, \mathfrak{C}_{\leq})$ is called the order induced c-space corresponding to the linearly ordered set $(X, \leq)$. In this paper, we point out the features of the c-space $(X, \mathfrak{C}_{\leq})$, induced by the ordering of elements of X. In digital topology, the topological approach to digital images in studying digital processing is mainly concerned with connected sets. The c-structure obtained from the usual ordering of $\mathbb{Z}$ coincides with the set of all connected sets of the digital line topology disclose the relevance of our study.

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of all natural numbers, set of all integers, set of all rationals, and the set of all real numbers respectively. The basic set-theoretic notions used in this paper are adopted from [4, 6]. A partially ordered set $(X, \leq)$ is said to be complete if every nonempty subset of X which is bounded above has a supremum. A subset $A \subseteq X$ is said to be dense in X if for every $x, y \in X$ with $x < y$, there exists $a \in A$ such that $x < a < y$. Let $(X, \leq)$ and $(Y, \leq')$ be two partially ordered sets. A function $f : X \to Y$ is said to be order preserving if for every $x, y \in X$, $x \leq y \Rightarrow f(x) \leq' f(y)$. Now, f is said to be order reversing if for every $x, y \in X$, $x \leq y \Rightarrow f(y) \leq' f(x)$. The partially ordered sets X and Y are said to order isomorphic if there exists a bijection $f : X \to Y$ such that both f and f^{-1} are order preserving. The partial order $\leq$ is said to be linear order if for all $x, y \in X$ either $x \leq y$ or $y \leq x$. A subset

A of X is said to be an interval if $x, y \in A$ and $z \in X$ be such that $x \leq z \leq y$, then $z \in A$. For $a, b \in X$, $[a, b] = \{x \in X : a \leq x \leq b\}$.

Let $(X, \mathfrak{C})$ be a c-space and $Y \subseteq X$, we define $\mathfrak{C}_Y = \{C \in \mathfrak{C} : C \subseteq Y\}$. Then $(Y, \mathfrak{C}_Y)$ is a c-space, called sub c-space [5, page 5] on Y. Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$, then the smallest c-structure on X containing $\mathcal{B}$ is called the c-structure generated by $\mathcal{B}$ and is denoted by $< \mathcal{B} >$.

Proposition 1.1. *[5, page 6] The non-trivial connected sets of a c-structure generated by $\mathcal{B}$ are characterized by the condition that any two points of such a connected set C can be joined by a finite chain of basic connected sets (ie, elements of $\mathcal{B}$) in $\mathfrak{C}$. That is, for all $x, y \in C$, we can find elements $B_i,\ i = 0$ to n in $\mathcal{B}$ such that $B_i \subseteq C$, $B_i \cap B_{i+1} \neq \emptyset$ for $i = 0$ to $n-1$ and $x \in B_0,\ y \in B_n$.*

2 Functions of Order Induced c-spaces

Here we discuss the properties of functions of an order induced c-space, mainly the relation between order preserving and order reversing functions of a linearly ordered set and c-continuous functions of the corresponding order induced c-space.

Definition 2.1. [5, page 4] Let $(X, \mathfrak{C})$ and $(Y, \mathfrak{C}')$ be two c-spaces. A function $f : X \to Y$ is called c-continuous if $C \in \mathfrak{C} \Rightarrow f(C) \in \mathfrak{C}'$. Then f is said to be a c-isomorphism if f is a bijection and both f and f^{-1} are c-continuous. The c-spaces are said to be c-isomorphic if there exists a c-isomorphism between them.

Theorem 2.2. *Let $(X, \mathfrak{C}_{\leq})$ be an order induced c-space and f be a c-continuous function on X. If $a, b \in X$ and $y \in X$ satisfies $f(a) < y < f(b)$, then there exists $x \in X$ between a and b such that $f(x) = y$.*

Proof. Consider the c-continuous function f on $(X, \mathfrak{C}_{\leq})$. Let $a, b \in X$ and $y \in X$ be such that $f(a) < y < f(b)$. Now let $C = [a, b]$ if $a < b$, otherwise take $C = [b, a]$. Then $C \in \mathfrak{C}_{\leq}$ implies $f(C) \in \mathfrak{C}_{\leq}$. Hence $f(a), f(b) \in f(C)$ and $f(a) < y < f(b)$ implies $y \in f(C)$. That is, there exists $x \in C \subseteq X$ such that $f(x) = y$. □

There are order preserving functions and order reversing functions that are not c-continuous.

Example 2.3. Consider the linearly ordered set $(\mathbb{Z}, \leq)$, where $\leq$ is the usual ordering of integers and the functions f, g on $(\mathbb{Z}, \leq)$ defined by $f(x) = 2x$ and $g(x) = -2x$. Then f is order preserving and g is order reversing. Here $A = \{1, 2\} \in \mathfrak{C}_{\leq}$, the order induced c-structure on $(\mathbb{Z}, \leq)$. But $f(A) = \{2, 4\} \notin \mathfrak{C}_{\leq}$ and $g(A) = \{-2, -4\} \notin \mathfrak{C}_{\leq}$. Therefore f and g are not c-continuous on the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{\leq})$.

Theorem 2.4. *If a bijection $f : (X, \leq) \to (Y, \leq')$ is order preserving or order reversing, then $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\leq'})$ is c-continuous.*

Proof. Suppose the bijection $f : X \to Y$ is order preserving and let $C \in \mathfrak{C}_{\leq}$. To prove $f(C) \in \mathfrak{C}_{\leq'}$, let $f(a), f(b) \in f(C)$ and $z \in Y$ such that $f(a) \leq' z \leq' f(b)$. Then $a, b \in C$ and also $a \leq b$. If not, $b < a$ implies $f(b) <' f(a)$, which is not true. Since f is surjective, there exists $w \in X$ such that $f(w) = z$. If $w < a$, then $z = f(w) <' f(a)$, a contradiction. Similar will happen if $b < w$. Therefore $a \leq w \leq b$. Hence it follows that $w \in C$ and $z = f(w) \in f(C)$. Thus f is c-continuous.

Now suppose the bijection $f : X \to Y$ is order reversing. Consider the dual $(Y, \geq')$ of the linearly ordered set $(Y, \leq')$. If we consider f as a function from $(X, \leq)$ to $(Y, \geq')$, it is order preserving. Then we have $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\geq'})$ is c-continuous. Since $\mathfrak{C}_{\leq'} = \mathfrak{C}_{\geq'}$, we get that $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\leq'})$ is a c-continuous function. □

Now the converse of Theorem 2.4 is not true. That is, there are c-continuous functions that are neither order preserving nor order reversing.

Example 2.5. Consider the linearly ordered set $(X, \leq)$, where $X = \{1, 2, 3, 4\}$ and $\leq$ is the usual ordering of numbers. Then the order induced c-structure on X is given by $\mathfrak{C}_{\leq} = \mathfrak{D} \cup \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. Let f be the function on X given by $f(1) = f(4) = 2$, $f(2) = f(3) = 1$. Then $f(C) \in \mathfrak{C}_{\leq}$, for every $C \in \mathfrak{C}_{\leq}$ implies f is c-continuous. But f is neither order preserving nor order reversing.

Theorem 2.6. *If $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\leq'})$ is a c-continuous one-one function then $f : (X, \leq) \to (Y, \leq')$ is either order preserving or order reversing.*

Proof. Consider a c-continuous one-one function $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\leq'})$ which is not order preserving. Then there exist $a, b \in X$ such that $a < b$ and $f(b) <' f(a)$. Assume that f is not order reversing. Then there exist $x, y \in X$ with $x < y$ such that $f(x) <' f(y)$. Then we have the following cases.

$\mathbf{Case(i)}: x < a$

Since $x \notin [a, b]$, we have $f(x) \notin [f(b), f(a)]$. Otherwise $f(b) \leq' f(x) \leq' f(a)$ and $f(a), f(b) \in f([a, b]) \in \mathfrak{C}_{\leq'}$ implies $f(x) \in f([a, b])$. Since f is one-one, we get $x \in [a, b]$, which is not true. Therefore either $f(x) <' f(b)$ or $f(a) <' f(x)$.

If $f(x) <' f(b)$, then $f(x), f(a) \in f([x, a]) \in \mathfrak{C}_{\leq'}$ and $f(x) <' f(b) <' f(a)$ implies $f(b) \in f([x, a])$. This implies $b \in [x, a]$, a contradiction.

Now let $f(a) <' f(x)$. If $y < a$, then $f(a) <' f(x) <' f(y)$ and $f(a), f(y) \in f([y, a]) \in \mathfrak{C}_{\leq'}$ implies $f(x) \in f([y, a])$. This implies $x \in [y, a]$, a contradiction. Similarly, if $a \leq y$ then we get the contradiction $x \in [a, y]$.

$\mathbf{Case(ii)}: b < y$

As in case(i), $y \notin [a, b]$ implies $f(y) \notin [f(b), f(a)]$. Therefore either $f(y) <' f(b)$ or $f(a) <' f(y)$. If $f(a) <' f(y)$, then $f(b) <' f(a) <' f(y)$ and $f(b), f(y) \in f([b, y]) \in \mathfrak{C}_{\leq'}$ implies $f(a) \in f([b, y])$. Hence $a \in [b, y]$, a contradiction.

Now let $f(y) <' f(b)$. If $x \leq b$, then $f(x) <' f(y) <' f(b)$ and $f(x), f(b) \in f([x, b]) \in \mathfrak{C}_{\leq'}$ implies $f(y) \in f([x, b])$. Hence $y \in [x, b]$, a contradiction. Similarly, if $b < x$ then we get the contradiction $y \in [b, x]$.

$\mathbf{Case(iii)}: a \leq x < y \leq b$

Since $x \in [a, b]$, we have $f(x) \in [f(b), f(a)]$. Otherwise, either $f(x) <' f(b)$ or $f(a) <' f(x)$. If $f(x) <' f(b)$, then $f(x), f(a) \in f([a, x]) \in \mathfrak{C}_{\leq'}$ and $f(x) <' f(b) <' f(a)$ implies $f(b) \in f([a, x])$. This implies $b \in [a, x]$, which is not true. If $f(a) <' f(x)$, then $f(b), f(x) \in f([x, b]) \in \mathfrak{C}_{\leq'}$ implies $f(a) \in f([x, b])$. This implies $a \in [x, b]$, which is not true. Similarly we have $f(y) \in [f(b), f(a)]$. Therefore $f(b) \leq' f(x) <' f(y) \leq' f(a)$. Then $f(x) <' f(y) \leq' f(a)$ and $f(x), f(a) \in f([a, x]) \in \mathfrak{C}_{\leq'}$ implies $f(y) \in f([a, x])$. Therefore $y \in [a, x]$, a contradiction.

Since we get contradictions in all cases, our assumption that $f(x) <' f(y)$ is wrong. Therefore $f(y) \leq' f(x)$ whenever $x \leq y$. Thus f is order reversing. □

Now using Theorem 2.6, we can easily deduce that the only c-continuous bijections on a finite order induced c-space $(X, \mathfrak{C}_{\leq})$, where $X = \{x_1, x_2, \ldots, x_n\}$ are the identity function and the function $f(x_r) = x_{n+1-r}$, for $r = 1, 2, \ldots, n$.

Remark 2.7. Theorem 2.6 is not true for c-continuous onto function. Consider the order induced c-spaces $(X, \mathfrak{C}_{\leq})$ and $(Y, \mathfrak{C}_{\leq'})$, where $X = \{1, 2, 3, 4\}$, $Y = \{1, 2\}$, $\leq$ and $\leq'$ are the usual ordering of integers on X and Y respectively. The c-continuous onto function $f : X \to Y$ given by $f(1) = f(4) = 1$, $f(2) = f(3) = 2$ is neither order preserving nor order reversing.

Theorem 2.8. *If $(X, \leq)$ and $(Y, \leq')$ are order isomorphic, then $(X, \mathfrak{C}_{\leq})$ and $(Y, \mathfrak{C}_{\leq'})$ are c-isomorphic. Conversely, if $(X, \mathfrak{C}_{\leq})$ and $(Y, \mathfrak{C}_{\leq'})$ are c-isomorphic, then either $(X, \leq)$ is order isomorphic to $(Y, \leq')$ or $(X, \leq)$ is order isomorphic to the dual of $(Y, \leq')$.*

Proof. Suppose the linearly ordered sets $(X, \leq)$ and $(Y, \leq')$ are order isomorphic. Then there exists a bijection $f : X \to Y$ such that both f and f^{-1} are order preserving. By Theorem 2.4, f and f^{-1} are c-continuous. Hence the order induced c-spaces $(X, \mathfrak{C}_{\leq})$ and $(Y, \mathfrak{C}_{\leq'})$ are c-isomorphic.

Conversely, suppose the order induced c-spaces $(X, \mathfrak{C}_{\leq})$ and $(Y, \mathfrak{C}_{\leq'})$ are c-isomorphic. Then there exists a c-isomorphism $f : (X, \mathfrak{C}_{\leq}) \to (Y, \mathfrak{C}_{\leq'})$. By Theorem 2.6, f is either order preserving or order reversing. Suppose $f : X \to Y$ is order preserving. To prove $f^{-1} : Y \to X$ is order preserving, let $y_1, y_2 \in Y$ with $y_1 <' y_2$. Then there exists $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. We have either $x_1 < x_2$ or $x_2 < x_1$. If $x_2 < x_1$, then $f(x_2) <' f(x_1)$, that is $y_2 <' y_1$, which is not true. Therefore $x_1 < x_2$, that is $f^{-1}(y_1) < f^{-1}(y_2)$. Thus f^{-1} is order preserving and hence $f : (X, \leq) \to (Y, \leq')$ is an order isomorphism. Now let f is order reversing. Then we can easily prove that, for every $y_1, y_2 \in Y$, $y_1 \leq' y_2$ implies $f^{-1}(y_2) \leq f^{-1}(y_1)$. Thus f^{-1} is order reversing. That is, $f : (X, \leq) \to (Y, \leq')$ is a bijection such that f and f^{-1} are order reversing. Therefore $(X, \leq)$ and the dual of $(Y, \leq')$ are order isomorphic. □

3 t-closed sets in an order induced c-space

Here we discuss t-closed subsets of an order induced c-space $(X, \mathfrak{C}_{\leq})$.

Definition 3.1. [5, page 3] Let $(X, \mathfrak{C})$ be a c-space and $A \subseteq X$. A point $x \in X$ is said to touch the set A if there is a nonempty $C \subseteq A$ such that $\{x\} \cup C \in \mathfrak{C}$. The set of all points touching the set A is denoted by $t(A)$. Then A is said to be t-closed if $t(A) = A$. For example, $\emptyset$ and X are t-closed in $(X, \mathfrak{C})$.

If $|X|$ is finite then $t(A) \neq A$, for every proper nontrivial subset A of X. That is, $\emptyset$ and X are the only t-closed subsets of $(X, \mathfrak{C}_{\leq})$. If $|X|$ is infinite then there may or may not exist proper nontrivial t-closed subsets. Let us go through the following example.

Example 3.2. Consider the order induced c-space $(\mathbb{N}, \mathfrak{C}_{\leq'})$, where the ordering $\leq'$ is given by

$$1 \leq' 3 \leq' 5 \leq' \dots \dots 6 \leq' 4 \leq' 2.$$

Let $A = \{1, 3, 5, \dots\}$, then $t(A) = A$. That is, $(\mathbb{N}, \mathfrak{C}_{\leq'})$ has a proper nontrivial t-closed subset. But $\emptyset$ and $\mathbb{Z}$ are the only t-closed sets of the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{\leq})$, where $\leq$ is the usual ordering of integers.

Theorem 3.3. *Let $(X, \mathfrak{C}_{\leq})$ be an order induced c-space. If $(X, \mathfrak{C}_{\leq})$ is 2-generated, then no proper nontrivial subset of X is t-closed.*

Proof. Consider the 2-generated order induced c-space $(X, \mathfrak{C}_{\leq})$. If $|X| \leq 2$, then there is nothing to prove. Now let $|X| > 2$ and A be a proper nontrivial subset of X. Choose $x \in A$ and $y \in A^c$, which always exists and let $x \leq y$. Then by Proposition 1.1, there exist finite number of elements $b_1, b_2, \dots, b_n \in X$ such that $x = b_1 < b_2 < \dots < b_n = y$ and b_{i+1} is the immediate successor of b_i, for $i = 1, 2, \dots, n-1$. Clearly $b_n \in A^c$. Let k be the smallest integer such that $b_k \in A^c$. Then $\{b_{k-1}, b_k\} \in \mathfrak{C}_{\leq}$ and $b_{k-1} \in A$ implies $b_k \in t(A)$. Therefore, $t(A) \neq A$, hence A is not t-closed. Similar will happen if $y < x$. Hence there does not exist any proper nontrivial t-closed sets in $(X, \mathfrak{C}_{\leq})$. □

Remark 3.4. Converse of the Theorem 3.3 is not true. Consider the order induced c-space $(X, \mathfrak{C}_{\leq'})$, where $X = \{0, 1, 2, 3, \dots\}$ and the ordering $\leq'$ is given by

$$1 \leq' 2 \leq' 3 \leq' \dots \leq' 0.$$

Here $\emptyset$ and X are the only t-closed sets of $(X, \mathfrak{C}_{\leq'})$, but it is not a 2-generated c-space.

Remark 3.5. Theorem 3.3 is not true for 2-generated c-spaces. Consider the 2-generated c-space $(X, \mathfrak{C})$, where $X = \{1, 2, 3, 4\}$ and $\mathfrak{C} = \mathfrak{D} \cup \{\{1, 2\}\}$. Let $A = \{1, 2\}$, then $t(A) = A$ implies A is t-closed in $(X, \mathfrak{C})$. Thus a 2-generated c-space may have a proper nontrivial t-closed subset.

For an order induced c-space, the closed and bounded interval $[a, b]$ need not be t-closed. Consider the following example.

Example 3.6. Consider the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{\leq})$, where $\leq$ is the usual ordering of integers. Here $[2, 4] = \{2, 3, 4\}$, but $t([2, 4]) = \{1, 2, 3, 4, 5\}$. Since $t([2, 4]) \neq [2, 4]$, the closed and bounded interval $[2, 4]$ is not t-closed.

In the next theorem, we give a necessary and sufficient condition for which every closed and bounded interval is t-closed in $(X, \mathfrak{C}_{\leq})$.

Theorem 3.7. *For an $(X, \mathfrak{C}_{\leq})$ be the order induced c-space corresponding to the linearly ordered set $(X, \leq)$, the following are equivalent.*

(i) $(X, \leq)$ *is dense.*

(ii) *For every* $a, b \in X$ *with* $a \leq b$*,* $[a, b]$ *is t-closed in* $(X, \mathfrak{C}_{\leq})$*.*

(iii) *For every* $x \in X$*,* $\{x\}$ *is t-closed in* $(X, \mathfrak{C}_{\leq})$*.*

Proof. $(i) \Rightarrow (ii)$: Suppose $(X, \leq)$ is a dense linearly ordered set. Assume $[a, b]$ is not t-closed for some $a, b \in X$, where $a \leq b$. Then $t([a, b]) \neq [a, b]$, hence there exists $x \in [a, b]^c$ such that $x \in t([a, b])$. Then there exists a nonempty subset $C \subseteq [a, b]$ such that $\{x\} \cup C \in \mathfrak{C}_{\leq}$. Choose an element $c_o \in C$, then clearly $a \leq c_o \leq b$. Since $x \in [a, b]^c$, either $x < a$ or $b < x$. Now let $x < a$. Since $(X, \leq)$ is dense, there exists $z \in X$ such that $x < z < a$. Then $x < z < a \leq c_0$ and $x, c_o \in \{x\} \cup C$ implies $z \in \{x\} \cup C$. Since $z \neq x$, we have $z \in C$. Hence $z \in [a, b]$, a contradiction since $z < a$. We get similar contradiction if $b < x$. Thus $[a, b]$ is t-closed in $(X, \mathfrak{C}_{\leq})$, for every $a, b \in X$ with $a \leq b$.

$(ii) \Rightarrow (iii)$: Suppose $[a, b]$ is t-closed, for every $a, b \in X$ with $a \leq b$. Take $a = b = x$, then we get $\{x\}$ is t-closed.

$(iii) \Rightarrow (i)$: Suppose $t(\{x\}) = \{x\}$, for every $x \in X$. Assume that $(X, \leq)$ is not a dense linearly ordered set. Then there exists $a, b \in X$ with $a < b$ such that there does not exist any element in X between a and b. This implies $\{a, b\} \in \mathfrak{C}_{\leq}$. Then $b \in t(\{a\})$, a contradiction. Therefore, for every $a, b \in X$ with $a < b$, there exists $z \in X$ such that $a < z < b$. That is, $(X, \leq)$ is dense. □

Remark 3.8. If $(X, \leq)$ is a dense linearly ordered set, then the order induced c-space $(X, \mathfrak{C}_{\leq})$ has proper nontrivial t-closed subsets.

Remark 3.9. The property of being dense in Theorem 3.8 cannot be replaced by completeness property. The order induced c-space $(\mathbb{Z}, \mathfrak{C}_{\leq})$ corresponding to the complete linearly ordered set $(\mathbb{Z}, \leq)$ has no proper nontrivial t-closed subsets, where $\leq$ is the usual ordering of integers.

Theorem 3.10. *Let* $(X, \leq)$ *be a dense linearly ordered set and* $(X, \mathfrak{C}_{\leq})$ *be the corresponding order induced c-space. For* $A \subseteq X$*, if* $t(A) = X$ *then* A *is dense in* $(X, \leq)$*.*

Proof. Suppose $A \subseteq X$ be such that $t(A) = X$. To prove A is dense in $(X, \leq)$, let $x, y \in X$ be such that $x < y$. We need to show that there exists $a \in A$ such that $x < a < y$. Since X is dense itself, there exists $z \in X$ such that $x < z < y$. If $z \in A$, then there is nothing to prove. Now let $z \in A^c$. Since $z \in t(A)$, there exists a nonempty subset $C \subseteq A$ such that $\{z\} \cup C \in \mathfrak{C}_{\leq}$. Choose an element $a_0 \in C \subseteq A$. If $x < a_0 < y$, then the proof is complete. Now suppose $a_0 \leq x$. Since X is dense itself, there exists $p \in X$ be such that $x < p < z$. Then $a_0 \leq x < p < z$ and $a_0, z \in \{z\} \cup C$ implies $p \in \{z\} \cup C$. Hence $p \in C \subseteq A$, since $p \neq z$. Similar will happen if $y \leq a_0$. That is, there always exists $p \in A$ such that $x < p < y$. Therefore A is dense in $(X, \leq)$. □

Remark 3.11. Converse of the Theorem 3.10 is not true. Consider the order induced c-space $(\mathbb{R}, \mathfrak{C}_{\leq})$ corresponding to the dense linearly ordered set $(\mathbb{R}, \leq)$, where $\leq$ is the usual ordering of numbers. Here $\mathbb{Q}$ is dense in $(\mathbb{R}, \leq)$, but $t(\mathbb{Q}) \neq \mathbb{R}$ in $(\mathbb{R}, \mathfrak{C}_{\leq})$.

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