Palestine Journal of Mathematics

Vol. 12(3)(2023) , 186–193

On another approach for characterization of moment sequences with determinants

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 44A60; Secondary 15A15, 11B83.

Keywords and phrases: Moment problem, Hankel determinants, generalized Fibonacci sequences.

The authors wish to express their gratitude to the anonymous referees for their careful reading of the manuscript.

Abstract In this paper, we aim to provide new and simple proof for the determinant characterization of moment sequences with discrete measures by investing in generalized Fibonacci sequences.

1 Introduction

Let $s = (s_n)_{n\geq 0}$ be an infinite sequence of real numbers. s is positive semidefinite, if for all $\xi_0, \xi_1, \dots, \xi_n \in \mathbb{N}$ and $n \in \mathbb{N}$, we have

$$\sum_{k,l=0}^{n} s_{k+l}\xi_k\xi_l \geq 0,$$

or equivalently $L_s(P^2) \geq 0$ for all $P \in \mathbb{R}[x]$, where L_s denotes the Riesz functional on $\mathbb{R}[x]$ defined by

$$L_s(x^n) = s_n, \quad n \in \mathbb{N}.$$

We recall the Hankel matrix $H_n(s)$ and the Hankel determinant $D_n(s)$ respectively by

$$H_n = \begin{pmatrix} s_0 & \cdots & s_n \\ \vdots & \vdots & \vdots \\ s_n & \cdots & s_{2n} \end{pmatrix} \textit{ and } \; D_n(s) = \det H_n(s).$$

The Hamburger moment problem associated with the sequence s concerns if there exists a positive Radon measure μ on $\mathbb{R}$ such that for all $n \in \mathbb{N}$ the integral $\int_{-\infty}^{+\infty} x^n d\mu$ converges and satisfies

$$s_n = \int_{-\infty}^{+\infty} x^n d\mu.$$

In the affirmative case, we say that s is a Hamburger moment sequence and that μ is a representing measure for s.

Assume that $s = (s_n)_{n\in\mathbb{N}}$ is the moment sequence of a positive measure μ on $\mathbb{R}$. Then, for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ we obtain

$$L_s(p^2) = \int (\sum_{k,l=0}^{n} a_k a_l x^{k+l}) d\mu = \sum_{k,l=0}^{n} a_k a_l s_{k+l} \geq 0.$$

So, the fact that the Hankel matrix $H_n(s)$ is positive semidefinite for each $n \in \mathbb{N}$, is a necessary condition for a sequence to be a moment sequence. Hamburger's Theorem(see [1] or [2]) states that this condition is also sufficient for the existence of a positive measure.

It is well known that a Hankel matrix H_n is positive definite if and only if all upper left submatrices have positive determinants [5, Theorem 7.2]. However, the condition that all the upper left submatrices have nonnegative determinant does not imply the positive semidefiniteness of H_n. It yields that the positive semidefiniteness is more difficult to check than the positive definiteness. The following example illustrates this fact.

Example 1.1. Let $s = (1, 2, 4, 8, 0, 0, 0, \ldots)$ be an infinite sequence of real numbers. We have

$$D_0 = 1, D_1 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, D_2 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 0 \end{vmatrix} = 0, D_3 = \begin{vmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 0 \\ 4 & 8 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{vmatrix} = 4096$$

and

$$D_4 = \begin{vmatrix} 1 & 2 & 4 & 8 & 0 \\ 2 & 4 & 8 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Hence, $D_k = 0$ for all $k \geq 0$. However, s is not positive semidefinite.

If $s = (s_n)_{n \geq 0}$ is a sequence of the moment, it is easy to see that if one of the finite Hankel matrices is singular, then all the following ones are also singular. Conversely, C. Berg and R. Szwarc [3, Theorem 1.1] proved that for a given infinite sequence of numbers $(s_n)_{n \geq 0}$, if the corresponding sequence of Hankel determinants $D_n = \det(H_n)$ satisfies

$$D_n > 0 \text{ for } n < n_0, \text{ and } D_n = 0 \text{ for } n \geq n_0, \tag{1.1}$$

then all Hankel matrices are positive semidefinite and in particular, $(s_n)_{n \geq 0}$ is the sequence of moments of a discrete measure concentrated in n_0 points of the real line.

In this paper, we aim to prove that if a sequence $s = (s_n)_{n \geq 0}$ verifies (1.1), then s is a generalized Fibonacci sequence, and this will allow us to provide a new and simple proof for the Theorem 1.1 in [3].

We recall that an r-generalized sequence $(s_n)_{n \geq 0}$, is defined by the initial conditions $(s_0, s_1, \ldots, s_{r-1})$ and the following linear recurrence relation of order r,

$$s_{n+1} = a_0 s_n + a_1 s_{n-1} + \ldots + a_{r-1} s_{n-r+1} \text{ for } n \geq r-1$$

where $a_0, a_1, \ldots, a_{r-1}$ and $r \in \mathbb{N}$ with $r \geq 2$ and $a_{r-1} \neq 0$.

The remainder of this paper is divided as follows. The next section provides some preliminaries that will be needed in this work. The third section is devoted to stating our main findings.

2 Preliminaries

In this section, we gather some results that will be used in the remainder of this paper.
Let $s = (s_j)_{j=0}^{j=2n}$ be a truncated sequence of $2n+1$ real numbers, called a $(2n)$-sequence, $(n \in \mathbb{N})$.
For all $k \leq n$ the Hankel matrix $H_k(s)$, is the matrix defined by

$$H_k(s) = (s_{i+j})_{i,j=0}^{k},$$

and $D_0, D_1, \ldots, D_n$ are the major minors of $H_n(s)$.

Definition 2.1. s is said to be positive definite if and only if:

$$\forall (a_0, \ldots, a_n) \in \mathbb{R}^{n+1},\ (a_0, \ldots, a_1) \neq (0, 0, \ldots) \Rightarrow \sum_{i,j=0}^{n} a_i a_j s_{i+j} > 0. \tag{2.1}$$

In terms of the Riesz functional L_s on $\mathbb{R}_{2n}[x]$ the condition (2.1) is equivalent to the requirement $L_s(P^2) > 0$ for all $p \in \mathbb{R}_n[x]$, $P \neq 0$.
Further, by Theorem 7.2 in [5] the condition (2.1) is equivalent to the condition

$$\forall k \in \{0, 1, \dots, n\}, D_k > 0$$

It is proved that if $s = (s_j)_{j=o}^{2n+1}$ is a real sequence such that the sequence $(s_j)_{j=o}^{2n}$ is positive definite, then by Lemma 9.1 in [4], s can be extended to a sequence $\tilde{s} = (s_j)_{j=0}^{2n+2}$ positive definite.
The map $<,>_{\tilde{s}}$ defined by $< P, Q >_{\tilde{s}} = L_{\tilde{s}}(P, Q)$ is a inner product over $\mathbb{R}_{n+1}[x]$. Bilinearity and symmetry are immediate and the positive definiteness follows from the fact that s is positive definite.
From the canonical basis $(1, x, \dots, x^{n+1})$ and using the Gram-Schmidt method, we construct an orthogonal basis formed of unit polynomials $(P_0, P_1, \dots, P_{n+1})$. This family, does not depend on s_{2n+2} (unless we want to normalize P_{n+1}).
The polynomials $(P_i)_{0 \leq i \leq n+1}$ are given by the formulas

$$P_0 = 1 \text{ and } P_k = \frac{1}{D_{k-1}} \begin{vmatrix} s_0 & \cdots & s_k \\ \vdots & \vdots & \vdots \\ s_{k-1} & \cdots & s_{2k-1} \\ 1 & \cdots & x^k \end{vmatrix}, \quad 1 \leq k \leq n+1.$$

In the sequel, if $s = (s_j)_{j=0}^{2n+1}$ is a real sequence such that $(s_j)_{j=0}^{2n}$ is positive definite then the family of unitary orthogonal polynomials $(P_0, \dots, P_{n+1})$ associated with the inner product $< P, Q >_{\tilde{s}} = L_{s\tilde{s}}(PQ)$, where $\tilde{s} = (s_j)_{j=0}^{2n+2}$ is a positive definite extension of s.
For simplicity, the polynomial P_{n+1} will be denoted by P.
In this work, we will use the following lemma, which follows from Theorems 9.4 and 9.6 in [4].

Lemma 2.2. *Let $s = (s_j)_{j=0}^{j=2n+1}$ be a real sequence such that $(s_j)_{j=o}^{2n}$ is positive definite, then s is a truncated moment sequence of the $(n+1)$-atomic measure:*

$$\mu_P = \sum_{j=1}^{n+1} m_j \delta_{\lambda_j},$$

where $m_j = L_s(\pi_j(x))$, $\pi_j(x) = \frac{P(x)}{P'(\lambda_j)(x-\lambda_j)}$ for all $1 \leq j \leq n+1$, $(\lambda_j)_{1 \leq j \leq n+1}$ are the roots of the polynomial P and δ_{λ_j} is the Dirac measure.

Proof. Let $f \in \mathbb{R}_{2n+1}[x]$, there exists $p_f, q_f \in \mathbb{R}_n[x)]$, such that $f = q_f P + p_f$.
By orthogonality, we have:

$$L_s(f) = L_s(q_f P + p_f) = L_s(p_f),$$

and since degree of p_f is lower or equal to n, then p_f coincides with its interpolating Lagrange polynomial at the points $((\lambda_i, p_f(\lambda_i))_{1 \leq i \leq n+1}$,

$$p_f = \sum_{j=1}^{n+1} \pi_j(x) p_f(\lambda_j).$$

Hence,

$$\begin{aligned} L_s(f) &= \sum_{j=1}^{n+1} p_f(\lambda_j) L_s(\pi_j(x)) \\ &= \sum_{j=1}^{n+1} f(\lambda_j) m_j \\ &= \int_{\mathbb{R}} f(x) d\mu_P(x). \end{aligned}$$

In particular, for all $0 \leq j \leq 2n+1$, $\quad s_j = \int_{\mathbb{R}} x^j d\mu_P(x)$. □

The following example illustrates this lemma.

Example 2.3. Let $s = (s_j)_{j=0}^{j=5} = (1, 1, 2, 6, 24, 104)$ be a 5-real sequence.
The sub-sequence $(s_j)_{j=0}^{j=4} = (1, 1, 2, 6, 24)$ is positive definite. We have

$$D_0 = D_1 = 1, D_2 = 4,$$
$$P_0 = 1, \quad P_1 = x - 1, \quad P_2 = x^2 - 4x + 2, \quad P = P_3 = x^3 - 5x^2 + 2x + 2,$$
$$\lambda_1 = 1, \quad \lambda_2 = 2 - \sqrt{6}, \quad \lambda_3 = 2 + \sqrt{6},$$
$$\text{and } m_1 = \frac{4}{5}, \qquad m_2 = \frac{6 + \sqrt{6}}{60}, \quad m_3 = \frac{6 - \sqrt{6}}{60}.$$

Thus the measure which represents $s = (s_j)_{j=0}^{j=5}$ is $\mu = \sum_{j=1}^{j=3} m_j \delta_{\lambda_j}$, so that

$$s_j = \int_{\mathbb{R}} x^j d\mu_P(x), \quad 0 \leq j \leq 5.$$

3 Main results

In this section, we present our main findings.

Theorem 3.1. *Consider an infinite sequence of numbers $s = (s_n)_{n \geq 0}$. If the corresponding sequence of Hankel determinants $D_n = \det(H_n)$ satisfies $D_n > 0$ for $n \leq r$ and $D_n = 0$ for $n > r$, then s is an $(r + 1)$-generalized Fibonacci sequence.*

To prove this theorem, we construct a finite measure μ, concentrated on $r + 1$ points which represents the truncated sequence $(s_0, s_1, \ldots, s_{2r+1})$ and we prove later, that the sequence s is represented by μ. For this, the following lemma will be useful.

Lemma 3.2. *Let $(s_j)_{j=0}^{j=2r+1}$ be a $(2r + 1)$-real sequence such that the sequence $(s_j)_{j=0}^{j=2r}$ is positive definite. If we denote the columns of $H_r(s)$ by $v_0, v_1, \ldots, v_r$ and if $v_{r+1} = (s_{r+1}, s_{r+2}, \ldots, s_{2r+1})^t$, the following assertions*

(i) $P = x^{r+1} - a_0 - a_1 x - \ldots - a_r x^r$.

(ii) $v_{r+1} = \sum_{i=0}^{i=r} a_i v_i$.

are equivalent for all $a_0, \ldots, a_r \in \mathbb{R}$.

Proof. Let $j \in \{0, 1, \ldots, r\}$, by orthogonality we have: $L_s(x^j P) = 0$.
Hence,

$$\forall j \in \{0, 1, \ldots, r\}, s_{r+j+1} - a_0 s_j - a_1 s_{j+1} - \ldots - a_r s_{j+r} = 0.$$

Thus, $v_{r+1} = \sum_{i=0}^{i=r} a_i v_i$.
Conversely, let λ be a root of the polynomial P, we have

$$P(\lambda) = \frac{1}{D_r} \begin{vmatrix} s_0 & \cdots & s_r & s_{r+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r} & s_{2r+1} \\ 1 & \cdots & \lambda^r & \lambda^{r+1} \end{vmatrix}.$$

Expanding according to the last row, we obtain

$$P(\lambda) = \frac{1}{D_r} [\lambda^{r+1} D_r + \sum_{k=0}^{r} (-1)^{k+1} \lambda^{r-k} D_{r,k}],$$

with

$$D_{r,k} = \begin{vmatrix} s_0 & \cdots & s_{r-k-1} & s_{r-k+1} & \cdots & s_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r-k-1} & s_{2r-k+1} & \cdots & s_{2r+1} \end{vmatrix}.$$

On the other hand,

$$v_{r+1} = \sum_{i=0}^{i=r} a_i v_i \Rightarrow v_{r+1} - \sum_{i=0, i\neq j}^{i=r} a_i v_i = a_j v_j.$$

Thus, replacing the last column v_{r+1} by $v_{r+1} - \sum\limits_{i=0, i\neq r-k}^{i=r} a_i v_i$, we obtain

$$D_{r,k} = \begin{vmatrix} s_0 & \cdots & s_{r-k-1} & s_{r-k+1} & \cdots & a_{r-k}s_{r-k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r-k-1} & s_{2r-k+1} & \cdots & a_{r-k}s_{2r-k} \end{vmatrix} = (-1)^k D_r a_{r-k}.$$

Hence,

$$\begin{aligned} P(\lambda) = 0 \Leftrightarrow \frac{1}{D_r}\left[\lambda^{r+1} D_r + \sum_{k=0}^{r} (-1)^{k+1} \lambda^{r-k} (-1)^k a_{r-k} D_r\right] = 0 \\ \Leftrightarrow \lambda^{r+1} + \sum_{k=0}^{r} (-1)^{2k+1} \lambda^{r-k} a_{r-k} = 0 \\ \Leftrightarrow \lambda^{r+1} = \sum_{k=0}^{r} \lambda^{r-k} a_{r-k} \\ \Leftrightarrow \lambda^{r+1} = \sum_{i=0}^{r} \lambda^{i} a_{i}. \end{aligned}$$

We deduce that λ is a root of P if and only if λ is a root of the polynomial $Q = x^{r+1} - \sum\limits_{i=0}^{i=r} a_i x^i$. The polynomials Q and P have the same degree and roots and are both unitary. Therefore $P = Q$ and the proof is ended. □

Now, we are in a position to provide proof of Theorem 3.1.

Proof. The sequence $(s_j)_{j=0}^{j=2r}$ is positive definite. Let us put:

$$P = P_{r+1} = x^{r+1} - \sum_{i=0}^{i=r} a_i x^i.$$

Then, using the previous Lemma 3.2, we have: $v_{r+1} = \sum\limits_{k=0}^{r} a_k v_k$. Hence,

$$\forall k \in \{r+1, r+2, \ldots, 2r+1\}, \quad s_k = \sum_{i=0}^{r} a_i s_{k+i-r-1}. \tag{3.1}$$

Let us show by induction that (3.1) is verified for all $k \geq 2r+1$.
For $k = 2r+1$, the proof is already done. Let $k > 2r+1$ and assume that (3.1) holds for all $k^{'} \in \{r+1, \ldots, k\}$ and let us show that,

$$s_{k+1} = \sum_{i=0}^{r} a_i s_{k+i-r}.$$

For all $k \geq 2r+1$, we have $k-r \geq r+1$, then $D_{k-r} = 0$, where

$$D_{k-r} = \begin{vmatrix} s_0 & \cdots & s_r & s_{r+1} & s_{r+2} & \cdots & s_{k-r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r} & s_{2r+1} & s_{2r+2} & \cdots & s_k \\ s_{r+1} & \cdots & s_{2r+1} & s_{2r+2} & s_{2r+3} & \cdots & s_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{k-r-1} & \cdots & s_{k-1} & s_k & s_{k+1} & \cdots & s_{2k-2r-1} \\ s_{k-r} & \cdots & s_k & s_{k+1} & s_{k+2} & \cdots & s_{2k-2r} \end{vmatrix}.$$

For $l \in \{r+1, \ldots, k-r\}$, let us replace the column v_l, by $v_l - \sum_{i=0}^{r} a_i v_{l+i-r-1}$ and we use the induction hypothesis. So we get

$$D_{k-r} = \begin{vmatrix} s_0 & \cdots & s_r & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r} & 0 & 0 & \cdots & 0 \\ s_{r+1} & \cdots & s_{2r+1} & 0 & 0 & \cdots & \alpha_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \alpha_2 \\ s_{k-r-1} & \cdots & s_{k-1} & 0 & \alpha_1 & \cdots & \cdots \\ s_{k-r} & \cdots & s_k & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-2r} \end{vmatrix},$$

with $\alpha_j = s_{k+j} - \sum_{i=0}^{i=r} a_i s_{k+j+i-r-1}$ for all $j \in \{1, \ldots, k-2r\}$.
This determinant takes the form

$$D_{k-r} = \begin{vmatrix} H_r(s) & 0 \\ C & A \end{vmatrix},$$

with

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 0 & \cdots & \alpha_1 & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_1 & \vdots & \cdots & \alpha_{k-2r-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{k-2r-1} & \alpha_{k-2r} \end{pmatrix}.$$

Thus, by performing a block calculation of the determinant D_{k-r}, we obtain

$$D_{k-r} = |A| \times |H_r(s)| = (-1)^{k-2r} \alpha_1^{k-2r} D_r.$$

Whence,

$$D_{k-r} = 0 \Rightarrow \alpha_1 = 0 \Rightarrow s_{k+1} = \sum_{i=0}^{r} a_i s_{k+i-r}.$$

Consequently, the sequence s is determined by the data of its $r+1$ first terms, $s_0, \ldots, s_r$ and by the following linear recurrence relation of order $r+1$:

$$s_k = a_0 s_{k-r-1} + a_1 s_{k-r} + \ldots + a_r s_{k-1}, \forall k \geq r+1.$$

This ends the proof of the theorem. □

The following corollary gives a new and simple proof of the Theorem 1.1 in [3].

Corollary 3.3. *If $s = (s_n)_{n \geq 0}$ satisfies the conditions of Theorem 3.1, then s is a sequence of moment of a discrete measure μ concentrated in $r+1$ points of the real line.*

Proof. The sequence $(s_j)_{j=0}^{2r}$ is definite positive. So, the sequence $(s_j)_{j=0}^{2r+1}$ is a moment sequence of the measure $(r+1)$-atomic $\mu_P = \sum\limits_{i=1}^{r+1} m_i \delta_{\lambda_i}$, where $(\lambda_j)_{1 \le j \le r+1}$ are the roots of $P = P_{r+1}$, and $m_j = L_s\left(\frac{P(x)}{P'(\lambda_j)(x-\lambda_j)}\right)$.
So, we have

$$\forall k \in \{0, \dots, 2r+1\}, s_k = \int_{\mathbb{R}} x^k d\mu_P(x). \tag{3.2}$$

To complete the proof, we will show, by induction on k, that

$$\forall k \ge 2r+2, s_k = \int_{\mathbb{R}} x^k d\mu_P(x).$$

For $k = 2r+2$, we have

$$\begin{aligned}
\int_{\mathbb{R}} x^{2r+2} d\mu_p(x) &= \sum_{i=1}^{r+1} m_i \lambda_i^{2r+2} \\
&= \sum_{i=1}^{r+1} m_i \lambda_i^{r+1} \lambda_i^{r+1} \\
&= \sum_{i=1}^{r+1} m_i \lambda_i^{r+1} \left[\sum_{k=0}^{r} a_k \lambda_i^k\right] \\
&= a_0 \sum_{i=1}^{r+1} m_i \lambda_i^{r+1} + \dots + a_r \sum_{i=1}^{r+1} m_i \lambda_i^{2r+1} \\
&= a_0 \int_R x^{r+1} d\mu_P(x) + \dots + a_r \int_R x^{2r+1} d\mu_P(x) \\
&= a_0 s_{r+1} + \dots + a_r s_{2r+1} \\
&= s_{2r+2}.
\end{aligned}$$

The last equality is obtained by the use of Theorem 3.1.
Let $k \ge 2r+2$, suppose that (3.2) holds for all $k' \le 2r+2$. We have

$$\begin{aligned}
\int_R x^{k+1} d\mu_P(x) &= \sum_{i=1}^{r+1} m_i \lambda_i^{k+1} \\
&= \sum_{i=1}^{r+1} m_i \lambda_i^{r+1} \lambda_i^{k-r} \\
&= \sum_{i=1}^{r+1} m_i \lambda_i^{k-r} \left(\sum_{j=0}^{r} a_j \lambda_i^j\right). \\
&= \sum_{j=0}^{r} a_j \sum_{i=1}^{r+1} m_i \lambda_i^{k-r+j} \\
&= \sum_{j=0}^{r} a_j \int_{\mathbb{R}} x^{k-r+j} d\mu_P(x) \\
&= \sum_{i=0}^{r} a_i s_{k-r+i} = s_{k+1}.
\end{aligned}$$

This achieves the proof. $\square$

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Received: 2022-01-15
Accepted: 2023-04-11