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On Lie ideals with generalized derivations and power values on prime rings

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Abstract Let R be a non-commutative prime ring of characteristic different from 2, U, C, L and H denote, the Utumi quotients ring, the extended centroid, a non-central Lie ideal, a non-zero generalized derivation of R respectively. If $[H(u), u]^n = [H(u), u]^m$ for all $u \in L$, with $1 < n \neq m \geq 1$, then one of the following holds:

(i) R satisfies the standard identity s_4, in four non-commutating variables and there exist $a \in U$ and $\alpha \in C$ such that $H(r) = ar + ra + \alpha r$ for all $r \in R$;

(ii) there exists $\lambda \in C$ such that $H(r) = \lambda r$ for all $r \in R$.

1 Introduction

In this article, R refers to a non-commutative prime ring with center $Z(R)$, U to its Utumi quotient ring, and $C = Z(U)$ to its center. For the definitions and associated attributes of these objects, we direct the reader to [2]. An additive mapping $d : R \to R$ is termed as a derivation if $d(rs) = d(r)s + rd(s)$ for all $r, s \in R$. If $H(rs) = H(r)s + rd(s)$ holds for all $r, s \in R$ and d is a derivation of R, then the additive mapping H is known as a generalized derivation on R. It goes without saying that every derivation is a generalized derivation. A map of the type $r \mapsto a_1 r + r a_2$ is an example of a generalized derivation, where a_1, a_2 are fixed elements in R. These type of mappings are known as generalized inner derivations. Due to Lanski [11], if d is a derivation such that $d(r)^n = 0$ for all $r \in L$, then d is zero, where n is a positive integer. Lee in [13] demonstrated an analogy result for generalized derivations. More specifically, Lee demonstrated that if H is a generalized derivation of R and L is a non-central Lie ideal of R, then R is commutative if $H(r)^n = 0$ for every $r \in L$. Carini and De Filippis [3] investigated a non-zero derivation d of R such that $[d(r), r]^n = 0$ for every $r \in L$ of R, where n is a positive integer, and they came to the conclusion that R is commutative if $char(R) \neq 2$.

Further, De Filippis extended this result in [5] by substituting a generalized derivation H for d, and concluded that there exist $\lambda \in C$ such that $H(r) = \lambda r$ for all $r \in R$, unless when R satisfies s_4 and there exists an element $a \in U$ such that $H(r) = ar + ra$ for all $r \in R$. Let R be a non-commutative prime ring with characteristic different from 2, as demonstrated by the author in article [16], H a non-zero generalized derivation of R. If $[H(u), u]^n = [H(u), u]$, for all $u \in L$, then either R satisfies the standard identity s_4 and there exist $a \in U$ and $\alpha \in C$ such that $H(r) = ar + ra + \alpha r$ or there exists $\lambda \in C$ such that $H(r) = \lambda r$ for all $r \in R$. In this paper, authors obtain same conclusion by considering the generalized identity $[H(u), u]^n = [H(u), u]^m$ for suitable condition on R. We revise a few widely accepted facts before moving on to the proof of our conclusion.

Lemma 1.1 ([13]). *Since each generalized derivation can be specifically extended to a generalized derivation of U, it is implicitly assumed that all generalized derivations of R are defined on the entire U.*

Lemma 1.2 ([12]). *The differential identity is fulfilled by R, I, and U if I is a two-sided ideal of R.*

Lemma 1.3 ([4]). *The generalized polynomial identity with coefficients in U is satisfied by R, I and U if I is a two-sided ideal of R.*

Lemma 1.4 ([8, 6]). *There exists a nonzero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$ if we assume that either R does not satisfy s_4 or $char(R) \neq 2$. Particularly, it follows that $[R, R] \subseteq L$ if R is a simple ring.*

2 Inner Generalized Derivations Case

In order to simplify this section, we will assume that H is an inner generalized derivation, with $H(r) = a_1 r + r a_2$ for every $r \in R$, where a_1, a_2 are constant components of U. As a result, we may rely on the assumption that R fulfills the following generalized identity:

$$P(r_1, r_2) = \big[a_1[r_1, r_2] + [r_1, r_2]a_2, [r_1, r_2]\big]^n - \big[a_1[r_1, r_2] + [r_1, r_2]a_2, [r_1, r_2]\big]^m,$$

for each $r_1, r_2 \in R$. In order to prove the result in this section, we also need the following:

Lemma 2.1. *By the main assumption of the paper we have that*

$$\big[a_1[r_1, r_2] + [r_1, r_2]a_2, [r_1, r_2]\big]^n - \big[a_1[r_1, r_2] + [r_1, r_2]a_2, [r_1, r_2]\big]^m = 0$$

for all $r_1, r_2 \in R$. Furthermore, for any inner automorphism ϕ of R one get that

$$\big[\phi(a)[r_1, r_2] + [r_1, r_2]\phi(b), [r_1, r_2]\big]^n - \big[\phi(a)[r_1, r_2] + [r_1, r_2]\phi(b), [r_1, r_2]\big]^m = 0$$

for all $r_1, r_2 \in R$. Trivially, $a_1, a_2, a_1 + a_2, a_1 - a_2$ are central in R iff $\phi(a_1), \phi(a_2), \phi(a_1 + a_2), \phi(a_1 - a_2)$ are central in R. Hence, we can replace a_1, a_2 respectively with $\phi(a_1), \phi(a_2)$.

Proposition 2.2. *Suppose that R is any non-commutative prime ring with $char(R) \neq 2$, $a_1, a_2 \in R$ such that $[a_1 u + u a_2, u]^n = [a_1 u + u a_2, u]^m$ for all $u \in [R, R]$, then either $a_1, a_2 \in C$ or R satisfies s_4 and $a_2 - a_1 \in C$, where $1 < n \neq m \geq 1$.*

Proof. The following generalized polynomial identities are satisfied by R:

$$P(r_1, r_2) = \big[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]\big]^n - \big[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]\big]^m.$$

Then, in accord with Beidar's [1, Theorem 2], U also satisfies this extended polynomial identity. If C is not finite, then $P(s_1, s_2) = 0$ for all $s_1, s_2 \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. As U and $U \otimes_C \overline{C}$ are centrally closed ([7], see Theorem 2.5, 3.5), replace R by U or $U \otimes_C \overline{C}$, depending on whether C is finite or infinite. Therefore, we assume that R is centrally closed over C which is either algebraic closed or finite. If $a_1, a_2 \in C$, then our job is done, so we consider that either $a_1 \notin C$ or $a_2 \notin C$. Using [4], $P(r_1, r_2)$ is non trivial generalized polynomial identity for R. Hence, by Martindale's theorem [14], R is primitive ring with non-zero socle S with C as associated division ring, then Jacobson theorem [9] says that a dense ring of linear transformations of a certain vector space V over C and R are isomorphic.

If $dim_C(V) = 2$, then $R \simeq M_2(C)$, the ring of 2×2 matrices over C. Say $a_2 - a_1 = \sum c_{ij} e_{ij}$ where $c_{ij} \in C$ and e_{ij} are usual unit matrices. Let $[r_1, r_2] = [e_{ii}, e_{ij}] = e_{ij}$ for $i \neq j$. Now, using this in our assumption, we get

$$X = [a e_{ij} + e_{ij} a_2, e_{ij}]^n - [a_1 e_{ij} + e_{ij} a_2, e_{ij}]^m = 0.$$

Hence, (i, j)-entry of X is zero. Using calculations, we get $c_{ij} = 0$ for all $i \neq j$. Therefore, $a_2 - a_1$ is a diagonal matrix. Let $\phi(r) = (1 + e_{ij}) r (1 - e_{ij})$ be an inner automorphisms, where

$i \neq j$, then by *Lemma* 2.1, $\phi(a_2 - a_1)$ is a diagonal matrix. So, (i, j)-entry of $\phi(a_2 - a_1)$ is zero.

$$\begin{aligned} 0 = [\phi(a_2 - a_1)]_{ij} &= \sum_{ij} c_{ij} e_{ij} \\ &= c_{jj} - c_{ii}. \end{aligned}$$

Hence, $a_2 - a_1 \in C$.

Assume $dim_c(V) \geq 3$, since $[R, R]$ satisfies

$$[H(u), u]^n - [H(u), u]^m = 0 \text{ for all } u \in [R, R].$$

$$[a_1 u + u a_2, u]^n - [a_1 u + u a_2, u]^m = 0 \text{ for all } u \in [R, R] \text{ and for some } a_1, a_2 \in U.$$

This implies that

$$[a_1 u^2 + ucu - u^2 a_2]^n - [a_1 u^2 + ucu - u^2 a_2]^m = 0 \text{ for all } u \in [R, R], \tag{2.1}$$

and for some $a_1, a_2 \in U$. Suppose that for some $v \in V$, $\{v, bv\}$ is linearly C-independent and then there exists $w \in V$ such that $\{v, a_2 v, w\}$ is linearly C-independent. Then, Jacobson's Theorem says that there exists $u_1, u_2 \in R$ such that

$$u_1 v = 0,\ u_2 v = v,\ u_1 a_2 v = w$$

$$u_2 a_2 v = bv,\ u_1 w = -2v,\ u_2 w = 0.$$

Therefore, $uv = 0,\ ua_2 v = w,\ uw = 2v$. Multiply from right side by v to (2.1) to obtain $m = n$, which contradicts. Hence, for all $v \in V$, we have $\{v, bv\}$ is linearly C-dependent and for all $v \in V$, we get $a_2 v = \alpha_v v$ for some $\alpha_v \in C$. Clearly, α_v is independent of the choice of v. Therefore, we can assume that $a_2 v = \alpha v$. Now, $[a_2, v]v = a_2(uv) - u(a_2 v) = \alpha(uv) - u(\alpha v) = 0$, since $\alpha \in C$. Which implies that $[a_2, u]v = 0$ for all $v \in V$, that is, $[a_2, u]V = (0)$. Since $[a_2, u]$ is acting faithfully as a linear transformation of vector space V, $[a_2, u] = 0$ for all $u \in R$. Then, $a_2 \in Z(R)$. Therefore, (2.1) becomes

$$[-cu^2 + ucu]^n - [-cu^2 + ucu]^m = 0 \text{ for all } u \in [R, R] \text{ and for some } a_1, a_2 \in U. \tag{2.2}$$

Again, for some $v \in V$, $\{v, cv\}$ is linearly C-independent and then there exists $w \in V$ such that $\{v, cv, w\}$ is linearly C-independent, then from Jacobson's Theorem, there exist $u_1, u_2 \in R$ such that

$$u_1 v = 0,\ u_2 v = v,\ u_1 cv = -2v$$

$$u_2 cv = 0,\ u_1 w = -v,\ u_2 w = v.$$

Hence, $uv = 0,\ ucv = 2v,\ uw = v$. Further, multiply right side by w in (2.2) to get $m = n$, that is a contradiction again. Therefore, for every $v \in V$, $\{v, cv\}$ is linearly C-dependent and using the same argument as we discussed the above, we find $c \in C$. Hence, we obtain the desired outcome. □

3 The General Case

We assume that H is a nonzero generalized derivation throughout this section. We can suppose that there exists $a \in U$ and d, a derivation on R such that $H(r) = ar + d(r)$ (see [13]) in order to demonstrate our core argument.

Theorem 3.1. *Suppose that R is a non-commutative prime ring with $char R \neq 2$ and H a nonzero generalized derivation of R, U stands for the Utumi quotient ring of R, C as the extended centroid of R. If $[H(u), u]^n = [H(u), u]^m$ for all $u \in L$, a non-central Lie ideal of R with $1 < n \neq m \geq 1$, then anyone of the following is true:*

(i) there exist $a \in U$ and $\alpha \in C$ such that $H(r) = ar + ra + \alpha r$ for all $r \in R$ and R satisfies the standard identity s_4;

(ii) there exists $\lambda \in C$ such that $H(r) = \lambda r$ for all $r \in R$.

Proof. Using Lemma 1.1, we can extend generalized derivation to Utumi quotient ring U. Furthermore, by Lemma 1.4, there exists a nonzero two sided ideal I of R such that $0 \neq [I, R] \subseteq L$. Next, using Lemma 2.1, I and U satisfy the same differential identity, therefore for all $r_1,\ r_2 \in U$,

$$[a[r_1, r_2] + d([r_1, r_2]), [r_1, r_2]]^n = [a[r_1, r_2] + d([r_1, r_2]), [r_1, r_2]]^m.$$

Considering Khachenko's theory ([10]), we reach the following two instances:

(i) d *as an inner derivation:* One can consider that $d(r) = [c, r]$ and $H(r) = (a + c)r - rc$, so that U satisfies

$$[(a + c)[r_1, r_2] - [r_1, r_2]c, [r_1, r_2]]^n = [(a + c)[r_1, r_2] - [r_1, r_2]c, [r_1, r_2]]^m.$$

Using Proposition 2.2, we arrived at desired conclusion.

(ii) d *as an outer derivation:* This case yields that for all $r_1, r_2 \in U$

$$[a[r_1, r_2] + d([r_1, r_2]), [r_1, r_2]]^n = [a[r_1, r_2] + d([r_1, r_2]), [r_1, r_2]]^m.$$

Using Kharchenko's theory, U satisfies the generalized polynomial identity:

$$[a[r_1, r_2] + [s_1, r_2] + [r_1, s_2], [r_1, r_2]]^n = [a[r_1, r_2] + [s_1, r_2] + [r_1, s_2], [r_1, r_2]]^m.$$

Particularly, $s_1 = s_2 = 0$ then from Proposition 2.2, we find $a \in C$. Therefore, we obtain the following

$$[[s_1, r_2] + [r_1, s_2], [r_1, r_2]]^n = [[s_1, r_2] + [r_1, s_2], [r_1, r_2]]^m. \tag{3.1}$$

Then, a well known result due to Posner [15], $U \simeq M_p(K)$, the ring of $p \times p$ matrices over field K. Trivially, for $m \geq 2$ R is non-commutative. So, we assume (3.1) with

$$r_1 = e_{ii},\ r_2 = e_{ji},\ s_1 = 0,\ s_2 = 2e_{ij},$$

then we obtain

$$(2^{n-m} - 1)(e_{jj} - e_{ii}) = 0.$$

Which leads to a contradiction. Therefore, we obtain the required conclusion. □

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