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ON PROJECTIVE RELATION OF GENERALIZED METRIC WITH CERTAIN (α, β) METRICS

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Abstract In this article, we have shown that two important pairs of (α, β)-metrics, namely, generalized (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, (where μ_1, μ_2 and μ_3 are constants) with Kropina metric and generalized (α, β)-metric with Randers metric, have same Douglas tensors if and only if both are Douglas metrics. Furthermore, we study the projective relation between two important (α, β)-metrics with dimension $n \geq 0$, where β and $\bar{\beta}$ are one-forms, while α and $\bar{\alpha}$ denote Riemannian metrics.

1 Introduction

The projective change of Finsler metric is a fascinating concept to investigate in Finsler geometry. On a manifold M, two Finsler metrics F and $\bar{F}$ are called projectively related if every geodesic of the first metric is also a geodesic of the second metric and vice-versa. Many authors [2, 3, 9, 10, 11, 12, 13], have investigated the projective relation between two Finsler metrics.

The paper [13] by Rapsack provides us with a very important and necessary result related to the projective change, dealing with necessary and sufficient conditions for projective relation. In the recent year many authors [5, 6, 7, 14], discussed the projective changes between special (α, β)-metric with Kropina metric in detail. In [8], Pradeep Kumar et al. have introduced a generalized (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ (where μ_1, μ_2 and μ_3 are constants) and discussed projective change.

The (α, β) metric is a substantial and significant class of Finsler metrics. It can be expressed as $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$. where α denotes Riemannian metric, β is 1-form and ϕ represents the positive C^∞ function. L. Berwald was the first author to introduce the Kropina metric $F = \frac{\alpha^2}{\beta}$ and was studied by V. K. Kropina [4]. However, the Kropina metric is a non-regular Finsler metric, but the Randers metric $F = \alpha + \beta$ is a regular Finsler metric. Kropina metric is regarded as a significant and elementary Finsler metric with a plethora of interesting and useful applications in physics, irreversible thermodynamics, dissipative mechanics, and electron optics in the presence of a magnetic field. They also have interesting applications in relativistic field theory, evolution, and developmental biology given in [1].

The aim of this article is to continue the investigation on the generalized (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ and to study the projective relation between two (α, β)-metrics. The results have been proved in two cases in section 3. In the first part, we have proved that both generalized metric F and Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics and also, we have proved F is projectively related to Kropina metric $\bar{F}$. Furthermore, we proved that the generalized metric F and Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ are Douglas metrics and are projectively related.

2 Preliminaries

Let $F^n = (M, F)$ be a Finsler space with a fundamental function $F(x, y)$ on a differential manifold M [1]. We have

* $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$,
* $C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{ij}$,
* $h_{ij} = g_{ij} - l_i l_j$,
* $\gamma^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}$,
* $G^i = \frac{1}{2}\gamma^i_{jk}y^i y^k$, $G^i_j = \dot{\partial}_i G^i$, $G^i_{jk} = \dot{\partial}_k G^i_j$, $G^i_{jkl} = \dot{\partial}_l G^i_{jk}$.

The theory of (α, β)-metric $F^n = (M, F(\alpha, \beta))$ is originated by M. Matsumoto in 1972 and analyzed by several authors [3, 5, 7]. Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if and only if [3]

$$G^i = \bar{G}^i + P(y)y^i, \tag{2.1}$$

here, $P(y)$ denotes a scalar-function on $TM\backslash\{0\}$ and homogeneous of degree one in y. The geodesic of F must satisfy the following ODE by certain Finsler metric $F := F(x, y)$

$$2G^i\left(x, \frac{dx}{dt}\right) = -\frac{d^2x^i}{dt^2},$$

here $G^i = G^i(x, y)$ is a geodesic coefficient, G^i is provided by

$$G^i = \frac{\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}}{4}g^{il}.$$

The following relation is satisfied by the positive C^∞ function $\phi = \phi(s)$, $|s| < b_0$

$$(b^2 - s^2)\phi^{''}(s) + \phi(s) - s\phi^{'}(s) > 0, \quad (|s| \leq b < b_0). \tag{2.2}$$

The $F = \alpha\phi(x)$, $s = \frac{\beta}{\alpha}$, is a regular (α, β)-metric, if $\| \beta_x \|_\alpha < b_0$, $\forall x \in M$ is satisfied by the Riemannian metric $\alpha^2 = a_{ij}y^i y^j$ and 1-form $\beta = b_i y^i$. Let $\bigtriangledown\beta = b_{i|j}dx^i \otimes dx^j$ be covariant derivative of β with respect to α. Denote $r_{ij} = \frac{1}{2}(b_{j|i} + b_{i|j})$ and $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$, where $b_{i|j}$ is known as coefficient of the covariant derivative of β in reference to α. Taking note of the fact that $s_{ij} = 0$ if and only if β is closed [5]. Let $s_j = b^i s^{ij}$, $s^i_j = a^{il}s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and $r_{00} = r_{ij}y^i y^j$.
The spray coefficient G^i, geodesic coefficient G^i_α of F and α are connected by [10].

$$G^i = \{-2\alpha Q s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\} + \alpha Q s^i_0 + G^i_\alpha, \tag{2.3}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Psi = \frac{1}{2}\frac{\phi''}{(b^2 - s^2)\phi'' + (\phi - s\phi')},$$

$$\Theta = \frac{\phi\phi' - s(\phi'\phi' + \phi\phi'')}{2\phi((b^2 - s^2)\phi'' + (\phi - s\phi'))}.$$

For the Kropina metric $F = \frac{\alpha^2}{\beta}$, it is very easy to see that it is not a regular (α, β)-metric but the relation $\phi(s) + (b^2 - s^2)\phi^{''}(s) - s\phi^{'}(s) > 0$ is still true for $|s| > 0$.

Definition 2.1. [3] Let $D := D^i_{jkl}\dot{\partial}_i \otimes dx^i \otimes dx^k \otimes dx^l$ is Douglas tensor.

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i + G^i\right), \tag{2.4}$$

here G^i is the spray coefficient of F. A Finsler metric is called Douglas metric if $D^i_{jkl} = 0$ [6].

Let

$$\bar{G}^i = \Psi\{-2\alpha Q s_0 + r_{00}\}b^i + \alpha Q s^i_0 + G^i_\alpha. \tag{2.5}$$

Then (2.3) reduces to

$$G^i = \bar{G}^i + \Theta\{r_{00} - 2\alpha Q s_0\}\frac{y^i}{\alpha}.$$

According to (2.1), G^i and $\bar{G}^i$ are projective equivalents and they are having similar Douglas tensors. Let

$$T^i = \Psi\{r_{00} - 2\alpha Q s_0\}b^i + \alpha Q s^i_0. \tag{2.6}$$

Then $\bar{G} = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} &= \bar{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{(n+1)}\frac{\partial T^m}{\partial y^m}y^i + G^i - \frac{1}{(n+1)}\frac{\partial G^m_\alpha}{\partial y^m}y^i\right). \end{aligned}$$

The Douglas tensor of an (α,β)-metric is given by

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \tag{2.7}$$

We use $\alpha_{y^k} = \alpha^{-1}$; $s_{y^k} = \alpha^{-2}(b_k\alpha - s y_k)$ to simplify (2.7), where $y_i = a_{il}y^l$ and α_{y^k} means $\frac{\partial \alpha}{\partial y^k}$. Then

$$[\alpha Q S^m_0]_{y^m} = \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s^m_0 + \alpha^{-1}y_m Q s^m_0 = Q' s_0,$$

and

$$[\Psi(r_{00} - 2Q\alpha s_0)b^m]^m_y = -\alpha^{-1}\Psi'[r_{00} - 2Q\alpha s_0](s^2 - b^2) + 2\Psi[r_0 + Q'(s^2 - b^2)s_0 - Qss_0],$$

where $r_0 = r_i y^i$ and $r_j = b^i r_{ij}$. Thus using (2.6), we have

$$T^m_{y^m} = 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0] + Q's_0 - \Psi'\alpha^{-1}(s^2 - b^2)[r_{00} - 2Q\alpha s_0], \tag{2.8}$$

Now, assume that the Douglas tensor is the same for F and $\bar{F}$. Hence, $D^i_{jkl} = \bar{D}^i_{jkl}$. From (2.4) and (2.7), we can write

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\right) = 0.$$

Then there exist a class of a scalar function $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i, \tag{2.9}$$

where $H^i_{00} = H^i_{jk}y^i y^k$, T^i and $T^m_{y^m}$ are given by (2.6) and (2.8).

3 Projective relation between two classes of (α, β)-metric

For a Finsler space $F^n = (M, F)$, the metric $F = F(x, y)$ is a Finsler metric provided $\| \beta \| < b_0$ and the geodesic coefficients of Finsler metric are obtained by Eq. (2.3).

(a). **Generalized (α, β)-metric** $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$.

For the generalized metric F, equation (2.2) can be used to demonstrate that F is a usual Finsler metric, given $\| \beta_x \|_\alpha < 1$, for any $x \in M$.

The geodesic coefficients are provided by (2.3) as follows

$$\begin{aligned} Q &= \frac{\mu_2 + 2\mu_3 s}{\mu_1 - \mu_3 s^2}, \\ \Psi &= \frac{\mu_3}{(\mu_1 + 2\mu_3 b^2 - 3\mu_3 s^2)}, \\ \Theta &= \frac{\mu_1\mu_2 - 3\mu_2\mu_3 s^2 - \mu_3^2 s^3}{2(\mu_1 + \mu_2 s + \mu_3 s^2)(\mu_1 + 2\mu_3 b^2 - 3\mu_3 s^2)}. \end{aligned} \tag{3.1}$$

(b). **Kropina metric** $\bar{F} = \dfrac{\bar{\alpha}^2}{\bar{\beta}}$.

For the Kropina metric $\bar{F}$, the geodesic coefficients are provided by (2.3) that yields

$$\begin{aligned} \bar{\Theta} &= -\frac{s}{\bar{b}^2}, \\ \bar{Q} &= -\frac{1}{2s}, \\ \bar{\Psi} &= \frac{1}{2\bar{b}^2}. \end{aligned} \tag{3.2}$$

(c). **Randers metric** $\bar{F} = \bar{\alpha} + \bar{\beta}$.

For the Randers metric $\bar{F}$, the geodesic coefficients are provided by (2.3) that yields

$$\begin{aligned} \bar{Q} &= 1, \\ \bar{\Psi} &= 0, \\ \bar{\Theta} &= \frac{1}{(2 + 2s)}. \end{aligned} \tag{3.3}$$

Now, we discuss the projective relation between generalized (α, β)-metric with Kropina and Randers metrics.

3.1 Projective relation between $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$.

This section is about the projective relation between generalized metric F and Kropina metric $\bar{F}$ on the same fundamental manifold M of dimension $n \geq 3$.
Now, we have the following theorem

Theorem 3.1. *Let* $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$ *be a generalized* (α, β)*-metric and* $\bar{F} = \dfrac{\bar{\alpha}^2}{\bar{\beta}}$ *be a Kropina- metric on* n*-dim manifold* M $(n \geq 3)$*, where* β *and* $\bar{\beta}$ *are two non-zero 1-forms,* α *and* $\bar{\alpha}$ *are two Riemannian metrics. Then Finsler metric* F *and* $\bar{F}$ *are Douglas metrics if and only if both are Douglas tensors.*

Proof. Let Finsler metric F and $\bar{F}$ be Douglas metrics with Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$ (by the definition of Douglas metric), i.e., the pair F and $\bar{F}$ are having identical Douglas tensor.
Conversely, suppose F, as well as $\bar{F}$, have the identical Douglas tensors, at that time Eq. (2.9) holds.
Substituting (3.1) and (3.2) in (2.9), we get

$$H^i_{00} = \frac{(A^i\alpha^{11} + B^i\alpha^{10} + C^i\alpha^9 + D^i\alpha^8 + E^i\alpha^7 + F^i\alpha^6 + G^i\alpha^5 + H^i\alpha^4 + I^i\alpha^3 + J^i\alpha^2 + K^i\alpha + L^i)}{M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i} + \frac{\bar{A}^i\alpha^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}}, \tag{3.4}$$

where

$$
\begin{aligned}
A^i =&\mu^2(\mu_1+2\mu_3b^2)(-2\mu_3b^is_0(\mu_2+2\mu_3)\beta+2\mu_2\mu_3b^2s^i_0),\\
B^i =&\mu_1^2(\mu_1+2\mu_3b^2)((\mu_1\mu_2+2\mu_1\mu_2\beta+4\mu_3^2b^2\beta)s^i_0-2\mu_1\mu_3r_0\lambda y^i+\mu_1\mu_3b^ir_{00}),\\
C^i =&4\mu_1\mu_2\mu_3^2(2\mu_1+\mu_2b^2)(b^2s^i_0\beta^2+(2\mu_1\mu_3\beta^3+1)b^is_0)+\beta^2[-3\mu_1^2\mu_2\mu_3(\mu_1\\
&+2\mu_3b^2)]s^i_0+\beta[-2\mu_1^2\mu_2\mu_3(\mu_1+2\mu_3b^2)]s_0\lambda y^i,\\
D^i =&\beta^3[-\mu_1\mu_3^2(\mu_1+2\mu_3b^2)(9\mu_1+4\mu_3b^2)]+\beta^2[-2\mu_1^2\mu_2\mu_3(2\mu_1+\mu_3b^2)]s^i_0+\\
&\beta^2[-\mu_1\mu_3^2b^i(\mu_1+2\mu_3b^2)+a\mu_1^2+4\mu_1\mu_3b^2]r_{00}+\beta^2[4\mu_1^2\mu_3^2(2\mu_1+\mu_3b^2)]\\
&r_0\lambda y^i+\beta(-6\mu_1^3\mu_3^2b^2r_{00}\lambda y^i)+2\mu_1^2\mu_3(-\mu_1(\mu_1+2\mu_3b^2)+6\mu_3b^2(\mu_2+\mu_3)\beta\\
&+2\mu_3\beta^2(\mu_1+\mu_3b^2))s_0\lambda y^i+\beta^2[4\mu_1^2\mu_3^2(\mu_1+2\mu_3b^2)]\lambda y^i,\\
E^i =&\beta^5[2\mu_3^2b^i(\mu_2+2\mu_3)(\mu_1+2\mu_3b^2+6\mu_1\mu_3)s_0]+\beta^4[2\mu_2\mu_3^2\{\mu_1(b^2+6\mu_1)+\\
&\mu_3b^2(9\mu_1+2b^2)\}]+\beta^3[-\mu_1\mu_2\mu_2^3(22\mu_1+8\mu_3b^2)\lambda y^i],\\
F^i =&\beta^5[2\mu_3^2(\mu_1+2\mu_3b^2)(\mu_1+6\mu_1\mu_3^2+2\mu_3b^2)]s^i_0+\beta^4[\mu_3((\mu_1+2\mu_3b^2)+6\mu_3^2)\\
&\{\mu_1(\mu_2s^i_0+\mu_3b^ir_{00})\}]+2\mu_1\mu_3^3(2\mu_1+\mu_3b^2)(6s^i_0\beta^3+b^ir_{00}\beta^2)-\beta^4[12\mu_1\mu_3^2\\
&(\mu_1+\mu_3b^2)r_0\lambda y^i]-\beta^4[16\mu_1\mu_3^3(2\mu_1+\mu_3b^2)s_0\lambda y^i]-\beta^4(12\mu_1^2\mu_3^2\lambda y^i)+\beta^3\\
&[6\mu_1^2\mu_3^2(\mu_1+3\mu_3b^2)r_{00}\lambda y^i]-\beta^3[24\mu_1\mu_3^2b^2(\mu_2+1)+4\mu_1^2\mu_3^2+8\mu_1^2\mu_3^2b^2]\lambda s_0y^i\\
&-\beta^2[2\mu_1^2\mu_3^2(\mu_1-2\mu_3b^2)-12\mu_1^2\mu_3^2+8\mu_1\mu_3^4b^4]s_0\lambda y^i+\beta[2\mu_1^2\mu_3^2(\mu_1+4\mu_3b^2)\\
&+8\mu_1\mu_3^4b^4]s_0\lambda y^i,\\
G^i =&\beta^6[6\mu_3^3b^i(\mu_2+\mu_3)s_0]-\beta^6[3\mu_2\mu_3^2(\mu_1+4\mu_3b^2+6\mu_1\mu_3)s^i_0]+\beta^5(36\mu_1\mu_2\mu_3^3)\\
&s_0\lambda y^i-\beta^3[8\mu_1\mu_2\mu_3^2(\mu_1-\mu_3b^2)+12\mu_1\mu_2\mu_3^2(b^2-6)+4\mu_2\mu_3^3b^4]s_0\lambda y^i+\beta^2\\
&[2\mu_1\mu_2\mu_3^3(1+4b^2)+4\mu_2\mu_3^3b^2+\mu_3+2b^2]\lambda y^i,\\
H^i =&\beta^7[-6\mu_3^3(\mu_1+2\mu_2b^2)\{\mu_1^2(\mu_1+2\mu_3b^2)+6\mu_1^3\mu_3+1\}]s^i_0+\beta^6[-3\mu_1\mu_2\mu_3^2s^i_0\\
&-\mu_1\mu_3^3\{\mu_1b^i(\mu_1+2\mu_3b^2)^2+6\mu_1\mu_3^2(\mu_1+2\mu_3b^2)+3b^i\}]r_{00}+\beta^6(24\mu_1\mu_3^4)\\
&s_0\lambda y^i-\beta^5[6\mu_3^3(1+12\mu_1^2)+6\mu_1\mu_3^3b^2(1+2n)]r_{00}\lambda y^i+\beta^5[12\mu_3^3b^2(\mu_3-\mu_2)\\
&s_0\lambda y^i]-\beta^4[18\mu_1^2\mu_3^3-12\mu_1\mu_3^3(\mu_1+2\mu_3b^2)]s_0\lambda y^i-\beta^3[4\mu_1\mu_3^3(3+4\mu_3b^2)\\
&-2\mu_1^2\mu_3^3-8\mu_3^4b^4]\lambda y^i,\\
I^i =&\beta^8(9\mu_2\mu_3^3s^i_0)-\beta^7[6\mu_2\mu_3^3(\mu_2+2)]s_0\lambda y^i-\beta^5[-6\mu_2\mu_3^3(\mu_1+b^2)+18\mu_1\mu_2\mu_3^2]\\
&s_0\lambda y^i-\beta^4[12\mu_2\mu_3^3(1+b^2)]s_0\lambda y^i,\\
J^i =&3\mu_3^4(6\beta^9s^i_0+6\beta^8r_{00})-\beta^8[6\mu_3^4(r_0+2s_0)\lambda y^i]-\beta^7[6\mu_3^4b^2+6\mu_1\mu_3^3(1-2\mu_3)]\\
&r_{00}\lambda y^i-\beta^6\mu_3^4(10\mu_1-32\mu_3b^2)s_0\lambda y^i+\beta^5[18\mu_1\mu_3^4-12\mu_3^4(1+2b^2)]s_0\lambda y^i,\\
K^i =&18\mu_2\mu_3^4\beta^6s_0\lambda y^i,
\end{aligned}
$$

$$
\begin{aligned}
L^i =& 18\mu_3^5\beta^7 s_0\lambda y^i - 12\mu_3^5\beta^8 - 6\mu_3^3\beta^9 r_{00}\lambda y^i,\\
M^i =& \mu_1^3(\mu_1 + 2\mu_3 b^2)^2,\\
N^i =& \beta^2[-\mu_1^2\mu_3(\mu_1 + 2\mu_3 b^2)(9\mu_1 + 6\mu_3 b^2)],\\
O^i =& \beta^4[\mu_1\mu_3\{3\mu_1\mu_3(7\mu_1 + 8\mu_3 b^2) + (\mu_1 + 2\mu_3 b^2)(\mu_1 + 2\mu_3 b^2\\
& + 4\mu_3(2\mu_1 + \mu_3 b^2))\}],\\
P^i =& -\mu_1[6\mu_1 + 12\mu_1\mu_3 b^2 + 18\mu_1^2\mu_3 + 3\mu_1\mu_3(7\mu_1 + 8\mu_3 b^2) + (\mu_1 + 2\mu_3 b^2)],\\
Q^i =& 15\mu_1\mu_3^3 + 3\mu_3^4(6\mu_1 + 4b^2),\\
R^i =& -9\mu_3^4\beta^1 0,\\
\bar{A}^i =& \bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0,\\
\bar{B}^i =& \bar{\beta}[2\lambda\bar{y}^i(\bar{r}_0 + \bar{s}_0) - \bar{b}^i\bar{r}_{00}],\\
& \text{and}\\
\lambda =& \frac{1}{n+1}.
\end{aligned} \tag{3.5}
$$

Thus, (3.4) is equivalent to

$$
\begin{aligned}
&\left(A^i\alpha^{11} + B^i\alpha^{10} + C^i\alpha^9 + D^i\alpha^8 + E^i\alpha^7 + F^i\alpha^6 + G^i\alpha^5 + H^i\alpha^4 + I^i\alpha^3 + J^i\alpha^2\right.\\
&\left.+K^i\alpha + L^i\right) \times \left(2\bar{b}^2\bar{\beta}\right) + \left(\bar{A}^i\alpha^2 + \bar{B}^i\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2\right.\\
&\left.+R^i\right) = H^i_{00}\left(2\bar{b}^2\bar{\beta}\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right).
\end{aligned} \tag{3.6}
$$

Replacing (y^i) by $(-y^i)$ in (3.6) yields

$$
\begin{aligned}
&\left(-A^i\alpha^{11} + B^i\alpha^{10} - C^i\alpha^9 + D^i\alpha^8 - E^i\alpha^7 + F^i\alpha^6 - G^i\alpha^5 + H^i\alpha^4 - I^i\alpha^3 + J^i\alpha^2\right.\\
&\left.-K^i\alpha + L^i\right) \times \left(-2\bar{b}^2\bar{\beta}\right) - \left(\bar{A}^i\alpha^2 + \bar{B}^i\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2\right.\\
&\left.+R^i\right) = -H^i_{00}\left(2\bar{b}^2\bar{\beta}\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right).
\end{aligned} \tag{3.7}
$$

Adding (3.6) and (3.7), we get

$$
\left(A^i\alpha^{11} + C^i\alpha^9 + E^i\alpha^7 + G^i\alpha^5 + I^i\alpha^3 + K^i\alpha\right) \times \left(2\bar{b}^2\bar{\beta}\right) = 0.
$$

Rearranging the above equation

$$
\left(A^i\alpha^{11} + C^i\alpha^9 + E^i\alpha^7 + G^i\alpha^5 + I^i\alpha^3 + K^i\alpha\right) = 0. \tag{3.8}
$$

Using (3.8), equation (3.4) is equivalent to

$$
\begin{aligned}
&\left(B^i\alpha^{10} + D^i\alpha^8 + F^i\alpha^6 + H^i\alpha^4 + J^i\alpha^2 + L^i\right)\left(2\bar{b}^2\bar{\beta}\right)\\
&+\left(\bar{A}^i\alpha^2 + \bar{B}^i\right) \times \left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right)\\
&= H^i_{00}\left(2\bar{b}^2\bar{\beta}\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right).
\end{aligned} \tag{3.9}
$$

From (3.9), we observe that $\bar{A}^i\alpha^2\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right)$ is divided by $\bar{\beta}$. Since, $\beta = \varphi\bar{\beta}$, then $\bar{A}^i\alpha^2 M^i\alpha^{10}$ can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime as related to α and $\bar{\alpha}$, $\bar{A}^i = \bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$.
Hence,

$$
\bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0 = \bar{\beta}\psi^i. \tag{3.10}
$$

Contracting (3.10) by $\bar{y}_i = \bar{a}_{ij}$, we get $\psi^i(\bar{y}_i) = -\bar{s}_0$. Since $\bar{y}_i$ is arbitrary vector, we get $\psi^i(x) = -\bar{s}^i$. Then we have

$$
\bar{s}_{ij} = \frac{1}{\bar{b}^2}\left[\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\right], \tag{3.11}
$$

provided $\bar{b}^2 \neq 0$. By Lemma 2.3(see,[7]), if $n = 2$ then $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric. As a result, the fact that F and $\bar{F}$ have identical Douglas tensors indicates that they follow Douglas metrics. □

Hence, the proof.
Now, we state and prove the following,

Theorem 3.2. *Let $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ be a generalized (α,β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on n-dim manifold M $(n \geq 3)$. The Finsler metric F is projectively connected to $\bar{F}$ if and only if the following equation holds true*

$$G^i_\alpha + \tau\frac{\mu_3}{\mu_1}\alpha^2 b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + r_{00}\bar{b}^i\right) + \theta y^i, \tag{3.12}$$

where β and $\bar{\beta}$ are non-zero 1-forms while α and $\bar{\alpha}$ are Riemannian metrics. $b^i = a^{ij}b_j$, $\bar{b}^i = \bar{a}^{ij}\bar{b}_j$, $\bar{b}^2 = \| \bar{\beta} \|^2_{\bar{\alpha}}$, scalar $\tau = \tau(x)$ and $\theta = \theta_i y^i$ is a one-form on M.

Proof. Let the Douglas tensor is invariant in the projective relation between two Finsler metrics. Since, F is projectively related to $\bar{F}$, moreover both have the identical Douglas tensor.

From theorem (3.1), we have proved that together (α,β)-metrics F as well as $\bar{F}$ are Douglas metrics.

We know that generalized (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, is Douglas metric if and only if

$$b_{i|j} = 2\tau\left[\left(1 + 2\frac{2\mu_3}{\mu_1}b^2\right)a_{ij} - \frac{3\mu_3}{\mu_1}b_i b_j\right]. \tag{3.13}$$

Within this instance, β is closed.
Substituting (3.13) and (3.1) in (2.3), we get

$$G^i = G^i_\alpha + \tau\left[\frac{\mu_1\mu_2\alpha^3 - 4\mu_3^2\beta^3 - 3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2 + 2\mu_2\alpha\beta + \mu_3\beta^2\right)}\right]y^i + \tau\frac{\mu_3\alpha^2}{\mu_1}b^i. \tag{3.14}$$

Also, substituting (3.15) and (3.6) in (2.3), we get

$$\bar{G}^i = \bar{G}^i_\alpha - \frac{1}{2\bar{b}^2}\left\{-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + \frac{2\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}y^i\right\}. \tag{3.15}$$

In view of F is projectively related to $\bar{F}$, there is a scalar $P = P(x,y)$ on $TM\backslash\{0\}$ as a result

$$G^i = \bar{G}^i + Py^i. \tag{3.16}$$

By using (3.14), (3.15) and (3.16), we have

$$\begin{aligned}\left\{P - \tau\left[\frac{\mu_1\mu_2\alpha^3 - 4\mu_3^2\beta^3 - 3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2 + 2\mu_2\alpha\beta + \mu_3\beta^2\right)}\right] - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right\}y^i = \\ G^i_\alpha - \bar{G}^i_\alpha + \tau\frac{\mu_3\alpha^2}{\mu_1}b^i - \frac{1}{2\bar{b}^2}\left(-\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right).\end{aligned} \tag{3.17}$$

The RHS of (3.17) is quadratic. Then there exist $\theta = \theta_i(x)y^i$ is a 1-form on M such that

$$\left\{P - \tau\left[\frac{\mu_1\mu_2\alpha^3 - 4\mu_3^2\beta^3 - 3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2 + 2\mu_2\alpha\beta + \mu_3\beta^2\right)}\right] - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right\} = \theta. \tag{3.18}$$

Thus, we have

$$G^i_\alpha + \tau\frac{\mu_3\alpha^2}{\mu_1}b^i = \bar{G}^i_\alpha + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i. \tag{3.19}$$

Conversely, from (3.14), (3.15) and (3.19), we get

$$G^i = \bar{G}^i + \left[\theta + \frac{\mu_1\mu_2\alpha^3 - 4\mu_3^2\beta^3 - 3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2 + 2\mu_2\alpha\beta + \mu_3\beta^2\right)}\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right] y^i. \tag{3.20}$$

From (3.16), we have

$$P = \theta + \frac{\mu_1\mu_2\alpha^3 - 4\mu_3^2\beta^3 - 3\mu_2\mu_3\alpha\beta^2}{\mu_1\left(2\mu_1\alpha^2 + 2\mu_2\alpha\beta + \mu_3\beta^2\right)}\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right) y^i. \tag{3.21}$$

This concludes that F is projectively related to $\bar{F}$. Hence proved. □

From theorem 3.1 and theorem 3.2, we obtain the following result,

Corollary 3.3. *The Finsler metric* $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ *and* $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, *on a* n*-dim manifold* M $(n \geq 3)$, *where* β *and* $\bar{\beta}$ *denote non-zero 1-forms and* α *and* $\bar{\alpha}$ *denote Reimannian metrics. The Finsler metric* F *is projectively related to* $\bar{F}$ *if and only if* $G^i_\alpha = \bar{G}^i_\alpha + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i - \tau\frac{\mu_3\alpha^2}{\mu_1}b^i$, *and* $\bar{s}_{ij} = \frac{1}{\bar{b}^2}\left[\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\right]$.

3.2 Projective relation between $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$.

Now, we obtain the projective relation between generalized metric F and Randers metric $\bar{F}$ on the same underlying manifold M of dimension $n \geq 3$.
First, we state the following.

Theorem 3.4. *Let* $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ *be a generalized metric and* $\bar{F} := \bar{\alpha} + \bar{\beta}$ *be a Randers metric on* n*-dim.* M $(n \geq 3)$, *where* β *and* $\bar{\beta}$ *are non-zero 1-forms while* α *and* $\bar{\alpha}$ *are Riemannian metrics. The Finsler metrics* F *and* $\bar{F}$ *have the same Douglas tensor if and only if both are Douglas metrics.*

Proof. Let Finsler metrics F and $\bar{F}$ are Douglas metrics with Douglas tensors D^i_{jkl} and $\bar{D}^i_{jkl}$. Then $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$ (by the definition of Douglas metric), i.e., the pair F as well as $\bar{F}$ are having the identical Douglas tensor.
Conversely, if suppose F and $\bar{F}$ have identical Douglas tensor, we claim equation (2.9).
Substituting (3.1) and (3.3) in (2.9), we get

$$\begin{aligned} H^i_{00} = &\frac{(A^i\alpha^{11} + B^i\alpha^{10} + C^i\alpha^9 + D^i\alpha^8 + E^i\alpha^7 + F^i\alpha^6 + G^i\alpha^5 + H^i\alpha^4 + I^i\alpha^3}{M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i} \\ &\frac{+J^i\alpha^2 + K^i\alpha + L^i)}{} - (\bar{\alpha} \times \bar{s}^i_0), \end{aligned} \tag{3.22}$$

where all the coefficients of different powers of α are stated in equation (3.9).
Furthermore, equation (3.22) can be re-written as

$$\begin{aligned} &\left(A^i\alpha^{11} + B^i\alpha^{10} + C^i\alpha^9 + D^i\alpha^8 + E^i\alpha^7 + F^i\alpha^6 + G^i\alpha^5 + H^i\alpha^4 + I^i\alpha^3 + J^i\alpha^2\right. \\ &\left. + K^i\alpha + L^i\right) = \left(H^i_{00} + (\bar{\alpha} \times \bar{s}^i_0)\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right). \end{aligned} \tag{3.23}$$

Replacing y^i by $-y^i$, we have

$$\begin{aligned} &\left(-A^i\alpha^{11} + B^i\alpha^{10} - C^i\alpha^9 + D^i\alpha^8 - E^i\alpha^7 + F^i\alpha^6 - G^i\alpha^5 + H^i\alpha^4 - I^i\alpha^3 + J^i\alpha^2\right. \\ &\left. - K^i\alpha + L^i\right) = \left(H^i_{00} - (\bar{\alpha} \times \bar{s}^i_0)\right)\left(M^i\alpha^{10} + N^i\alpha^8 + O^i\alpha^6 + P^i\alpha^4 + Q^i\alpha^2 + R^i\right). \end{aligned} \tag{3.24}$$

Subtracting (3.24) from (3.23), we get

$$\begin{aligned}&\left(A^i\alpha^{11}+C^i\alpha^9+E^i\alpha^7+G^i\alpha^5+I^i\alpha^3+K^i\alpha\right)\\&=\left(\bar{\alpha}\times\bar{s}^i_0\right)\left(M^i\alpha^{10}+N^i\alpha^8+O^i\alpha^6+P^i\alpha^4+Q^i\alpha^2+R^i\right).\end{aligned}\tag{3.25}$$

Adding (3.23) and (3.24), we get

$$\begin{aligned}&\left(B^i\alpha^{10}+D^i\alpha^8+F^i\alpha^6+H^i\alpha^4+J^i\alpha^2+L^i\right)\\&=H^i_{00}\left(M^i\alpha^{10}+N^i\alpha^8+O^i\alpha^6+P^i\alpha^4+Q^i\alpha^2+R^i\right).\end{aligned}\tag{3.26}$$

From (3.25), we can see that $\bar{\alpha}R^i\bar{s}^i_0$ containing the element α. Now, we divide the proof into two different cases, and the result is as follows.

Case 1: Suppose $\bar{\alpha}\neq\delta(x)\alpha$, then α^2 is one of the term in $\bar{\alpha}R^i\bar{s}^i_0=-9\mu_3^4\bar{\alpha}\beta^{10}\bar{s}^i_0$. But β^2 has no factor α^2. Then the only possibility is that $\beta\bar{s}^i_0$ has the factor α^2. This is because β^{10} and α are relatively prime polynomials of y^i. When $n\geq 3$, suppose $\tau^i\neq 0$, then

$$2\geq rank(b_k\bar{s}^i_j)+rank(b_j\bar{s}^i_k)>rank(b_k\bar{s}^i_j+b_j\bar{s}^i_k)=rank(2\tau^i\alpha_{jk})\geq 3.\tag{3.27}$$

Equation (3.27) satisfies only when $\tau^i=0$. Then $\bar{s}^i_0=0$, which implies $\bar{\beta}$ is closed.

Case 2: Suppose $\bar{\alpha}=\delta(x)\alpha$, then (3.25) becomes

$$\begin{aligned}&\left(A^i\alpha^{10}+C^i\alpha^8+E^i\alpha^6+G^i\alpha^4+I^i\alpha^2+K^i\right)\alpha\\&=\left(M^i\alpha^{10}+N^i\alpha^8+O^i\alpha^6+P^i\alpha^4+Q^i\alpha^2+R^i\right)\alpha\delta(x)\bar{s}^i_0+\delta(x)\bar{s}^i_0R^i,\end{aligned}\tag{3.28}$$

We observe that $\delta(x)\bar{s}^i_0R^i=-9\mu_3^4\beta^{10}\bar{s}^i_0\delta(x)$ has the factor of α. Since, deg$(x)\neq 0$ then $\beta^{10}\bar{s}^i_0$ has the factor α implying $\bar{s}^i_0=0$, This shows that β is closed.

It is known that the Randers metric $\bar{F}=\bar{\alpha}+\bar{\beta}$ is a Douglas metric. Now, from the above result, we can conclude that F and $\bar{F}$ are Douglas metrics. □

Hence the proof.
Now we can state the following results.

Theorem 3.5. *Let $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\beta^2}{\alpha}$ be a generalized metric and $\bar{F}=\bar{\alpha}+\bar{\beta}$ be a Randers metric on n-dim. M $(n\geq 0)$. The Finsler metric F is projectively related to $\bar{F}$ if and only if the following equations holds true*

$$\begin{aligned}G^i_\alpha&=\bar{G}^i_{\bar{\alpha}}+Py^i,\\b_{i|j}&=0,\\d\beta&=0,\end{aligned}\tag{3.29}$$

where $b=\parallel\beta\parallel_\alpha$ and P is a scalar function on TM $\{0\}$.

Proof. We know that for two Finsler metrics, the Douglas tensor is projectively invariant. Therefore F and $\bar{F}$ are identical Douglas metrics if and only if they are projectively related. In view of Theorem 3.3, we have both F and $\bar{F}$ are Douglas metrics. Consider a generalized (α,β)-metric is a Douglas metric if n=2 then β is parallel to α. Thus

$$b_{i|j}=0.\tag{3.30}$$

Substituting (3.30) and (3.1) in (2.3), we get

$$G^i=G^I_\alpha.$$

Since, F is projectively equivalent to $\bar{F}$ and β is closed, then $\bar{F}$ is Douglas metric that is

$$\bar{s}_{ij} = 0. \tag{3.31}$$

Substituting (3.31) and (3.3) in (2.3), we get

$$\bar{G}^{ij} = G^i_\alpha + \frac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})} y^i. \tag{3.32}$$

Suppose F is projectively equivalent to $\bar{F}$, we have

$$G^i = \bar{G}^{ij} + Py^i. \tag{3.33}$$

From (3.30), (3.32) and (3.33), we get

$$G^i_\alpha = \bar{G}^i_\alpha + \left\{\frac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})} + P\right\} y^i. \tag{3.34}$$

We know that the RHS of (3.34) is quadratic. Then it follows that 1-form $\theta = \theta_i y^i$ on M such that

$$G^i = \bar{G}^i_{\bar{\alpha}} + \theta y^i, \tag{3.35}$$

where $\theta = P + \dfrac{\bar{r}_{00}}{2(\bar{\alpha}+\bar{\beta})}$. From (3.30), (3.32) and (3.34), α is projectively related to $\bar{\alpha}$. Conversely, if $\bar{\beta}$ is closed. It can be proved that α is projectively related to $\bar{\alpha}$. Substituting (3.30) and (3.1) in (2.3), we get (3.32). Also, (3.32), (3.3) and (2.3) yields (3.33). Now, from (3.31), (3.33) and (3.35), we get $G^i = \bar{G}^i_{\bar{\alpha}} + \theta y^i$. Hence, F is projectively related to $\bar{F}$.

Also, if $\bar{F}$ is locally-Minkowskian, $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}_i y^i$ is 1-form with $\bar{b}_i$ is constant. Then

$$\begin{aligned} &G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + Py^i, \\ &b_{i|j} = 0. \end{aligned} \tag{3.36}$$

□

Thus, we state the following

Corollary 3.6. *Finsler metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$ is projectively equivalent to $\bar{F} = \bar{\alpha} + \bar{\beta}$ if and only if F is projectively flat. Furthermore, if F is projectively flat, the equation* (3.36) *holds.*

4 Conclusion

In this article, we have obtained the projective relation between two important (α,β)-metrics. At first, we have considered generalized (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$ with Kropina metric $\bar{F} = \dfrac{\bar{\alpha}^2}{\bar{\beta}}$. Subsequently, we have considered generalized (α,β)-metric F with Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$. Where α and $\bar{\alpha}$ represents Riemannian metrics and β and $\bar{\beta}$ denotes non-zero 1-forms. This study provides two important results of two different (α,β)-metric pairs under the projective change.

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