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SEMI-NIL CLEAN RINGS

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Abstract In this paper, we introduce a ring R to be a semi-nil clean ring if every element of R is a sum of a nilpotent element and a periodic element. This ring is a kind of generalization of the nil-semi clean ring, which is available in the literature. In fact, for commutative rings, the semi-nil clean and nil-clean rings are the same. We give several characterizations and examples for the semi-nil clean rings.

1 Introduction

In [9], Nicolson defined a ring R to be a clean ring if each element of R is the sum of a unit and an idempotent element. Clean rings received much attraction after the work of Anderson and Camilo [5]. Based on this concept of defining a new ring, authors defined various new rings such as weakly clean rings [2], n-clean rings [10], semiclean rings[11] etc. All these rings have been defined on unit elements. Recently, researchers got their interest in rings which are defined on nilpotent elements. In this direction, Alexander J. Diesl [3] defined a ring R to be a nil clean ring if each element of R can be written as a sum of a nilpotent element and an idempotent element, and if nilpotent element and idempotent element commutes, then the ring is called strongly nil clean ring. The articles [3] and [8] extensively discussed nil clean rings and strongly nil clean rings. In [7], Peter V. Danchev and W.Wm. McGovern defined the notion of a weakly nil clean ring which is a generalization of nil clean ring as similar to weakly clean rings the generalization of clean rings. They provided various characterizations as well as various properties of weakly nil clean rings. In particular, they characterized the weakly nil clean group ring and proved that the Nagata ring and power series rings are never weakly nil clean.

Now taking inspiration from above, we define a semi-nil clean element as follows:

Definition 1.1. An element of a ring R is said to be semi-nil clean if it can be written as a sum of a nilpotent element and a periodic element.

If all the elements of a ring R are semi-nil clean, then the ring R is called a semi-nil clean ring. All finite rings are semi-nil clean rings. There are many examples of infinite rings that are semi-nil clean. For example, consider $R = \mathbb{Z}_2[x_1, x_2, \cdots]/(x_1^3 - x_1^2, x_2^2, x_3^2, \cdots)$, then R is semi-nil clean ring. Another example is the product of infinite copies of $\mathbb{Z}_2$ which is an infinite semi-nil clean. We would like to mention here that the article [1] discusses the class of nil-semi clean rings that came to our knowledge when we were preparing this article. The paper [1] derived several results of nil-semi clean rings. It is interesting to note that the periodic elements other than the non-zero nilpotent elements are semi-idempotent elements which are discussed in [1]. Therefore, the nil-semi clean rings are a particular case of semi-nil clean rings. Since the set of nilpotent elements forms an ideal in a commutative ring, the notion of semi-nil clean and nil-semi clean are the same for the commutative rings. These two notions of rings are different only in the case of non-commutative rings. In Section 2, we give an example of a ring in which the notion of the semi-nil clean element and the nil-semi clean element are different. We discuss the behaviour of semi-nil clean property on fundamental ring constructions of homomorphic images, polynomial adjunctions and direct products in detail. We provide a characterization for indecomposable semi-nil rings of prime characteristics. Finally, we talk about the ring of Morita context concerning the semi-nil clean property.

Throughout the paper, R denotes an associative ring unless otherwise mentioned. The set of periodic elements, nilpotent elements and the Jacobson radical are denoted by $Per(R), N(R)$ and $J(R)$ respectively.

2 Properties of semi-nil clean rings

We start this section by giving an example of a ring in which there is a distinction between the notion of semi-nil clean element and nil-semi clean element.

Example 2.1. Let $R = M_2(\mathbb{Z}[x])$ and the element $A = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$. It is easy to see that any nilpotent element in R is of the form $\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$ where $a(x) + d(x) = 0$ and $a(x)d(x) - b(x)c(x) = 0$. It is easy to see that the element A is semi-nil clean. Now suppose A is a nil-semi clean element in the ring R. Then $A - B$ is a periodic element in R for some nilpotent element $B \in R$, but not a nilpotent element in R. This shows that $A - B$ is either a unit or a zero divisor other than zero element in R. Thus $\det(A-B) = -x^2 + (b(x)+c(x))x$ is a unit or a zero divisor which is not zero in $\mathbb{Z}[x]$. This is a contradiction. Hence A is not nil-semi clean.

The class of semi-nil clean rings contains the class of nil clean rings, moreover it contains the class of weakly nil clean rings, i.e.,

$$\text{Nil clean rings} \subsetneq \text{Weakly nil clean rings} \subsetneq \text{semi-nil clean rings}.$$

We can have the strict inclusion in the above relation by using the following simple examples of rings,

(i) $\mathbb{Z}_3$ is weakly nil clean but not nil clean ring [7],

(ii) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is semi-nil clean ring but not weakly nil clean [7].

Similar to the case of part **1** of [1, Propositin 2.5], every homomorphic image of a semi-nil clean ring is again a semi-nil clean ring. At this point, we should ask one question, whether the ring R is semi-nil clean or not, if all non-trivial homomorphic images are semi-nil clean. The answer to the above-raised question is that ring R need not be semi-nil clean. For example, consider the ring of integers $\mathbb{Z}$. It is not a semi-nil clean ring, but its non-trivial homomorphic images are of the form $\mathbb{Z}_m$ where $m \geq 2$ is an integer are semi-nil clean. The following proposition provides the condition on an ideal I of a ring R so that the result R is semi-nil clean if and only if R/I is semi-nil clean follows.

Proposition 2.2. *Let I be a nil ideal of a ring R and periodic elements can be lifted modulo I. Then R is semi-nil clean if and only if R/I is semi-nil clean.*

Proof. ($\Rightarrow$) This is trivial. ($\Leftarrow$) Let R/I be a semi-nil clean ring and $x \in R$. Then $\overline{x} = \overline{a} + \overline{p}$ where $\overline{a} \in N(R/I)$ and $\overline{p} \in Per(R/I)$. This implies $x = p + (a+b)$ where $b \in I$. Since periodic elements can be lifted modulo I, we can assume that $p \in Per(R)$. Again, since $\overline{a} \in N(R/I)$, $\overline{a}^n = \overline{0}$ for some $n \in \mathbb{N}$, and thus $a \in I$. Therefore $(a + b) \in I$. Thus x is a semi-nil clean element, and hence R is a semi-nil clean ring. □

The following example shows that we can not let down the condition on I of being a nil ideal in the above proposition.

Example 2.3. Let $R = \mathbb{Z}[x]/(x^2)$ and $I = (x, 2)$. Clearly $I \nsubseteq N(R)$ as $N(R) = (x)$. Then $R/I \cong \mathbb{Z}_2$ is semi-nil clean, but R is not a semi-nil clean ring.

The homomorphic image of an indecomposable ring need not be an indecomposable ring. But it is interesting that the indecomposability of a ring R can be transferred to the homomorphic images of R if the ring R is a semi-nil clean ring. The following theorem discusses the homomorphic images of the indecomposable semi-nil clean ring.

Proposition 2.4. *Every homomorphic image of an indecomposable semi-nil clean ring is an indecomposable semi-nil clean ring.*

Proof. Since R is an indecomposable ring, $Per(R) = N(R) \cup tU(R)$ where $tU(R) = \{x \in R : x^n = 1 \text{ for some } n \in \mathbb{N}\}$. Again, since R is also a semi-nil clean ring, $R = N(R) \cup tU(R)$. It is easy to see that if $R = N(R) \cup tU(R)$, then R is an indecomposable ring. Thus a ring R is an indecomposable semi-nil clean if and only if $R = N(R) \cup tU(R)$. Let R' be a homomophic image of an indecomposable semi-nil ring R. Let $y \in R'$. Then there exists $x \in R$ such that $f(x) = y$ where $f : R \mapsto R'$ is a surjective homomorphism. Since either $x \in N(R)$ or $x \in tU(R)$, $y \in N(R') \cup tU(R')$. Hence R' is an indecomposable semi-nil ring. □

In the following theorem, we investigate the properties of semi-nil clean rings and discuss how semi-nil clean property act on the polynomial, power series adjunctions and the Nagata ring. Few results already proved in the paper [1].

Theorem 2.5. *Let R be a commutative ring with unity.*
(i) *If R is semi-nil clean ring, then $J(R) = N(R)$. Converse need not be true.*
(ii) *A semi-nil clean ring is zero dimensional and hence a clean ring.*
(iii) *For a ring R, the polynomial ring $R[x]$ is never a semi-nil clean ring.*
(iv) *For a ring R, the power series ring $R[[x]]$ is never a semi-nil clean ring.*
(v) *For a ring R, the Nagata ring $R(x)$ is never a semi-nil clean ring.*
(vi) *Let $n \in \mathbb{N}$. Then R is semi-nil clean if and only if $R[[x]]/(x^n)$ is semi-nil clean if and only if $R[x]/(x^n)$ is semi-nil clean.*

Proof. (i): The proof can be seen in [1, Theorem 3.1]. For converse, let $R = \mathbb{Z}_2[x]$. Clearly, $J(R) = N(R)$ but R is not semi-nil clean as $N(R) = \{0\}$ and $Per(R) = \{0, 1, -1\}$. Furthermore, the result is also true for 2-primal ring, a ring in which the set of nilpotent elements is equal to the prime radical of the ring.
(ii): Let P be a prime ideal of a semi-nil clean ring R. Then R/P is a semi-nil clean domain. Therefore, every element of R/P is a periodic element and hence, has a multiplicative inverse. This implies P is a maximal ideal of R. Hence, R is zero dimensional. By [5, corollary 11], every zero dimensional ring is clean.
(iii): Suppose $R[x]$ is a semi-nil clean ring. Then $R[x]/N(R)[x]$ is a semi-nil clean ring since it is a homomorphic image of a semi-nil clean ring. It is easy to see that $R[x]/N(R)[x] \cong R'[x]$, where $R' = R/N(R)$ and $N(R') = 0$. Thus by part **(3)** of [4, Theorem 2.1], we have $Per(R'[x]) = Per(R')$. Since $R'[x]$ is a semi-nil clean ring with $N(R') = 0$, $x \in Per(R'[x])$. This gives us $x \in Per(R')$, which is a contradictoin. Hence $R[x]$ can never be a semi-nil clean ring.
(iv): Suppose $R[[x]]$ is a semi-nil clean ring. Let P be any prime ideal in R. Then $P[[x]]$ is a prime ideal in $R[[x]]$ and then $R[[x]]/P[[x]]$ is semi-nil clean domain. Therefore it is a field. We have $R[[x]]/P[[x]] \cong (R/P)[[x]]$ and $(R/P)[[x]]$ is not a field, this leads to a contradiction. Hence $R[[x]]$ never a semi-nil clean ring.
(v): Let R be a ring. The Nagata ring is the localization of a polynomial ring $R[x]$ over the set of primitive polynomials, and it is denoted by $R(x)$. In view of $R(x)/N(R(x)) \cong (R/N(R))(x)$, the similar argument as given in part **(iii)** proves the result.
(vi): Let R be a semi-nil clean ring. Again, let $f = (\sum_{i=0}^{n-1} a_i x^i) + (x^n) \in R[[x]]/(x^n)$ where $a_i's \in R$. Since R is semi-nil clean, $a_0 = a + b$ where $a \in N(R)$ and $b \in Per(R)$. Then $f = (b+(x^n))+(a+(\sum_{i=0}^{n-1} a_i x^i)+(x^n))$ where former is periodic and later is nilpotent. Hence $R[[x]]/(x^n)$ is a semi-nil clean ring. Converse part follows from the fact $R \cong R'/(\overline{x})$ where $R' = R[[x]]/(x^n)$ and $(\overline{x}) = (x + (x^n))$. With the standard result $R[[x]]/(x^n) \cong R[x]/(x^n)$, we complete the proof. □

Corollary 2.6. *Let R be a 2-primal ring. Then R is semi-nil clean if and only if $R/J(R)$ is semi-nil clean, $J(R)$ is nil and periodic elements can be lifted modulo $J(R)$.*

Proof. $(\Rightarrow)$: Let R be a 2-primal semi-nil clean ring. By part **(i)** of Theorem 2.5, $J(R)$ is nil. Since $R/J(R)$ is the homomorphic image of a semi-nil clean ring, $R/J(R)$ is semi-nil clean. Let $a + J(R) \in Per(R/J(R))$. Since R is a semi-nil clean ring, $a = b + c$ where $b \in N(R)$ and $c \in Per(R)$. This gives $a + J(R) = c + J(R)$. Hence periodic elements lifted by modulo $J(R)$. On the other hand, by Proposition 2.2, R is semi-nil clean. □

The next theorem gives a characterization of a commutative indecomposable ring of a prime characteristic q.

Theorem 2.7. *Let R be a commutative indecomposable ring of characteristic q, a prime integer. Then R is semi-nil clean if and only if $R/N(R)$ is a field with each element is periodic.*

Proof. Let R be an indecomposable ring. Then $Per(R) = N(R) \cup tU(R)$.
($\Rightarrow$) Let $\overline{0} \neq \overline{a} \in R/N(R)$. Since R is a semi-nil clean ring, $a = x + p$ where $x \in N(R)$ and $p \in Per(R)$. Obviously, $p \in tU(R)$ as if $p \in N(R)$, then $a \in N(R)$ and hence $\overline{a} = \overline{0}$, which is a contradiction. Therefore $\overline{a} = \overline{p} \in U(R/N(R))$. Hence $R/N(R)$ is a field with each element is periodic. Conversely, if $x \in N(R)$, then we are done. If $x \in R - N(R)$, then $\overline{x} \neq \overline{0}$ in $R/N(R)$. This gives there exists $l > 0$ such that $\overline{x}^l = \overline{1}$ in $R/N(R)$. This implies $x^l = 1 + y$ where $y \in N(R)$ is such that $y^n = 0$ for some $n \in \mathbb{N}$. Now if $n \leq q$, then $(x^l)^q = 1$. If $n \geq q$, then $n \leq q^n$ and therefore $(x^l)^{q^n} = 1$. Hence $x \in Per(R)$. This completes the proof. □

Consider a ring $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, which is a decomposable semi-nil clean ring. Clearly, $R/N(R)$ is not a field. This shows that the indecomposability of the ring R is an essential condition of Theorem 2.7.

Remark 2.8. If R is a reduced semi-nil clean ring, then every unit element has a finite order.

The following theorem shows the class of semi-nil clean ring is contained in the class of semiclean ring defined in [11]. Part **3** of [1, Theorem 2.5], shows that every nil-semi clean ring is a semiclean ring, we also provide the same proof here for semi-nil clean ring for the sake of completeness.

Theorem 2.9. *Every semi-nil clean ring is a semiclean ring. Converse is true for UU-rings, rings in which every unit element u can be expressed as $u = 1 + \eta$ where η is a nilpotent element of the ring.*

Proof. Let R be a semi-nil clean ring and $x \in R$. Then $x - 1 = a + p$ where $a \in N(R)$ and $p \in Per(R)$. This implies $x = (1 + a) + p$. Hence R is semiclean since $1 + a \in U(R)$. Conversely, let R be a semiclean UU-ring and $x \in R$. Then $1 - x = u + b$ where $u \in U(R)$ and $b \in Per(R)$. Since R is UU-ring, $u = 1 + \eta$ where $\eta \in N(R)$. Then $x = -\eta - b$ where $-\eta \in N(R)$ and $-b \in Per(R)$. Hence R is semi-nil clean. □

In general, the converse of Theorems 2.9 is not true. For example, the ring $R = \mathbb{Z}_2[[x]]$ is clean and hence semiclean, but not semi-nil clean by part **(iv)** of Theorem 2.5.

Theorem 2.10. *Let $(R_i)_{i=1}^n$ be rings. Then the product $R = \prod_{i=1}^n R_i$ is semi-nil clean if and only if each R_i is semi-nil clean.*

Proof. Suppose each R_i be semi-nil clean ring. Let $a = (a_1, a_2, \ldots, a_n) \in R = \prod_{i=1}^n R_i$. For each $i, a_i = b_i + p_i$ where $b_i \in N(R_i)$ and $p_i \in Per(R_i)$. It is easy to see that $(b_1, b_2, \ldots, b_n) \in N(R)$ and by [6, Lemma 2.4], $(p_1, p_2, \ldots, p_n) \in Per(R)$. Hence R is semi-nil clean. Converse is trivial in view of $R_i \cong (\prod_{i=1}^{n} R_i)/(\prod_{j=1, j\neq i}^{n} R_j)$. □

In general, infinite direct product of semi-nil clean rings is not semi-nil clean. The same example which is given in [3] for nil clean ring i.e., $R = \prod_{n\in\mathbb{N}} \mathbb{Z}_{2^n}$ works for semi-nil clean ring.

Let R_1 and R_2 be rings with unity, M be an R_1-R_2- bimodule and N be an R_2-R_1-bimodule. Then the algebraic structure $(R_1, R_2, M, N, \phi, \psi)$ is said to be Morita context where $\phi : M \otimes_{R_2} N \to R_1$ and $\psi : N \otimes_{R_1} M \to R_2$ are bimodule homomorphisms satisfying $b\phi(m \otimes n) = \psi(b \otimes m)n$ for all $b \in N$ and $a\psi(n \otimes m) = \phi(a \otimes n)m$ for all $a \in M$. The bimodule homomorphisms ϕ and ψ are called pairings. The collection $T = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in A, b \in B, m \in \right.$

M and $n \in N\}$ forms a ring under usual addition and multiplication defined by the following: $\begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} a' & m' \\ n' & b' \end{bmatrix} = \begin{bmatrix} aa' + \phi(m \otimes n') & am' + mb' \\ na' + bn' & \psi(n \otimes m') + bb' \end{bmatrix}$. This ring is called the ring of Morita context. If pairings are zero-morphisms, then T is called Morita context with zero pairings. It is interesting to note that if $R_1 = R_2 = R$ and $N = 0$, then the ring of Morita context T becomes the idealization $R(M)$ of the ring R and the R-module M. The next lemma is useful to prove Theorem 2.12, which discuss the semi-nil clean property of the ring of Morita context T.

Lemma 2.11. *Let T be a ring of Morita context with zero pairings. Then*

$$N(T) = \left\{ \begin{bmatrix} x_1 & m \\ n & x_2 \end{bmatrix} : x_1 \in N(R_1), x_2 \in N(R_2), m \in M \text{ and } n \in N \right\}.$$

Proof. The proof is also straightforward. □

Theorem 2.12. *Let $T = \begin{bmatrix} R_1 & M \\ N & R_2 \end{bmatrix}$ be a ring of Morita context with zero pairings. Then T is semi-nil clean if and only if R_1 and R_2 are semi-nil clean.*

Proof. Let R_1 and R_2 be semi-nil clean and $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in T$ be any element. Then one can write that $\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} x_1 & m \\ n & x_2 \end{bmatrix} + \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ where $x_1 \in N(R_1), x_2 \in N(R_2), p_1 \in Per(R_1)$ and $p_2 \in Per(R_2)$. By Lemma 2.11, $\begin{bmatrix} x_1 & m \\ n & x_2 \end{bmatrix} \in N(T)$, and clearly, $\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \in Per(T)$. Hence T is semi-nil clean.

Conversely, Let T be a semi-nil clean ring. Let $a \in R_1$. Then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in T$. Thus by assumption one can write $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 & m_1 \\ n_1 & x_2 \end{bmatrix} + \begin{bmatrix} p_1 & m_1' \\ n_1' & p_2 \end{bmatrix}$ where $\begin{bmatrix} x_1 & m \\ n & x_2 \end{bmatrix} \in N(T)$ and $\begin{bmatrix} p_1 & m_1' \\ n_1' & p_2 \end{bmatrix} \in Per(T)$. This gives $a = x_1 + p_1$ where $x_1 \in N(R_1)$ and $x_2 \in N(R_2)$. Hence R_1 is a semi-nil clean ring. Similarly, one can prove R_2 is semi-nil clean. □

Corollary 2.13. (i) *Let R_1 and R_2 be rings and M be a R_1-R_2 bimodule. Then $T' = \begin{bmatrix} R_1 & M \\ 0 & R_2 \end{bmatrix}$ is semi-nil clean if and only if R_1 and R_2 are semi-nil clean.*
(ii) *The idealization $R(M)$ of a ring R and an R-module M is a semi-nil clean ring if and only if R is a semi-nil clean ring.*
(iii) *Let R be an abelian ring and e be an idempotent in R. Then R is a semi-nil clean ring if and only if eRe and $(1-e)R(1-e)$ are semi-nil clean rings.*
(iv) *Let R be an abelian ring and $e_1, e_2, \ldots, e_n$ be orthogonal idempotents such that $e_1 + \cdots + e_n = 1$. Then R is a semi-nil clean ring if and only if for each $i = 1, 2, \ldots, n$, e_iRe_i is a semi-nil clean ring .*
(v) *The $n \times n$ upper (lower) triangular matrix ring over a ring R is semi-nil clean if and only if so is R.*

Proof. (i) It is evident that T' is obtained by replacing $N = 0$ in Theorem 2.12 i.e., T' is the ring of Morita context with zero pairing. Hence result is followed by Theorem 2.12.
(ii) Proof follows from part **(i)**.
(iii) Since R is an abelian ring, $R \cong \begin{bmatrix} eRe & 0 \\ 0 & (1-e)R(1-e) \end{bmatrix}$. Hence proof is followed by

part **(i)**.
(iv) Proof is straightforward by Part **(iii)** and induction.
(v) Let $TM_n(R)$ be the ring of all $n \times n$ upper triangular matrix over R. We prove this by induction on n. Let us suppose that result is true for $n-1$ and clearly one can write $TM_n(R) = \begin{bmatrix} R & M \\ 0 & TM_{n-1} \end{bmatrix}$ where $M = \{(a_1, \dots, a_{n-1}) : a_i \in R\}$. Clearly, M is a $(R, TM_n(R))$- module, therefore the result follows by part **(i)**. Similarly, we can prove that the ring of lower triangular matrices over a ring R is sem-inil clean if and only if so is R. Converse is straightforward. □

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