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Vertex (n, k)-Choosability in Signed Graphs

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Abstract A signed graph is a graph in which each edge is labelled with $+1$ or -1. In this paper we define the vertex (n, k)-choosability of signed graphs which provides an extension of the vertex (n, k)-choosability of the underlying graph. Also we define the vertex choice number of a signed graph and prove some of its properties and provide bounds in terms of the vertex choice number of the underlying graph and other signed graph parameters. Also we find a characterisation for balance in bipartite signed graphs in terms of its vertex choice number.

1 Introduction

Let $G = (V, E)$, be a graph of order n and size m, where $V(G) = \{1, 2, ..., n\}$. A signed graph $\Sigma = (G, \sigma)$ is a graph $G = (V, E)$ together with a signature function $\sigma : E \to \{+1, -1\}$. Unless mentioned otherwise, all the underlying graphs considered in this paper are simple, finite, connected and undirected. For any terms and definitions not defined specifically in this paper, the reader may refer to [6] and [10].

An edge e of a signed graph Σ is said to be positive (or negative) if $\sigma(e) = +1$ (or $\sigma(e) = -1$). A signed graph $\Sigma = (G, \sigma)$ is balanced if every cycle contains an even number of negative edges. That is, a signed graph Σ is said to be *balanced* if for all cycles C in Σ the product $\sigma(C) = \prod_{e \in E(C)} \sigma(e) = 1$ and is called *antibalanced* if $-\Sigma$ is balanced.

For a signed graph $\Sigma = (G, \sigma)$ switching at a vertex v means changing the sign of each edge incident with v. In general switching at a vertex subset X of $V(\Sigma)$ means changing the sign of every edge with exactly one end in X. Two signed graphs are switching equivalent if one can be obtained from the other by a switching at some vertex subset. The operation of switching does not essentially change the signed graph, because it preserves the sign of each cycle. It is well known that a signed graph is balanced if and only if it is switching equivalent to an all positive graph. The following theorem is known as *Harary's bipartition theorem* gives a characterisation for balance in signed graphs.

Theorem 1.1 (Harary's bipartition theorem [5]). *A signed graph Σ is balanced if and only if there is a bipartition of its vertex set, $V = V_1 \cup V_2$, such that every positive edge is induced by V_1 or V_2 while every negative edge has one endpoint in V_1 and one in V_2.*

The concept of *list assignment* (or a *list coloring)* of graphs was studied in [1] and by invoking the concept of list-assignments of graphs, the concept of $(a : b)$-choosability was defined and studied in [3].

Definition 1.2. [3] A graph $G = (V, E)$ is $(a : b)$-choosable, if for every family of sets $\{S(v) : v \in V\}$ of cardinality a, there exist subsets $C(v) \subset S(v)$, where $|C(v)| = b$ for every $v \in V$, and $C(u) \cap C(v) = \emptyset$, whenever $u, v \in V$ are adjacent.

The vertex set oriented $(a : b)$-choosability, known as the vertex (n, k)-choosability of graphs is defined and its relation with other graph parameters are studied in [4], [7] and [8].

Definition 1.3. [4] A graph $G = (V, E)$ of order n is said to be *vertex (n, k)-choosable*, if there exists a collection of subsets $\{S_k(v) : v \in V\}$ of $V(G)$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$.

Definition 1.4. [4] The maximum value of k for which the given graph G is vertex (n, k)-choosable is called *vertex choice number of* G, and is denoted by $\mathcal{V}_{ch}(G)$.

In this paper we study the vertex (n, k)-choosability in signed graphs which provides an extension of the vertex (n, k)-choosability of an unsigned graph. Also we define the vertex choice number of a signed graph and provide bounds in terms of other graph(signed graph) parameters.

2 Definition and Main Results

In this section, we extend the notion of vertex (n, k)-choosability of graphs into signed graphs. Also, the vertex choice number of a signed graph is defined and we give a characterization for balance in bipartite signed graphs in terms of its vertex choice number.

Definition 2.1. Let A be a non-empty set of integers. Then, A is said to be symmetric, if $A = -A$, otherwise the set is said to be non-symmetric.

Definition 2.2. A signed graph $\Sigma = (V, E, \sigma)$ of order n is said to be *vertex (n, k)-choosable*, if there exists a collection of non-symmetric sets $\{S_k(v) : v \in V\}$, where $S_k(v)$ is a subset of $\{\pm 1, \pm 2, \ldots, \pm \frac{n}{2}\}$ if n is even, (or $S_k(v)$ is a subset of $\{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\}$, if n is odd) of cardinality k, such that

$$S_k(u) \cap (\sigma(uv) S_k(v)) = \emptyset \text{ for all } uv \in E(\Sigma).$$

Equivalently

$$(\sigma(uv) S_k(u)) \cap S_k(v) = \emptyset \text{ for all } uv \in E(\Sigma).$$

OR

$$S_k(u) \cap S_k(v) = \emptyset \text{ whenever } uv \in E^+(\Sigma)$$

and

$$S_k(u) \cap -(S_k(v)) = \emptyset \text{ whenever } uv \in E^-(\Sigma).$$

Definition 2.3. The maximum value of k for which the given signed graph $\Sigma = (G, \sigma)$ is vertex (n, k)-choosable is called *vertex choice number of* Σ, and is denoted by $\mathcal{V}_{ch}(\Sigma)$.

First we prove that the vertex (n, k)-choosability of Σ is switching invariant.

Theorem 2.4. *The vertex choice number of a signed graph Σ is switching invariant.*

Proof. Suppose that $\mathcal{V}_{ch}(\Sigma) = k$ and Σ is switched to Σ^ζ.
When we switch the signed graph Σ to Σ^ζ, together we switch the sets assigned to its vertices in the following way $\zeta(v) S_k(v) = S'_k(v)$. That is, if the vertex v is switched then the set assigned to v is also switched to $\zeta(v) S_k(v)$.
Let $\zeta(u) = \zeta(v)$. Then, the sign of the edge uv will be the same in both Σ and Σ^ζ.
Hence,

$$S_k(u) \cap S_k(v) = \emptyset \implies S'_k(u) \cap S'_k(v) = \emptyset$$

and

$$S_k(u) \cap -(S_k(v)) = \emptyset \implies S'_k(u) \cap -(S'_k(v)) = \emptyset$$

If $\zeta(u) \neq \zeta(v)$. Then the sign of the edge uv will be reversed.

$$S_k(u) \cap S_k(v) = \emptyset \implies S'_k(u) \cap -(S'_k(v)) = \emptyset$$

and

$$S_k(u) \cap -(S_k(v)) = \emptyset \implies S'_k(u) \cap S'_k(v) = \emptyset.$$

Then, the non-symmetric collection $\{S'_k(v) : v \in V\}$, can be assigned to the vertices of Σ^ζ. Since, Σ^ζ can be switched to Σ, we can say that $\mathcal{V}_{ch}(\Sigma^\zeta) = k$. □

Corollary 2.5. *For any balanced signed graph $\Sigma = (G, \sigma)$, $\mathcal{V}_{ch}(G) = \mathcal{V}_{ch}(\Sigma)$.*

Proof. Since Σ is balanced, Σ can be switched to all-positive signed graph G. Then, by Theorem 2.4, $\mathcal{V}_{ch}(G) = \mathcal{V}_{ch}(\Sigma)$. □

Theorem 2.6. *Let $\Sigma = (G, \sigma)$ be an odd cycle with $\mathcal{V}_{ch}(\Sigma) = k$. Then, Σ is balanced implies there exists at least one vertex $v \in V(\Sigma)$, such that $0 \in S_k(v)$.*

Proof. Let $\Sigma = (G, \sigma)$ be a balanced odd cycle of order n, with $V(\Sigma) = \{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\}$. Then, by Corollary 2.5, $\mathcal{V}_{ch}(\Sigma) = \mathcal{V}_{ch}(G) = \frac{n-1}{2}$. Suppose that there does not exist a vertex $v \in V(\Sigma)$ such that $0 \in S_k(v)$. Then, at a time we can find only two disjoint non-symmetric sets of order $\frac{n-1}{2}$, let it be $S_k(i)$ and $S_k(j)$. Switch Σ to Σ^ζ so that $\sigma(12)$ and $\sigma(23)$ are the only negative edges in Σ^ζ. Let $S_k(1) = S_k(2) = S_k(3) = S_k(i)$. Then, the only possible way to label the remaining $n-3$ vertices in Σ are as follows:
$S_k(4) = S_k(6) = \cdots = S_k(n-1) = S_k(j)$ and $S_k(5) = S_k(7) = \cdots = S_k(n) = S_k(i) = S_k(1)$. Since, $\sigma(n1)$ is positive, $S_k(1)$ and $S_k(n)$ should be disjoint, a contradiction. Therefore, there should be a vertex $v \in V(\Sigma)$ such that $0 \in S_k(v)$. □

Definition 2.7 ([9]). *The balanced clique number* of Σ, denoted by $\omega_b(\Sigma)$, is the maximum order of a balanced complete subgraph.

Proposition 2.8. *Let $\Sigma = (G, \sigma)$ be a signed graph of odd order with $\mathcal{V}_{ch}(\Sigma) = k$ and $\omega_b(\Sigma) = 2$. Then, for every balanced odd cycles C_m in Σ, there should be at least one vertex $v \in V(C_m)$, such that $0 \in S_k(v)$.*

Theorem 2.9. *A bipartite signed graph $\Sigma = (G, \sigma)$ is balanced if and only if $\mathcal{V}_{ch}(G) = \mathcal{V}_{ch}(\Sigma)$.*

Proof. First part follows from Corollary 2.5.
For the converse part, let Σ be a bipartite signed graph of order n with $\mathcal{V}_{ch}(\Sigma) = \mathcal{V}_{ch}(G) = \lfloor \frac{n}{2} \rfloor$. Suppose that Σ is not balanced. Then, we can find a negative even cycle in Σ, say, $C_m = v_1 v_2 \ldots v_m$. Switch Σ to Σ^ζ so that C_m contains only one negative edge, say, $v_1 v_2$. Since, $\omega_b(\Sigma) = 2$ and Σ contains no odd cycles, by Proposition 2.8 we have, it is not necessary to consider symmetric sets containing 0. Then, there are only two disjoint non-symmetric sets of cardinality $\lfloor \frac{n}{2} \rfloor$, say, $S_k(i)$ and $S_k(j)$. Since, $\sigma(12) = -1$, $S_k(1) = S_k(2) = S_k(i)$. The only possible way to label the remaining $m-2$ vertices is, $S_k(3) = S_k(5) = \cdots = S_k(m-1) = S_k(j)$ and $S_k(4) = S_k(6) = \cdots = S_k(m) = S_k(i) = S_k(1)$. Since, $\sigma(m1) = +1$, by our definition, $S_k(m)$ should be disjoint from $S_k(1)$, a contradiction. Therefore, Σ should be balanced. □

Lemma 2.10. *The vertex choice number of a full negative signed graph Σ of order n is $\lfloor \frac{n}{2} \rfloor$.*

Proof. Suppose that Σ is a full negative signed graph of order n. Let $e = ij$ be any edge in Σ. Since e is negative, choose a non-symmetric set $S_k(i)$ so that $S_k(i) \cap -(S_k(i)) = \emptyset$. Then, we can assign the same set to j also, that is $S_k(j) = S_k(i)$. In a similar manner every vertices adjacent to i or j can be labeled by the same set $S_k(i)$. Since, Σ is full negative, if we consider any edge, the end vertices can be labeled by the same set. Therefore, the maximum order k of the non-symmetric set we can choose from $\{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\}$, (if n is odd) so that $S_k(i) \cap -(S_k(i)) = \emptyset$ is $\lfloor \frac{n}{2} \rfloor$. Similarly, when n is even the maximum order k of the non-symmetric set we can choose from $\{\pm 1, \pm 2, \ldots, \pm \frac{n}{2}\}$, so that $S_k(i) \cap -(S_k(i)) = \emptyset$ is $\frac{n}{2}$. □

Corollary 2.11. *For every $k \geq 1$, there exists a signed simple graph $\Sigma = (G, \sigma)$ with $\mathcal{V}_{ch}(\Sigma) = k$.*

Proof. For any full negative signed graph $\Sigma = (G, \sigma)$ on $2k$ vertices, the vertex choice number will be k. □

Theorem 2.12. *For a non-cyclic signed graph Σ of order n, $\mathcal{V}_{ch}(\Sigma) = \lfloor \frac{n}{2} \rfloor$ if and only if Σ is antibalanced.*

Proof. Let Σ be an antibalanced non-cyclic signed graph of order n. Then, Σ is switching equivalent to the all negative signature of Σ, and by Lemma 2.10, $\mathcal{V}_{ch}(\Sigma) = \lfloor \frac{n}{2} \rfloor$.
Conversely, suppose that Σ is a non-cyclic signed graph of order n with $\mathcal{V}_{ch}(\Sigma) = \lfloor \frac{n}{2} \rfloor$. Let $S_k(i)$ and $S_k(j)$ be two non-symmetric sets of order $\lfloor \frac{n}{2} \rfloor$. Then, we can find a bipartition of the vertex

set of Σ, $V(\Sigma) = V_1 \cup V_2$, where all the vertices in V_1 is labeled by $S_k(i)$ and all the vertices in V_2 is labeled by $S_k(j)$. Then, the negative edges in Σ will be either in V_1 or in V_2 and the positive edges has one point in V_1 and one in V_2. If we consider $-\Sigma$, we can see that V_1 and V_2 will give a Harary bipartition of $-\Sigma$. Then, by Theorem 1.1 we get $-\Sigma$ is balanced, which implies Σ is antibalanced. □

Theorem 2.13. *Let $\Sigma = (G, \sigma)$ be a signed graph of order n. Then, $\mathcal{V}_{ch}(G) \leq \mathcal{V}_{ch}(\Sigma) \leq \left\lfloor \frac{n}{2} \right\rfloor$. The lower bound is attained when Σ is balanced and the upper bound is attained when Σ is antibalanced.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph of order n. To label the end vertices of an edge $e = ij$ in Σ, either $S_k(i) \cap S_k(j) = \emptyset$ or $S_k(i) \cap -(S_k(j)) = \emptyset$. That is, there should be at least two disjoint non-symmetric sets of order k. Thus, the maximum possible value of k so that Σ is vertex (n, k)-choosable is $\left\lfloor \frac{n}{2} \right\rfloor$.
Since, the signature of an antibalanced signed graph Σ is switching equivalent to the all negative signature, we have by Lemma 2.10, the vertex choice number of Σ will be $\left\lfloor \frac{n}{2} \right\rfloor$. Hence, the upper bound for $\mathcal{V}_{ch}(\Sigma)$ is attained when Σ is antibalanced. By Corollary 2.5, the lower bound is attained when Σ is balanced.

□

Remark 2.14. Let Σ be a signed complete graph on n vertices, then

- $\mathcal{V}_{ch}(\Sigma) = 1$ if and only if Σ is balanced.
- $\mathcal{V}_{ch}(\Sigma) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if Σ is antibalanced.

Let $\Sigma = (G, \sigma)$ and $A = (a_{ij})$ be the *adjacency matrix* of G. Then, the adjacency matrix of Σ is defined as the matrix $A(\Sigma) = (a^{\sigma}_{ij})$, where $a^{\sigma}_{ij} = \sigma(ij)a_{ij}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of $A(\Sigma)$ in non-increasing order

Theorem 2.15 ([9]). *Let Σ be a signed graph on n vertices. Then*

$$\omega_b(\Sigma) \geq \frac{n}{n - \lambda_1}.$$

Theorem 2.16. *For a signed graph Σ of order n, $\mathcal{V}_{ch}(\Sigma) \leq \left\lfloor \frac{n}{\omega_b(\Sigma)} \right\rfloor$.*

Proof. Let Σ be a signed graph of order n, with $\omega_b(\Sigma) = m$. Let K_m be a balanced complete subgraph of order m in Σ. Switch Σ to Σ^{ζ}, in such a way that K_m contains no negative edges in Σ^{ζ}. Since, all the vertices in K_m are connected by positive edges we should have at least m disjoint non-symmetric sets to label the vertices of K_m. After labeling the vertices of the balanced complete subgraph of maximum order, we can label the remaining vertices in Σ using the sets labeled to the vertices of K_m. Hence, there should be at least m disjoint non-symmetric sets to label all the vertices of Σ. Therefore, $\mathcal{V}_{ch}(\Sigma) \leq \left\lfloor \frac{n}{m} \right\rfloor$. □

Now, we provide an upper bound for the largest adjacency eigenvalue of Σ in terms of its vertex choice number.

Theorem 2.17. *Let Σ be a signed graph of order n and λ_{max} be the largest eigenvalue of Σ. Then, $\lambda_{max} \leq n - \mathcal{V}_{ch}(\Sigma)$.*

Proof. The theorem follows from Theorem 2.15 and Theorem 2.16. □

References

[1] B. Bollobás, A. J. Harris, List coloring of graphs, *Graphs Comb.*, **1**, 115–127 (1985).

[2] Eckhard Steffen, Alexander Vogel, Concepts of signed graph coloring, European Journal of Combinatorics, DOI : 10.1016/j.ejc.2020.103226 (2021).

[3] P. Erdös, A.L. Rubin, H. Taylor, Choosability in graphs, *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing*, Congr. Numer., XXVI,, 125–157 (1979).

[4] K. A. Germina, P. Soorya, Vertex (n, k)-choosability of graphs, Italian Journal of Pure and Applied Mathematics, **43**, 905-911 (2020).

[5] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. **2**, 143–146 (1953–1954).

[6] F. Harary, *Graph theory*, Narosa Publ. House, New Delhi, (2001).

[7] P. Soorya, K. A. Germina, On the vertex choice number and chromatic number of graphs, International journal of scientific and technology research, **9**, 4140-4142 (2020).

[8] P. Soorya, K. A. Germina, N. K Sudev, On α-vertex choosability of graphs, National Academy Science Letters, **44** :343-346 (2021).

[9] Wei Wang, Zhidan Yan, Jianguo Qian, Eigenvalues and chromatic number of a signed graph, Linear Algebra and its Applications, DOI: 10.1016/j.laa.2021.02.018 (2021)·

[10] T. Zaslavsky, Signed graphs, Discrete Appl. Math. **4**, 47–74 (1982). Erratum, Discrete Appl. Math. **5**, 248 (1983).

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