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GRAPHS INDUCED BY VECTOR SPACES

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Abstract In this paper, we introduce a way of illustrating a vector space in the vector graph in a way up to isomorphism there is a precise correspondence between such a graph and the vector space. Then, we find some relation between the graph properties and the space dimension. We also explore the clique and chromatic number and connectivity.

1 Introduction

Maybe defining an abstract notion in mathematics, needs some obvious illustration, but exploring its deeper properties requires a profounder or even other kinds. A junior mathematician may think about a group, ring, or a vector space as a generalization of Euclidean spaces and suggests some of its properties, but some professionals though not only has such an image, but also think about some other visualizations.

Cayley [8], for instance, brought another illustration of a group as a graph in which its vertices were the group elements and edges designed by the binary operation of the group, which helped not only himself, but also other group theorists to find some new properties of a group.

Another correspondence between theory and graph theory was brought by Beck [7] and was adjusted by Anderson et. al. [4], whose vertices are all zero divisor elements of a ring.

Recently, some authors tried to define a graph illustration for other kinds of mathematical aspects such as rings [3], topological spaces [6], measure spaces [5], lattices [2],measure functions [1] and semigroups [9].

These works, motivated us to define a new kind of graph associated to a vector space, which has some precise relation between the notions, up to isomorphism, for both finite and infinite dimensional vector spaces.

For a vector space S, we consider a subset of linear maps on S as the vertex set in which f and g are adjacent if and only if both of $f \circ g$ and $g \circ f$ will be the zero map.

At first, some basic properties of such a graph is explored such as adjacency, distance, and finding some cut vertices. After that the relation between a vector space and its associated graph is investigated.

2 Definitions and Examples

Let S be a vector space and P be the set of all linear maps over S. $\Gamma(S)$, the graph induced by S, is a graph with the vertex set V, where V is a subset of P containing all maps which are neither surjective nor injective with the following properties:

- $ker(f) \nsubseteq Im(f)$.
- $Im(f) \nsubseteq ker(f)$.

For $f, g \in \nu(\Gamma)$, there is an edge between f and g if and only if $f \circ g = g \circ f = 0$ and we denote these connected vertices by $f \leftrightarrow g$.

The condition $Im(f) \nsubseteq ker(f)$ implies that the graph has no loops, and the condition $ker(f) \nsubseteq Im(f)$ implies the ellimination of the isolated vertices.

We denote the set of vertices and edges of Γ by $\nu(\Gamma)$ and $e(\Gamma)$ respectively. For the rest of the paper, we will assume that the vector space has dimension more than one, since one-dimensional vector spaces induce a null graph.

Lemma 2.1. *$f, g \in \nu(\Gamma)$ are connected if and only if $Im(f) \subseteq ker(g)$ and $Im(g) \subseteq ker(f)$.*

Theorem 2.2. *Let $f, g \in \nu(\Gamma)$,Then $d(f, g) = 2$ if and only if:*

1. $d(f, g) \neq 1$.
2. $ker(f) \cap ker(g) \neq \{0\}$.
3. $< Im(f) \cup Im(g) > \neq S$.
4. $ker(f) \cap ker(g) \nsubseteq < Im(f) \cup Im(g) >$.

Proof. $\Rightarrow$)
If $d(f, g) = 2$ then for some vertex h, we will have $f \leftrightarrow h \leftrightarrow g$. So $Im(h) \subseteq ker(f) \cap ker(g)$ and $< Im(f) \cup Im(g) > \subseteq ker(h)$, hence conditions 2 and 3 must be satisfied. Assume that $ker(f) \cap ker(g) \subseteq < Im(f) \cup Im(g) >$, so we have:

$$Im(h) \subseteq ker(f) \cap ker(g) \subseteq < Im(f) \cup Im(g) > \subseteq ker(h),$$

which is a contradiction.
$\Leftarrow$)
By 4 we have:

$$\exists \alpha \neq 0 : \alpha \in ker(f) \cap ker(g) \setminus < Im(f) \cup Im(g) > .$$

We define:

$$h(x) = \begin{cases} c\alpha & x = c\alpha . c \in F \\ 0 & otherwise \end{cases} .$$

One can easily see that $f \leftrightarrow h \leftrightarrow g$. □

For a vector space S, $H \subset S$ is called a hyperspace of S if for some functional $f : S \to \mathbb{R}$ we have:

$$\forall x \in H : f(x) = 0.$$

Lemma 2.3. *A vertex in $\nu(\Gamma)$ is connected to a vertex that its kernel is a hyperspace of S.*

Proof. Let f be a vertex and $\beta \in ker(f) \setminus Im(f)$. We define h as follows:

$$h(\alpha) = \begin{cases} \alpha & \alpha = c\beta . c \in F \\ 0 & otherwise \end{cases} ,$$

so $f \leftrightarrow h$. □

A vertex f is called a hyper vertex if its kernel is a hyperspace of S and it is called a minimal vertex if its image is a hyperspace of S.

Lemma 2.4. *If f, g are hyper vertices then $d(f, g) \leq 2$.*

Proof. If $f \leftrightarrow g$ the lemma is proved. Otherwise, let $\beta \neq 0$ be in $ker(f) \cap ker(g) - < Im(f) \cup Im(f) \cup Im(g) >$. Define h as follows:

$$h(\alpha) = \begin{cases} \alpha & \alpha = c\beta . c \in F \\ 0 & otherwise \end{cases} ,$$

thus $f \leftrightarrow h \leftrightarrow g$. □

Corollary 2.5. *The girth of Γ is 3.*

3 Complete and Bipartite Graphs

In this section we investigate the complete and bipartite subgraphs of $\Gamma(S)$.

Theorem 3.1. *If S is an infinite dimensional vector space and $\Gamma(S)$ is the corresponding graph, for any natural number n there exists a complete subgraph K_n of $\Gamma(S)$.*

Proof. Let $P = \{\alpha_i\}_{i\in I}$ be a basis for S. We define n vertices $f_i : i = 1, 2, \ldots, n$ over P as follows:

$$f_i(x) = \begin{cases} \alpha_i & x = \alpha_i \\ 0 & x \neq \alpha_i \end{cases}.$$

The subgraph of $\Gamma(S)$, consisting the vertices $\{f_i\}_{i\in I}$ is the desired subgraph. □

Corollary 3.2. *If S is an infinite dimensional vector space, the clique number and the chromatic number of Γ is infinity. Also there exists an n-regular subgraph for any natural number n.*

Theorem 3.3. *Let S be a finite dimensional vector space. The dimension of S is n if and only if the largest complete subgraph of Γ is K_n.*

Proof. $\Rightarrow$)

If S is an $n-$dimensional vector space then n vertices defined in Theorem 3.1 construct K_n. We prove that Γ does not have K_{n+1} as its subgraph. If not, there exist $n + 1$ distinct vertices like $f_1, f_2, \ldots, f_{n+1}$ such that for any distinct $i, j \in \{1, 2, \ldots, n + 1\}$,

$$f_i \leftrightarrow f_j.$$

Let $K_i = ker(f_i), i = 1, 2, \ldots, n + 1$. For any $i \neq j$, $K_i \nsubseteq K_j$ otherwise

$$f_i \leftrightarrow f_j \Rightarrow Im(f_j) \subseteq K_i \subseteq K_j,$$

which is a contradiction. $\bigcap_{j=1}^{n+1} K_j$ is an $r-$dimensional subspace of S ($r \geq 0$) hence $\bigcap_{j=1}^{n+1} K_j$ is a subspace of S with dimension greater or equal to $r + 1$ because

$$K_i \nsubseteq \bigcap_{j=1}^{n+1} K_j.$$

Also $\bigcap_{j=1}^{n+1} K_j$ is a subspace of S with dimension greater or equal to $r + 2$ and so on.

So we can see dimension that $K_{n+1} = \bigcap_{j=1}^{n+1} K_J$ is a subspace of S with dimension greater or equal to $r + n$ and it is a contradiction, since f_{n+1} is a vertex.

$\Leftarrow$)

If S is an $m-$dimensional vector space and Γ has K_n as a largest complete subgraph then it is easy to show that m must be equal to n. □

Corollary 3.4. *If S is a $n-$dimensional vector space, the clique number of $\Gamma(S)$ is n and the chromatic number of $\Gamma(S)$ is greater than or equal to n.*

Corollary 3.5. *If $dim(S) \geq 5$, Γ is not a planar graph.*

Proposition 3.6. *There is not any complete subgraph K_n, $n \geq 3$ which contains a minimal vertex. Also, any minimal vertex lies in a star subgraph S_n for $n \in \mathbb{N}$.*

Proof. Let f be a minimal vertex. There exists $\alpha in S$ such that $kern(f) =< \{\alpha\} >$.

So, there exists a family of g_i's which $f \leftrightarrow g_i$ for all i. Since $ker(g_i) = Im(f)$ is a hyperspace and $Im(g_i) =< \{\alpha\} >$, there is not any pair of (i, j) such that $g_i \leftrightarrow g_j$ so there is not any complete subgraph which contains f. Moreover,the definition of g_i implies that the vertex f is in a star subspace S_n, $n \in \mathbb{N}$.

□

Theorem 3.7. *Let S be an $n-$dimensional vector space and $f \in \nu(\Gamma)$. f is a hyper vertex if and only if there exists a complete subgraph K_n which contains f as one of its vertices.*

Proof. $\Rightarrow$)

If f is a hyper vertex, theorem 3.3 implies that there exists a complete subgraph K_n which contains f.

$\Leftarrow$)

Let K_n be a complete subgraph of Γ and $\{f_1, f_2, f_3, \ldots, f_n\}$ be its vertices. Without loss of generality, soppuse that f_1 is a vertex which is not a hyper vertex. So

$$Im(f_1) \geq 2$$

and

$$Im(f_1) \subseteq ker(f_2).$$

Let $\beta_0, \beta 1 \in Im(f_1)$ be linearly independent vectors. Since $Im(f_1) \subseteq ker(f_2)$ and $Im(f_2) \nsubseteq ker(f_2)$, there exists $\beta_2 \in Im(f_2)$ where $\{\beta_0, \beta_1, \beta_2\}$ is an independent set. Continuing the previous process, there exists $\beta_3 \in Im(f_n)$ where $\{\beta_0, \beta_1, \beta_2, \ldots, \beta_n\}$ is an independent set, which is a contradiction. □

Theorem 3.8. *Let $\{f_i\}_{i=1}^n$ be distinct vertices such that $< \bigcup_{i=1}^n Im(f_i) > \neq S$, $\bigcap_{i=1}^n ker(f_i) \neq \{0\}$ and $\bigcap_{i=1}^n ker(f_i) \nsubseteq < \bigcup_{i=1}^n Im(f_i) >$. Then for any natural number m, there exist m distinct vertices like $g_1, g_2, \ldots, g_m$ such that the families of f_n's and $\{g_j\}_{j=1}^m$ induces a complete bipartite subgraph $K_{n,m}$ of Γ.*

Proof. Let $\alpha \in \bigcap_{i=1}^n ker(f_i) - < \bigcup_{i=1}^n Im(f_i) >$. We define m hyper vertices g_i for $1 \leq i \leq m$ as follows:

$$g_i(x) = \begin{cases} ic\alpha & x = c\alpha, c \in F \\ 0 & otherwise \end{cases}.$$

So, for $1 \leq i \leq n$ and $1 \leq j \leq m$:

$$f_i \longleftrightarrow g_j.$$

□

Corollary 3.9. *For any natural number n, there exists a complete bipartite subgraph $K_{n,n}$. Specially, there exists $K_{3,3}$ so Γ is not a planar graph.*

Let $S = \mathbb{R}^2$ and $P = \{i, j\}$ be a standard basis for S. For $1 \leq k \leq n$ define the family of f_k's over P as follows:

$$f_k(x) = \begin{cases} 0 & x = ci, c \in \mathbb{R} \\ k(cj) & x = cj, c \in \mathbb{R} \end{cases}.$$

The family of g_k's construct a complete bipartite subgraph $K_{n,n}$ with f_k, where

$$g_k(x) = \begin{cases} 0 & x = cj, c \in \mathbb{R} \\ k(ci) & x = cj, c \in \mathbb{R} \end{cases}.$$

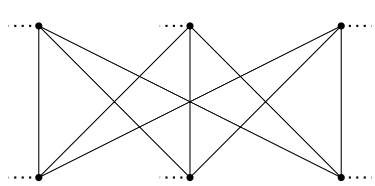

Figure 1.

Theorem 3.10. *Let $f_1 \in \nu(\Gamma)$ and m and n be two natural numbers. There exists a complete bipartite subgraph of Γ such as $k_{m,n}$ which contains f_1.*

Proof. Since f_1 is a vertex so there are nonzero elements of S such as α and β which satisfies

$$\beta \in ker(f_1) \setminus Im(f_1),$$

and

$$\alpha \in Im(f_1) \setminus ker(f_1).$$

For $2 \leq i \leq n$ and $1 \leq j \leq m$ define f_i and g_j as follow

$$f_i(x) = \begin{cases} ix & x = c\alpha, c \in F \\ f_i(x) & otherwise \end{cases},$$

and

$$g_j(x) = \begin{cases} jx & x = c\beta, c \in F \\ 0 & otherwise \end{cases}.$$

So, for any $1 \leq i \leq n$ and $1 \leq j \leq m$ we have $f_i \leftrightarrow g_j$. □

Theorem 3.11. $\Gamma(S)$ *is connected and the diameter of* Γ *is* 4.

Proof. Let f and g be two vertices. According to Lemma 2.3, $f \leftrightarrow h$ and $g \leftrightarrow h'$ where h and h' are hyper vertices. Thus according to Lemma 2.4, $d(h, h') \leq 2$ so

$$f \leftrightarrow h \leftrightarrow l \leftrightarrow h' \leftrightarrow g,$$

this completes the proof. □

Let $S = \mathbb{R}^4$ and $P = \{i, j, k, \omega\}$ be a basis of S. Consider the following vertices:

$$l_1 = \begin{cases} i \to 0 \\ j \to j \\ k \to k \\ \omega \to \omega \end{cases} \quad f = \begin{cases} i \to i \\ j \to 0 \\ k \to 0 \\ \omega \to 0 \end{cases} \quad h = \begin{cases} i \to 0 \\ j \to j \\ k \to 0 \\ \omega \to 0 \end{cases}$$

$$g = \begin{cases} i \to i + \omega \\ j \to 0 \\ k \to 0 \\ \omega \to 0 \end{cases} \quad l_2 = \begin{cases} i + \omega \to 0 \\ j \to j \\ k \to k \\ \omega \to \omega \end{cases}.$$

So, it can be easily seen that:

$$l_1 \leftrightarrow f \leftrightarrow h \leftrightarrow g \leftrightarrow l_2,$$

and this is the shortest path between l_1 and l_2, so $d(l_1, l_2) = 4$.

Theorem 3.12. $\Gamma(S)$ *does not have any cut vertex.*

Proof. Let f be a cut vertex, so there exists vertices g and h such that: $g \leftrightarrow f \leftrightarrow h$ and $d(g, h) = 2$ in $\Gamma(S)$ and $d(g, h) = \infty$ in $\Gamma(S) \setminus \{f\}$. Since g and h are connected to f:

$$Im(g) \cup Im(h) \subseteq ker(f),$$

and

$$Im(f) \subseteq ker(g) \cap ker(h).$$

Let $P = \{\beta_i\}_{i \in \mathbb{N}}$ be a basis of $Im(f)$ and $f(\alpha_1) = \beta_1$ and $f(\alpha_2) = \beta_2$. We define:

$$f'(x) = \begin{cases} c\beta_2 & x = c\alpha_1 & 0 \neq c \in F \\ c\beta_1 & x = c\alpha_2 & 0 \neq c \in F \\ f(x) & otherwise \end{cases}.$$

So, $g \leftrightarrow f' \leftrightarrow h$ which is a contradiction.

If $Im(f)$ is a one-dimensional subspace of S, we can use the vertex $f'(x) = 2f(x)$. □

Definition 3.13. Let S be a vector space and $P = \{f|f : S \leftrightarrow S\}$. We define an equivalence relation $\sim$ over P as follows:

$$f \sim g \qquad \text{if and only if} \qquad Im(g) = Im(f) \wedge ker(g) = ker(f)$$

$f \in \frac{P}{\sim}$ is a vertex of the graph Γ' if it has the following conditions:

- f is neither surjective nor injective.
- $ker(f) \nsubseteq Im(f)$ and $Im(f) \nsubseteq ker(f)$.

If $f, g \in \nu(\Gamma')$, $f \leftrightarrow g$ if and only if $f \circ g = g \circ f = 0$.

Theorem 3.14. *Every minimal vertex of* Γ' *is connected to a cut vertex.*

Proof. Let g be a minimal vertex in Γ'. So, there exists a basis of S like $P = \{\alpha_i\}_{i \in \mathbb{N}}$ such that g can be defined over P:

$$g(x) = \begin{cases} 0 & x = c\alpha_j \quad 0 \neq c \in F \quad , \quad \alpha_j \in F, \\ g(x) \neq 0 & otherwise \end{cases}$$

where $\alpha_j \notin Im(g)$. This vertex is only connected to hyper vertex f which $\{\alpha_j\}$ is a basis of $Im(f)$ so, f is cut vertex. □

Proposition 3.15. $f \in \nu(\Gamma')$ *is a cut vertex if and only if it is a hyper vertex.*

Proof. $\Rightarrow$)
Let f be a cut vertex and it is not a hyper vertex. Let g and h be vertices such that $g \leftrightarrow f \leftrightarrow g$ and $d(g, h) = \infty$ in $\Gamma(S) \setminus f$. By Theorem3.11, there exists a path between g and h via a hyper vertice, so $d(g, h) < \infty$ which is a contradiction.

$\Leftarrow$) If f is a hyper vertex, there exists $\alpha \in S$ such that $f(\alpha) = \beta \neq 0$ where $\beta \notin ker(f)$, so there exists $g \in \nu(\Gamma')$ such that $f \leftrightarrow g$, where g is a minimal vertex and $ker(g) =< \{\beta\} >$. If g is connected to a hyper vertex f', since $f' \sim f$, f is a cut vertex. □

4 Dual Space

Let S and S^* be two isomorphic vector spaces. Let $B = \{\alpha_i\}_{i=1}^n$ be a basis of S and $B^* = \{\gamma_i\}_{i=1}^n$ be a basis of S^*. Define the linear transformation $T : S \to S^*$ as follows:

$$T(0) = 0 \qquad and \qquad T(\alpha_i) = \gamma_i \qquad 1 \leq i \leq n.$$

Lemma 4.1. *Let* $P = \{f|f : S \leftrightarrow S\}$, $f(\alpha) = \beta$, *and* $T(\alpha) = \gamma$ *where* $\alpha, \beta \in S$ *and* $\gamma \in S^*$. *For any* $f \in P$ *the transformation* T *introduces a unique linear function* f^* *as follow:*

$$\forall \gamma \in S^* : f^*(\gamma) = T(\beta).$$

Proof. The function f^* is well defined because if $\gamma_1 = \gamma_2$ then

$$T^{-1}(\gamma_1) = T^{-1}(\gamma_2) = 0,$$

so $f(\alpha) = \beta$ implies:

$$f^*(\gamma_1) = f^*(\gamma_2) = T(\beta).$$

Also f^* is linear because $f^*(0) = 0$ and if $f^*(\gamma_1 + \gamma_2) = y$ let

$$\begin{array}{llll} \gamma_1 + \gamma_2 = \gamma \quad , & \gamma = T(\alpha) \quad , & \gamma_1 = T(\alpha_1) \quad , & \gamma_2 = T(\alpha_2) \\ f^*(\gamma_1) = y_1 \quad , & f^*(\gamma_2) = y_2 \quad , & T(x_1) = y_1 \quad , & T(x_2) = y_2. \end{array}$$

First:

$$\alpha = \alpha_1 + \alpha_2 \Leftrightarrow T(\alpha) = T(\alpha_1) + T(\alpha_2) \Leftrightarrow \gamma = \gamma_1 + \gamma_2.$$

Since $\gamma = T(\alpha)$,

$$f^*(\gamma) = T(f(\alpha)) = T(f(\alpha_1)) + T(f(\alpha_2)) = f^*(\gamma_1) + f^*(\gamma_2),$$

also,

$$f^*(c\gamma) = T(cf(\alpha)) = cT(f(\alpha)) = cf^*(\gamma).$$

If $f^*(\gamma) = g^*(\gamma)$ then there exists $\beta, \beta' \in S$ such that $f^*(\gamma) = T(\beta)$ and $g^*(\gamma) = T(\beta')$. This completes the proof.

□

The one-to-one transformation introduces a set of vertices in S^* which is called the induced graph Γ^*.

Theorem 4.2. *The induced graph Γ^* is well defined.*

Proof. Let $f \in \nu(\Gamma)$, so $ker(f) \neq \{0\}$ and therefore there exists $0 \neq \alpha \in S$ such that $f(\alpha) = 0$ and

$$\gamma = T(\alpha) \neq 0,$$

so

$$f^*(\gamma) = T(0) = 0.$$

Therefore, $ker(f^*) \neq \{0\}$ and f^* is not surjective or injective.

It is easy to show that $ker(f^*) \nsubseteq Im(f^*)$ and $Im(f^*) \nsubseteq ker(f^*)$. □

Theorem 4.3. *Under the previous assumptions, we have*

$$\Gamma \cong \Gamma^*.$$

Proof. The previous theorem proved that $\nu(\Gamma) \cong \nu(\Gamma^*)$. Let $f \leftrightarrow g$ in Γ and $0 \neq y = f^*(\gamma) \in Im(f^*)$. So, there exists $T(\beta) \in S^*$ such that $f^*(\gamma) = T(\beta)$.

Let $T(\alpha) = \gamma$, we prove that $f(\alpha) = \beta$, since otherwise

$$f(\alpha) = \beta' \neq \beta \qquad and \qquad T(\alpha) = \gamma,$$

then

$$f^*(\gamma) = T(\beta') \neq T(\beta),$$

which is a contradiction. Now $f(\alpha) = \beta$ implies that

$$\beta \in Im(f) \subseteq ker(g),$$

hence $g(\beta) = 0$ and $T(\beta) = g$. We have $g^*(y) = 0$ and consequently $Im(f^*) \subseteq ker(g^*)$. The proof of $Im(g^*) \subseteq ker(f^*)$ is similar. Also it is easy to show that if $f^*, g^* \in \nu(\Gamma)$ are connected, then $f \leftrightarrow g$ in Γ. □

Corollary 4.4. *Let S be a finite dimensional vector space and S^* be its dual space.*

For any basis of S such as $B = \{\alpha_i\}_{i=1}^n$, there exists a basis $B^* = \{\gamma_i\}_{i=1}^n$ of S^* such that

$$\gamma_i(\alpha_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

so the induced graph Γ^* on S^* is isomorphic to Γ.

Theorem 4.5. *Let S and S' be two finite-dimensional vector spaces over a field $\mathbb{F}$ and $\Gamma(S)$ and $\Gamma(S')$ be their corresponding graphs respectively. If $\Gamma(S) \cong \Gamma(S')$ then S is isomorphic to S'.*

Proof. Since both graphs have the same K_n, Theorem 3.3 implies that the dimensional of S and S' should be n. Hence $S \cong S'$. □

Corollary 4.6. *Let S be an $n-$dimensional vector space. For any basis of S, Theorem 3.3 implies that there exists a complete subgraph K_n of Γ. Also, for any complete subgraph of Γ which its vertices is $\{f_1, f_2, \ldots, f_n\}$, Theorem 3.7 implies that there exists*

$$P = \{\beta_1, \beta_2, \ldots, \beta_n\},$$

where for $1 \leq i \leq n$

$$Im(f_i) =< \{\beta_i\} >,$$

and $< P >= S$, so P is a basis for S. Therefore the set of basis of S is in one-to-one correspondence to a set of complete subgraphs K_n of S.

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