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Multivalued transformations and representations of semihypergroups

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Abstract The ordinary Cayley's theorem describes groups and semigroups in terms of their representations by unary transformations (functions). In group theory, the Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group acting on G. There is also a theorem in semigroup theory stating that every semigroup can be embedded into a full transformation semigroup. In this paper, we generalize these theorems to semihypergroups through the new concepts of multivalued full transformations. Moreover, subclasses of multivalued full transformations, left(right) hypertranslations and inner left(right) hypertranslations are studied.

1 Introduction

Let X be a nonempty set. We recall that $\mathrm{sym}(X)$ denotes the group of all permutations on X (bijections from X to X) under the usual composition of functions and it is called a *symmetric group*. Any subgroup of $\mathrm{sym}(X)$ is called a *permutation group*. The Cayley's theorem is a significant theorem in group theory, it was introduced in 1878 by Arthur Cayley. The theorem was stated that every abstract group G is isomorphic to a subgroup of the symmetric group. With respect to an algebraic structure, there is no distinction between abstract groups and permutation groups. It is widely known that, if G has order n, then it is isomorphic to a permutation group. It is also true for infinite group; every group is isomorphic to a symmetric group. Current topics of representations in other structures were given in [16].

The algebraic theory of semigroups has been well investigated during the second half of the twentieth century. Many concepts in study of semigroups were directly motivated by their analogous in group theory. Just as groups arise most ordinary as groups of permutations of some sets. Hence semigroups arise from more general mappings of a set into itself. The analogous of a group of all permutations of X is the *full transformation semigroup* $T(X)$, the semigroup of all mappings from X into X under composition of mappings. It is commonly seen that $T(X)$ is a semigroup [9] and the corresponding statement for semigroups has essentially the same form, i.e., every semigroup is isomorphic to a transformation semigroup. This emphasizes that the understanding of a semigroup of mappings in semigroup theory is of imperative signification. Transformation semigroups are one of the most primary mathematical tools. It also occurs in theoretical computer science, where properties of languages depend on algebraic properties of various semigroups of transformations. Generalizations of a semigroup of transformations were studied by the authors in [10, 11, 12, 13, 14] and Nongmanee and Leeratanavalee in [18]. Actually, Tamura [21, 22] discussed the translations on a semigroup. By a *left translation* of a semigroup S, we mean a transformation $\lambda : S \to S$ satisfying $\lambda(xy) = \lambda(x)y$ for all x, y in S. Otherwise, a transformation ρ of S is called a *right translation* of S if $\rho(xy) = x\rho(y)$ for all x, y in S. We denote the set of all left translations and right translations of a semigroup S by $\Lambda(S)$ and $P(S)$, respectively. For each element a of a groupoid S, a mapping $\lambda_a : S \to S$ defined by $\lambda_a(x) = ax$ for all $x \in S$ is associated. We call λ_a the *inner left translation of* S corresponding to the element a of S. Similarly, the mapping $\rho_a : S \to S$ defined by $\rho_a(x) = xa$

for all $x \in S$. We call ρ_a the *inner right translation of* S. The set of all inner left translations and inner right translations are denoted by $\Lambda_0(S)$ and $P_0(S)$, respectively. It is observed that in any semigroup S containing a left(right) identity element e, every left(right) translation is inner. In fact, we have $\lambda(x) = \lambda(ex) = \lambda(e)x = ax$ where $a = \lambda(e)$. Normally, it is clear that $\Lambda(S), \Lambda_0(S), P(S), P_0(S)$ are semigroups with respect to the usual composition of transformations. Moreover, a characterization of any groupoid to be a semigroup was stated. Indeed, a binary groupoid is a semigroup if and only if any inner left(right) translation of is a left(right) translation.

For any semigroup $(S, \cdot)$, if S does not contain any identity element, the set S^1 means the semigroup S with an identity adjoined, i.e., $S^1 := S \cup \{1\}$, where $1 \notin S$ and the binary operation $*$ on S^1 is defined as follows: For each $a, b \in S^1$

$$a * b = \begin{cases} a \cdot b, & a, b \in S; \\ a, & b = 1; \\ b, & a = 1. \end{cases}$$

It is easy to show that $*$ is associative, thus $(S^1, *)$ is a semigroup. Hence $\rho : S \to T(S^1)$ is a monomorphism from a semigroup S to the semigroup of transformations on S^1. The above statement may be called the *Cayley's theorem for semigroups*.

Algebraic hyperstructures are generalizations of classical algebraic structures. Recent topics and developments of algebraic hyperstructures in different directions were collected in [6, 19]. One of the studies on hyperstructures is the theory of semihypergroups which was started by Marty in 1934 [15]. Indeed, semihypergroups are the simplest algebraic hyperstructures which pose the properties of associativity. It is commonly seen that every semigroup is a semihypergroup. In a semigroup, the composition of two elements is an element, while the composition of two elements in a semihypergroup is a nonempty set (the formal definition of semihypergroups will be recalled in the next section). Also, it has many foundations and applications in probability, automata, geometry, lattices, graphs, hypergraphs, and other branches of science such as chemistry, biology, and physics. For more details, see [2, 7]. Algebraic structures of hyperrings were further explored in [3, 8]. Daengsean and Leeratanavalee [4, 5] studied ordered n-ary semihypergroups. Characterizations of various types of hyperideals in semihypergroups and semihyperrings may be seen in [1, 17, 20].

The lack of certain algebraic properties, especially the Cayley's theorem on a semihypergroup, poses a critical challenge in extending results from semigroups and full transformation semigroups to semihypergroups. The notions in this paper are inspired by the reasons and results mentioned above and by the ongoing research in this area. The new concepts of the multivalued full transformations are presented. We propose certain subclasses of multivalued full transformations, called left(right) hypertranslations and inner left(right) hypertranslations, which can be regarded as extended versions of left(right) translations and inner left(right) translations in semigroups. We then state and prove the Cayley's theorem for semihypergroups. Finally, we conclude with some open problems and potential directions which we intend to study on and explore in the near future.

2 Preliminaries

Throughout this paper, we use some basic concepts and definitions which belong to common terminology in semihypergroup theory. For details, one can refer the reader to the monograph [7].

Let H be a nonempty set. A mapping $\circ : H \times H \to P^*(H)$, where $P^*(H)$ denotes the family of all nonempty subsets of H, is called a *hyperoperation* on H. The pair $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A, B are two nonempty subsets of H and $x \in H$, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \; A \circ x = A \circ \{x\} \text{ and } \{x\} \circ A = x \circ A.$$

Generally, when nothing opposes it, there is no distinction between the elements and their corresponding singletons.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, that is

$$\bigcup_{v \in x \circ y} v \circ z = \bigcup_{u \in y \circ z} x \circ u.$$

Here are some examples of semihypergroups.

Example 2.1. ([7]) Let $H = \{a, b, c, d\}$ and $\circ$ be a hyperoperation defined by the following table.

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
c	a	b	c	d
d	a	b	c	d

Then $(H, \circ)$ is a semihypergroup.

Example 2.2. ([7]) Suppose that a hyperoperation on $H = \{x, y, z, t\}$ is defined by the following table.

$\circ$	x	y	z	t
x	x	$\{x, y\}$	$\{x, z\}$	H
y	y	y	$\{y, t\}$	$\{y, t\}$
z	z	$\{z, t\}$	z	$\{z, t\}$
t	t	t	t	t

Then $(H, \circ)$ is a semihypergroup.

Definition 2.3. ([7]) An element 1 in a semihypergroup $(H, \circ)$ is called a *left scalar identity* or a *right scalar identity* provided that for every $x \in H$ we have $1 \circ x = \{x\}$, or $x \circ 1 = \{x\}$, respectively. An element 1, which is a left and a right scalar identity, is called a *scalar identity*.

If H contains some left scalar identity 1_l and some right scalar identity 1_r we have $\{1_l\} = 1_l \circ 1_r = \{1_r\}$ and hence these two elements coincide. Hence in this case, H contains a scalar identity element. It is possible for a semihypergroup to contain neither left nor right scalar identity. Example 2.2 is an illustration.

The absence of a scalar identity can be easily improved in the following way.

Definition 2.4. Let $(H, \circ)$ be a semihypergroup. In general, H does not necessary contain the scalar identity. Then we let 1 be a new element which is not in H. We construct a new set $H \cup \{1\}$ and define a hyperoperation $\star$ on $H \cup \{1\}$ by

$$\begin{aligned} a \star b &= a \circ b \text{ if } a, b \in H, \\ a \star 1 &= \{a\} = 1 \star a \text{ for all } a \in H, \\ 1 \star 1 &= \{1\}. \end{aligned}$$

By direct computation, it is not difficult to verify that $(H \cup \{1\}, \star)$ is a semihypergroup with the scalar identity 1. For any semihypergroup H, we define

$$H^1 = \begin{cases} H & \text{if } H \text{ has a scalar identity,} \\ H \cup \{1\} & \text{otherwise.} \end{cases}$$

Therefore, H^1 is the semihypergroup obtained by adjoining a scalar identity to H if necessary. Furthermore, from the definition of $\star$ we have that 1 is the scalar identity element in H^1. Moreover, the restriction of the hyperoperation $\star$ to H coincides with the original hyperoperation $\circ$. Hence H is a subsemihypergroup of H^1.

3 Multivalued full transformations and their subclasses

In this section, we first introduce an algebraic structure which is called a semigroup of multivalued full transformations and then give several examples and discuss some properties.

Let X be a nonempty set. A multivalued mapping f from X to $P^*(X)$ can be described by listing the elements of X in the first row and the image of each element under f directly below it in the second row. Hence, f is an array of the form

$$f = \begin{pmatrix} x \\ f(x) \end{pmatrix}_{x \in X}, \tag{3.1}$$

where $f(x) \subseteq X$. This is generally denoted by $f : X \to P^*(X)$. The set $f(x)$ is called the image of the mapping f at x. The set of all multivalued full transformations of X is denoted by $T(X, P^*(X))$.

We now describe certain multivalued full transformation semigroups on some sets in the following example.

Example 3.1. Let $A = \{a, b\}$. Here is the list of all elements of $T(A, P^*(A))$: $\alpha_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} a & b \\ a & a \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} a & b \\ b & b \end{pmatrix}$, $\alpha_5 = \begin{pmatrix} a & b \\ a & A \end{pmatrix}$, $\alpha_6 = \begin{pmatrix} a & b \\ b & A \end{pmatrix}$, $\alpha_7 = \begin{pmatrix} a & b \\ A & a \end{pmatrix}$, $\alpha_8 = \begin{pmatrix} a & b \\ A & b \end{pmatrix}$, $\alpha_9 = \begin{pmatrix} a & b \\ A & A \end{pmatrix}$. In this list $\alpha_1, \ldots, \alpha_4$ are full transformations and hence $\alpha_1, \ldots, \alpha_4$ are all elements in $T(A)$.

Let now X be a nonempty set and let $f, g : X \to P^*(X)$ be two multivalued mappings. We can define the product $\bullet$ or the composition $f \bullet g$ of f and g by the following rule: For all $x \in X$,

$$(f \bullet g)(x) = \bigcup_{y \in g(x)} f(y).$$

The composition $f \bullet g$ is also a multivalued mapping from X to $P^*(X)$. Furthermore, the product $f \bullet g$ of elements of $T(X, P^*(X))$ is the composition from right-to-left order, i.e., g followed by f; that is the multivalued transformations on X given by $x \mapsto (f \bullet g)(x)$.

For instance, let $f = \begin{pmatrix} 1 & 2 & 3 \\ \{1,2\} & 3 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ \{1,3\} & \{2,3\} & 1 \end{pmatrix}$ be two elements of $T(\{1,2,3\}, P^*(\{1,2,3\}))$. Then under the composition $f \bullet g$, we have

$$(f \bullet g)(1) = \bigcup_{y \in g(1)} f(y) = \bigcup_{y \in \{1,3\}} f(y) = f(1) \cup f(3) = \{1,2\},$$

$$(f \bullet g)(2) = \bigcup_{y \in g(2)} f(y) = \bigcup_{y \in \{2,3\}} f(y) = f(2) \cup f(3) = \{2,3\},$$

$$(f \bullet g)(3) = \bigcup_{y \in g(3)} f(y) = \bigcup_{y \in \{1\}} f(y) = f(1) = \{1,2\},$$

which means

$$f \bullet g = \begin{pmatrix} 1 & 2 & 3 \\ \{1,2\} & 3 & 2 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 & 3 \\ \{1,3\} & \{2,3\} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \{1,2\} & \{2,3\} & \{1,2\} \end{pmatrix}.$$

On the other hand, we obtain

$$g \bullet f = \begin{pmatrix} 1 & 2 & 3 \\ \{1,3\} & \{2,3\} & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 & 3 \\ \{1,2\} & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \{1,2,3\} & 1 & \{2,3\} \end{pmatrix}.$$

This example shows that the operation $\bullet$ needs not be commutative.

Proposition 3.2. *The composition of multivalued transformations is associative, i.e., if f, g and h are multivalued transformations, then $(f \bullet g) \bullet h = f \bullet (g \bullet h)$.*

Proof. It easy to see that $\bullet$ is a binary operation on the set $T(X, P^*(X))$. We now let $f, g, h \in T(X, P^*(X))$ and $x \in X$. Then we have

$$\begin{aligned} ((f \bullet g) \bullet h)(x) &= \bigcup_{y \in h(x)} (f \bullet g)(y) \\ &= \bigcup_{y \in h(x), z \in g(y)} f(z) \\ &= \bigcup_{z \in \bigcup\limits_{y \in h(x)} g(y)} f(z) \\ &= \bigcup_{z \in (g \bullet h)(x)} f(z) \\ &= (f \bullet (g \bullet h))(x). \end{aligned}$$

□

By Proposition 3.2, the composition of multivalued full transformations in $T(X, P^*(X))$ is an associative operation. Hence we have the following theorem.

Theorem 3.3. *The set of all full multivalued transformations on the set X is a semigroup with respect to a composition of mutivalued full transformations.*

The following theorem shows that any semigroup of multivalued full transformations can be considered as an extension of a semigroup of full transformations in sense of injective homomorphic images.

Theorem 3.4. *The full transformations semigroup $(T(X), \circ)$ is embeddable into the semigroup of multivalued full transformations $(T(X, P^*(X)), \bullet)$.*

Proof. Define a mapping $\pi : T(X) \to T(X, P^*(X))$ by

$$\pi(f) = \begin{pmatrix} x \\ \{f(x)\} \end{pmatrix}_{x \in X} := \widehat{f}$$

for all $f \in T(X)$. By using the definition of the composition $\bullet$, it is not difficult to show the preservation of operations, i.e., $\pi(f \circ g) = \widehat{f \circ g} = \widehat{f} \bullet \widehat{g} = \pi(f) \bullet \pi(g)$. To do this, we now let $x \in X$ and then $(\widehat{f \circ g})(x) = \{(f \circ g)(x)\} = \{f(g(x))\} = \widehat{f}(g(x)) = \bigcup_{y \in \{g(x)\}} \widehat{f}(y) = \bigcup_{y \in \hat{g}(x)} \widehat{f}(y) = (\widehat{f} \bullet \widehat{g})(x)$. Finally, let $f, g \in T(X)$ and $x \in X$ be such that $\widehat{f}(x) = \widehat{g}(x)$. Then we obtain that $\{f(x)\} = \{g(x)\}$ and thus $f = g$. Hence π is injective. □

In the term of combinatorial result, we have the following proposition which is useful for determine the cardinality of $T(X, P^*(X))$.

Proposition 3.5. *Let X be a finite set which is not empty. Then the cardinality of $T(X, P^*(X))$ is $|P^*(X)|^{|X|}$.*

Proof. Each element $f \in T(X, P^*(X))$ is uniquely defined by (3.1). The image subset $f(x)$ of X can be independently chosen from the set of all nonempty subsets of X, $P^*(X)$. Hence, by the product rule implies $T(X, P^*(X)) = |P^*(X)|^{|X|}$. □

It is rather useful to present a small multivalued full transformations semigroup $T(X, P^*(X))$ by using its multiplication table, which is also called the *Cayley's table* for multivalued full transformations . Such table is a square matrix with $|T(X, P^*(X))|$ rows and $|T(X, P^*(X))|$ columns, which are indexed by the elements of the set $T(X, P^*(X))$. At the intersection of the f^{th} row and the g^{th} column, $f, g \in T(X, P^*(X))$, one writes the composition $f \bullet g$ defined above.

Example 3.6. Let $A = \{a, b\}$ be a set with two elements. Then the multivalued full transformations semigroup $T(A, P^*(A)) = \{\alpha_1, \ldots, \alpha_9\}$ (as in Example 3.1) has the following Cayley's table:

$\bullet$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
α_1	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
α_2	α_2	α_2	α_2	α_2	α_2	α_2	α_2	α_2	α_2
α_3	α_3	α_3	α_3	α_3	α_3	α_3	α_3	α_3	α_3
α_4	α_4	α_3	α_2	α_1	α_5	α_5	α_7	α_7	α_8
α_5	α_5	α_2	α_9	α_7	α_5	α_9	α_7	α_9	α_9
α_6	α_6	α_3	α_9	α_8	α_6	α_9	α_8	α_9	α_9
α_7	α_7	α_9	α_2	α_5	α_9	α_5	α_9	α_7	α_9
α_8	α_8	α_9	α_3	α_6	α_9	α_6	α_9	α_8	α_9
α_9	α_9	α_9	α_9	α_9	α_9	α_9	α_9	α_9	α_9

To show how semihypergroups are related to subclasses of multivalued full transformations on some set, we need more concepts.

Definition 3.7. Let $(H, \circ)$ be a semihypergroup. A mapping $\lambda \in T(H, P^*(H))$ is called a *left hypertranslation* of H if

$$\bigcup_{a \in x \circ y} \lambda(a) = \bigcup_{b \in \lambda(x)} b \circ y$$

for all $a, b, x, y \in H$. Analogously, a mapping $\rho \in T(H, P^*(H))$ is called a *right hypertranslation* of H if

$$\bigcup_{a \in x \circ y} \rho(a) = \bigcup_{b \in \rho(y)} x \circ b$$

Denoted by $\Lambda^*(H)$ and $\Omega^*(H)$ the set of all left hypertranslations and right hypertranslations on H, respectively.

Proposition 3.8. *For any semihypergroup $(H, \circ)$, the sets $\Lambda^*(H)$ and $\Omega^*(H)$ are subsemigroups of $T(H, P^*(H))$ with respect to $\bullet$.*

Proof. We show that $\Lambda^*(H)$ is a subsemigroup of $T(H, P^*(H))$. For this, let x, y be arbitrary elements in a semihypergroup $(H, \circ)$ and let f, g be mappings in $\Lambda^*(H)$. Then $\bigcup_{a \in x \circ y} (f \bullet g)(a) = \bigcup_{a \in x \circ y} (\bigcup_{b \in g(a)} f(b)) = \bigcup_{b \in \bigcup_{a \in x \circ y} g(a)} f(b) = \bigcup_{b \in \bigcup_{c \in g(x)} c \circ y} f(b) = \bigcup_{b \in g(x) \circ y} f(b) = \bigcup_{d \in f(g(x))} d \circ y = \bigcup_{d \in \bigcup_{a \in g(x)} f(a)} d \circ y = \bigcup_{d \in (f \bullet g)(x)} d \circ y$. This shows that $\Lambda^*(H)$ is a subsemigroup of $T(H, P^*(H))$. □

Definition 3.9. Let $(H, \circ)$ be a semihypergroup and $a \in H$. A mapping $\lambda_a \in T(H, P^*(H))$ is called an *inner left hypertranslation* of H if $\lambda_a(x) = a \circ x$ for all $x \in H$. For any subset A of H, a mapping $\lambda_A \in T(H, P^*(H))$ can be defined by $\lambda_A(x) = A \circ x$. Analogously, a mapping $\rho_a \in T(H, P^*(H))$ is called an inner *right hypertranslation* of H if $\rho_a(x) = x \circ a$. A mapping $\rho_A \in T(H, P^*(H))$ can be defined by $\rho_A(x) = x \circ A$. Denoted by $\Lambda_0^*(H)$ and $\Omega_0^*(H)$ the set of all inner left hypertranslations and inner right hypertranslations on H, respectively.

We are ready to prove the following proposition:

Proposition 3.10. *The sets $\Lambda_0^*(H)$ and $\Omega_0^*(H)$ are subsemigroups of $T(H, P^*(H))$ with respect to the operation $\bullet$.*

Proof. To prove that $\Lambda_0^*(H)$ is a subsemigroup of $T(H, P^*(H))$ with respect to an operation $\bullet$, we let $\lambda_a, \lambda_b \in \Lambda_0^*(H)$ for a, b in a semihypergroup H. Due to the associativity of $\circ$ on H, we have $(\lambda_a \bullet \lambda_b)(x) = \bigcup_{y \in \lambda_b(x)} \lambda_a(y) = \bigcup_{y \in b \circ x} a \circ y = \bigcup_{z \in a \circ b} z \circ x = (a \circ b) \circ x$ □

The following theorem gives a characterization for any multivalued full transformation on a semihypergroup to be a left hypertranslation.

Theorem 3.11. *A multivalued full transformation on a semihypergroup $(H, \circ)$ is a left hypertranslation if and only if it commutes with every inner right hypertranslation of H.*

Proof. Let f be a mapping in the set $T(H, P^*(H))$ of all multivalued full transformations on a semihypergroup H and let ρ_a be an inner right hypertranslation of H for $a \in H$. To show that $f \bullet \rho_a = \rho_a \bullet f$, we let $x \in H$. Since f is a left hypertranslation, we have that $(f \bullet \rho_a)(x) = \bigcup_{b \in \rho_a(x)} f(b) = \bigcup_{b \in x \circ a} f(b) = \bigcup_{c \in f(x)} c \circ a = \bigcup_{c \in f(x)} \rho_a(c) = (\rho_a \bullet f)(x)$, which implies $f \bullet \rho_a = \rho_a \bullet f$. Conversely, let x and y be any two elements in H. Then $\bigcup_{a \in x \circ y} f(a) = \bigcup_{a \in \rho_y(x)} f(a) = (f \bullet \rho_a)(x) = (\rho_a \bullet f)(x) = \bigcup_{b \in f(x)} \rho_y(b) = \bigcup_{b \in f(x)} b \circ y$. This shows that f is a left hypertranslation on H. □

Similarly, we obtain the following:

Corollary 3.12. *A multivalued full transformation on a semihypergroup $(H, \circ)$ is a right hypertranslation if and only if it commutes with every inner left hypertranslation of H.*

Next, we propose an important result that gives a characterization of semihypergroups in terms of a compostion between any two inner left hypertranslations.

Theorem 3.13. *A binary hypergroupoid $(H, \circ)$ is a semihypergroup if and only if $\lambda_{a \circ b} = \lambda_a \bullet \lambda_b$ for all $a, b \in H$.*

Proof. Let $a, b, x \in H$. Then $\lambda_{a \circ b}(x) = (a \circ b) \circ x = a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y = \bigcup_{y \in \lambda_b(x)} \lambda_a(y) = (\lambda_a \bullet \lambda_b)(x)$. For the converse, suppose that $(H, \circ)$ is a binary hypergroupoid and $a, b, c \in H$. It is not difficult to show that $\circ$ is associative. Indeed, we have

$$(a \circ b) \circ c = \lambda_{a \circ b}(c) = (\lambda_a \bullet \lambda_b)(c) = \bigcup_{y \in \lambda_b(c)} \lambda_a(y) = \bigcup_{y \in b \circ c} a \circ y = a \circ (b \circ c),$$

which completes the proof. □

Similarly, Corollary 3.14 is proved.

Corollary 3.14. *A binary hypergroupoid $(H, \circ)$ is a semihypergroup if and only if $\rho_{a \circ b} = \rho_b \bullet \rho_a$ for all $a, b \in H$.*

As an illustration of Theorem 3.13, we consider the semihypergroup which was given in Example 2.2. It is easy to see that each element in a semihypergroup $(H, \circ)$ can be induced the following inner left hypertranslations.

$$\lambda_x = \begin{pmatrix} x & y & z & t \\ x & \{x, y\} & \{x, z\} & H \end{pmatrix},$$
$$\lambda_y = \begin{pmatrix} x & y & z & t \\ y & y & \{y, t\} & \{y, t\} \end{pmatrix},$$
$$\lambda_z = \begin{pmatrix} x & y & z & t \\ z & \{z, t\} & z & \{z, t\} \end{pmatrix},$$
$$\lambda_t = \begin{pmatrix} x & y & z & t \\ t & t & t & t \end{pmatrix}.$$

Here is an example showing that the left inner hypertranslations $\lambda_x, \ldots, \lambda_t$ satisfy the property mentioned in Theorem 3.13. We now consider

$$\begin{aligned} \lambda_{x \circ z} &= \lambda_{\{x, z\}} \\ &= \begin{pmatrix} x & y & z & t \\ \{x, z\} \circ x & \{x, z\} \circ y & \{x, z\} \circ z & \{x, z\} \circ t \end{pmatrix} \\ &= \begin{pmatrix} x & y & z & t \\ \{x, z\} & H & \{x, z\} & H \end{pmatrix} \\ &= \begin{pmatrix} x & y & z & t \\ x & \{x, y\} & \{x, z\} & H \end{pmatrix} \bullet \begin{pmatrix} x & y & z & t \\ z & \{z, t\} & z & \{z, t\} \end{pmatrix} \\ &= \lambda_x \bullet \lambda_z. \end{aligned}$$

This shows that $\lambda_{x\circ z} = \lambda_x \bullet \lambda_z$.

We note that Theorem 3.13 is a powerful tool to prove some properties of the mapping from any abstract semihypergroup to some set of multivalued full transformations (we will set such mapping later). Moreover, this theorem also confirms that we can not apply the hyperoperation for the set $T(X, P^*(X))$ where $X \neq \emptyset$. In fact, if the hyperoperation on $T(X, P^*(X))$ was defined, then by Theorem 3.13, the left-hand side is a natural multivalued full transformation, but the result in the right-hand side is the set of multivalued full transformations which is impossible.

Moreover, necessary and sufficient conditions for any hypergroupoid to be a semihypergroup can be given in the following theorem.

Theorem 3.15. *A hypergroupoid $(H, \circ)$ is a semihypergroup if and only if every inner left hypertranslation on H is a left hypertranslation on H.*

Proof. Let $(H, \circ)$ be a semihypergroup and let λ_a be an inner left hypertranslation on H for an arbitrary element a in H. Since a hyperoperation $\circ$ on H is associative, we have

$$\bigcup_{z\in x\circ y} \lambda_a(z) = \bigcup_{z\in x\circ y} a\circ z = \bigcup_{w\in a\circ x} w\circ y = \bigcup_{w\in\lambda_a(x)} w\circ y,$$

which proves that λ_a is a left hypertranslation on H. Conversely, it follows from hypothesis that the associativity of $\circ$ is directly obtained. In fact, we let $x, y, z \in H$ and thus

$$(x\circ y)\circ z = \bigcup_{a\in x\circ y} a\circ z = \bigcup_{a\in\lambda_x(y)} a\circ z = \bigcup_{b\in y\circ z} \lambda_x(b) = \bigcup_{b\in y\circ z} x\circ b = x\circ(y\circ z).$$

□

Corollary 3.16. *A hypergroupoid $(H, \circ)$ is a semihypergroup if and only if every inner right hypertranslation on H is a right hypertranslation on H.*

Some properties of left hypertranslations and inner left hypertranslations on a semihypergroup are examined.

Theorem 3.17. *Let λ and λ_a be a left hypertranslation and an inner left hypertranslation on a semihypergroup $(H, \circ)$, respectively. Then $\lambda \bullet \lambda_a = \lambda_{\lambda_a}$ for all $a \in H$.*

Proof. Let x be an element in H. Then $(\lambda\bullet\lambda_a)(x) = \bigcup\limits_{y\in\lambda_a(x)} \lambda(y) = \bigcup\limits_{y\in a\circ x} \lambda(y) = \bigcup\limits_{z\in\lambda(a)} z\circ x = \lambda(a)\circ x = \lambda_{\lambda_a}(x)$. □

Similarly, we have the following corollary

Corollary 3.18. *Let ρ and ρ_a be a right hypertranslation and an inner right hypertranslation on a semihypergroup $(H, \circ)$, respectively. Then $\rho \bullet \rho_a = \rho_{\rho_a}$ for all $a \in H$.*

4 The representation theorem for semihypergroups

Using the concept of full multivalued transformations as we already introduced in the previous section, we now present an important result which shows a representation of each semihypergroup by a multivalued full transformation.

Theorem 4.1. *Every semihypergroup is isomorphically represented by a multivalued full transformation.*

Proof. We first show that there exists a multivalued full transformation induced by an element a of H. For this construction, consider the set $H^1 = H \cup \{1\}$ where 1 is a different element not containing in H. For every element a of H, we assign an $\lambda_a : H^1 \to P^*(H^1)$ by setting $\lambda_a(x) = a \circ x$ for all $x \in H^1$. Clearly, $\lambda_a \in \Lambda_0^*(H^1)$. Furthermore, a mapping $\lambda_A : H^1 \to P^*(H^1)$ is defined by $\lambda_A(x) = A \circ x$ for any subset A of H^1 and $x \in H^1$. Obviously, $\lambda_A(x) = \bigcup\limits_{a\in A} \lambda_a(x)$

for all $x \in H^1$. Define a mapping $\varphi : H \to \Lambda_0^*(H)$ by $\varphi(a) = \lambda_a$ for all $a \in H$. We prove that φ is an isomorphism from H to $\Lambda_0^*(H)$. By Lemma 3.13, we obtain that

$$\varphi(a \circ b) = \lambda_{a\circ b} = \lambda_a \bullet \lambda_b = \varphi(a) \bullet \varphi(b).$$

Thus φ is a homomorphism. Furthermore,

$$\lambda_a = \lambda_b \Rightarrow \lambda_a(1) = \lambda_b(1) \Rightarrow a \circ 1 = b \circ 1 \Rightarrow \{a\} = \{b\} \Rightarrow a = b.$$

Hence φ is injective. The proof is completed. □

We now give some examples.

Example 4.2. Consider a semihypergroup $H = \{p, q, r, s\}$ with respect to the following Cayley's table.

$\circ$	p	q	r	s
p	p	q	r	s
q	q	$\{p,r\}$	$\{q,r\}$	s
r	r	$\{q,r\}$	$\{p,q\}$	s
s	s	s	s	H

Obviously, H is a semihypergroup containing a scalar identity p. We now illustrate that H is isomorphic to some set of multivalued full transformations. To do this, we consider a mapping $\lambda_p : H \to P^*(H)$ such that $\lambda_p(p) = p \circ p = p$, $\lambda_p(q) = p \circ q = q$, $\lambda_p(r) = p \circ r = r$ and $\lambda_p(s) = p \circ s = s$. Thus $\lambda_p = \begin{pmatrix} p & q & r & s \\ p & q & r & s \end{pmatrix}$ and then

$$p \mapsto \lambda_p = \begin{pmatrix} p & q & r & s \\ p & q & r & s \end{pmatrix}.$$

Furthermore, a mapping $\lambda_q : H \to P^*(H)$ is defined by $\lambda_q(p) = p$, $\lambda_q(q) = \{p, r\}$, $\lambda_q(r) = \{q, r\}$ and $\lambda_q(s) = s$. Thus the multivalued full transformation which corresponding to an element q in H is $\lambda_q = \begin{pmatrix} p & q & r & s \\ q & \{p,r\} & \{q,r\} & s \end{pmatrix}$ and so

$$q \mapsto \lambda_q = \begin{pmatrix} p & q & r & s \\ q & \{p,r\} & \{q,r\} & s \end{pmatrix}.$$

For other elements in H, we obtain extended left regular representations for a semihypergroup H, i.e.

$$r \mapsto \lambda_r = \begin{pmatrix} p & q & r & s \\ r & \{q,r\} & \{p,q\} & s \end{pmatrix},$$

$$s \mapsto \lambda_s = \begin{pmatrix} p & q & r & s \\ s & s & s & H \end{pmatrix}.$$

It follows from Theorem 4.1 that $H \cong \{\lambda_p, \lambda_q, \lambda_r, \lambda_s\}$. Furthermore, the table for these representations is just like the original table with x renamed by λ_x, as seen in the following table:

$\bullet$	λ_p	λ_q	λ_r	λ_s
λ_p	λ_p	λ_q	λ_r	λ_s
λ_q	λ_q	$\lambda_{\{p,r\}}$	$\lambda_{\{q,r\}}$	λ_s
λ_r	λ_r	$\lambda_{\{q,r\}}$	$\lambda_{\{p,q\}}$	λ_s
λ_s	λ_s	λ_s	λ_s	λ_H

For the infinite semihypergroups, we now provide an interesting example as follows:

Example 4.3. Let $\mathbb{N}$ be the set of nonnegative integers. For every $x, y \in \mathbb{N}$, we define the following hyperoperation on $\mathbb{N}$ by $x \circ y = \{z \in \mathbb{N} \mid z \geq \max\{x, y\}\}$. Then $(\mathbb{N}, \circ)$ is a semihypergroup. Applying Theorem 4.1, we obtain that a semihypergroup $(\mathbb{N}, \circ)$ which was defined above is isomorphic to some set of multivalued full transformations by setting a mapping $\lambda_x : \mathbb{N}^1 \to \mathbb{N}^1$ such that $\lambda_x = \begin{pmatrix} x \\ x \circ y \end{pmatrix}_{y \in \mathbb{N}^1}$. Let A be the set of all such mappings. Then a mapping $\varphi : \mathbb{N} \to A$ can be defined by $\varphi(x) = \lambda_x$ for all $x \in \mathbb{N}^1$ and hence $\mathbb{N} \cong A$.

A homomorphism from a semihypergroup H into some $T(X, P^*(X))$ for some X is called a *representation* of H (by maps).

According to the main thoerem, we notice that for every element a of a semihypergroup H can be represented by a multivalued full transformation λ_a, the connection between the semihypergroups H and $T(H, P^*(H))$. Hence the representation φ which was introduced in the proof of Theorem 4.1 is called the *extended left regular semihypergroup representation*. It is necessary to ensure the injection of the representation.

The following corollary follows from Theorem 3.4.

Corollary 4.4. *Every semigroup can be embeded into a multivalued full transformation semigroup.*

5 Concluding remarks

In this paper, any mapping which takes from a nonempty set to its power set without a nonempty set, called a multivalued full transformation, was introduced. It can be noticed that this concept can be considered as a generalization of transformation semigroups. Subclasses of such mappings, for example, left hypertranslations and inner left hypertranslations were mentioned. Necessary and sufficient conditions for which each binary hypergroupoid formed a semihypergroup were determined. We further proved that every element of any abstract semihypergroup is (up to isomorphism) a multivalued full transformation. Hence, to study the semihypergroup H, we need only study multivalued full transformations in the set $T(H, P^*(H))$. This approach has an advantage because $T(H, P^*(H))$ consists of concrete mappings that can be analyzed using tools not obtainable in an abstract semihypergroup. However, it is not especially convenient for computation since $T(H, P^*(H))$ has order $|P^*(H)|^{|H|}$. It is too large in a comparison with $|H|$. For example, if $|H| = 5$, then $|T(H, P^*(H))| = |P^*(H)|^{|H|} = (2^5 - 1)^5 = 31^5 = 28,629,151$. In the near future, it is interesting to purpose that how we can define a generalization of Theorem 4.1 to reduce the size of the set of all multivalued full transformations.

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