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THE DUALITY PROBLEM FOR THE OWDC OPERATORS AND A NEW CLASS OF OW*DC OPERATORS

A. EL ALOUI and K. BOURAS

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Abstract In this paper, we investigate the duality problem for the class of OWDC operators. Precisely, we give a sufficient and necessary condition under which the order weak demicompactness of an operator implies the order weak demicompactness of its adjoint, and conversely. Also, we introduce a new class of OW*DC operators. In addition, we establish some properties of this class of operators on Banach lattices.

1 Introduction

The investigation of the demicompact operator class has a long history. It can be dated back to 1966, when Petryshyn used it to study the existence and construction of fixed points in noncompact mappings [12]. For more details on this subject, we refer the reader to the references [4, 7]. Recently, Krichen and O'regan [9] introduced the generalized notion of relative demicompact operators with respect to a given linear operator. In 2020, Benkhaled et al. [5] introduced the new class of order weakly demicompact (OWDC) operators. Furthermore, the authors established some properties of this class of operators. One of the goals of this paper is to study the duality problem for this new class of operators. First, we need to fix some notations and recall some definitions. All over this paper, X and Y will denote real Banach spaces, and E and F will denote real Banach lattices. The positive cone of E will be denoted by $E_+ = \{x \in E; 0 \leq x\}$. We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E [1]. Also, recall that a Banach lattice E is called to have the dual Schur property ($E \in (DSP)$), if each disjoint weak* null sequence $(f_n) \subset E'$ is norm null [10]. A norm bounded subset A of a Banach lattice E is said to be almost limited if every disjoint weak* null sequence $(f_n) \subset E'$ converges uniformly to zero [6]. An operator $T : X \to E$ is called almost limited if $T(B_X)$ is an almost limited set in E, equivalently, $\|T'(f_n)\| \to 0$ for every disjoint weak* null sequence (f_n) in E'. An operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^+$, the subset $T([0, x])$ is relatively weakly compact in F. An operator $T : E \longrightarrow E$ is said to be OWDC if, for every order bounded sequence (x_n) in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$ [5].

Recall that the class of OWDC operators is not a subclass of order weakly compact operators. For instance, the operator $-Id_{l^\infty}$ is OWDC. However, since the norm of l^∞ is not order continuous, $-Id_{l^\infty}$ is not order weakly compact operators. The duality problem of the latter class is studied in [3]. Furthermore, the class of OWDC operators does not satisfy the duality property [5]. That is, there is an OWDC operator T from E into E whose dual T' from E' into E' is not an OWDC operator, and conversely, there is an operator T from E into E that is not an OWDC operator while its dual T' from E' into E' is one. So, what are the conditions on the Banach lattices E for which every order weakly demicompact operator $T : E \longrightarrow E$ is order weakly compact? And what are the necessary and sufficient conditions on E that guarantee the direct and indirect duality property of the class of order weakly demicompact operators? The answers to these questions are given in the first part of this paper (see Theorem 2.1, Theorem 2.2 and Proposition 2.1). In the second part of this paper, we introduce a new class of operators, which we will call order weakly* demicompact (Definition 3.1). Then, we use our new class to generalize some results regarding the characterizations of the operators $T : E \to E$ whose dual T' is order weakly compact (resp, almost limited $T : E \to E$) operators (Proposition 3.1). Furthermore, we illustrate our analysis with some examples (see Examples, 3.1, 3.2, 3.3, 3.4). Also, the class of order weakly* demicompact operators does not satisfy the duality property; that is, there exist order weakly* demicompact operators whose adjoints are not order weakly* demicompact. Indeed, it is clear that $-Id_{l^1}$ is an order weakly* demicompact. But $(-Id_{l^1})' = -Id_{l^\infty}$ is not order weakly compact because the norm of l^∞ is not order continuous. The solution to this problem is given in Proposition 3.2. It should also be noted that an order weakly* demicompact operator is not necessarily almost limited. In fact, it is clear that $-Id_{l^1}$ is order weakly* demicompact. but $-Id_{l^1}$ is not almost limited. The solution to this problem is proved in Proposition 3.3. Lastly, we characterize Banach lattices on which all operators are order weakly* demicompact, and we prove that for a Banach lattice E, each operator T from E into E is order weakly* demicompact if and only if E' has the norm order continuous (see Theorem 3.1). This latest theorem allows us to characterize that a Banach lattice has the order continuous norm (see Corollary 3.1). In this work:

- OWDC(E, E) denotes the class of order weakly demicompact operators from a Banach lattice E into E,
- OW*DC(E, E) denotes the class of order weakly* demicompact operators from a Banach lattice E into E.

2 The duality problem for OWDC operators.

In this part, we will study two properties: The first one concerns direct duality, and the second one concerns indirect duality.

(i) The class OWDC(E, E) admits the property of direct duality (Dd), if

$$T \in \text{OWDC}(E, E) \Longrightarrow T' \in \text{OWDC}(E', E')$$

(ii) The class OWDC(E, E) admits the property of indirect duality (Di), if

$$T' \in \text{OWDC}(E', E') \Longrightarrow T \in \text{OWDC}(E, E)$$

2.1 Direct duality (Dd)

Theorem 2.1. Let E be Banach lattice. The following two propositions are equivalent:

(i) The class of OWDC(E, E) admits the property of direct duality (Dd).

(ii) E' has order continuous norm.

Proof. (1) $\Longrightarrow$ (2) By way of contradiction, let us assume that the norm of E' is not order continuous. To complete the proof, we need to look for an operator $T : E \to E$ that satisfies the following two pieces of information:

- T is order weakly demicompact,
- but its adjoint T' is not.

Then, it follows from Theorem 4.14 of Aliprantis and Burkinshaw [1] that there exists a disjoint sequence (u_n) of elements in E^+ with $\|u_n\| \leqslant 1$ for all n and there exists some $0 \leqslant \chi \in E'$ satisfying $\chi(u_n) = 1$ for all n.

Hence, by Lemma 2.6 of [2] there exists a positive disjoint sequence χ_n of E' with $\|\chi_n\| \leqslant 1$ such that

$$\chi_n(u_m) = \begin{cases} \chi(u_n) = 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

Note that $0 \leqslant \chi_n \leqslant \chi$ holds for all n.

Define two operators $S_1 : E \to \ell^1$ and $S_2 : \ell^1 \to E$ by

$$S_1(x) = (\chi_n(x))_{n=1}^{\infty} \ ; x \in E$$

$$S_2((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n u_n \ ; \ (\lambda_n) \in \ell^1$$

.

Since $\sum_{n=1}^{\infty} |\chi_n(x)| \leqslant \sum_{n=1}^{\infty} \chi_n(|x|) \leqslant \chi(|x|)$. It should be stressed that the operator $S_1(x)$ is well defined for every $x \in E$.

Note that, given $\sum_{n=1}^{\infty} \|\lambda_n u_n\| = \sum_{n=1}^{\infty} |\lambda_n| < \infty$, the series defining $S_2((\lambda_n))$ converges in norm for each $(\lambda_n) \in \ell^1$.

Next, we will consider the operator $T = S_2 \circ S_1 : E \to \ell^1 \to E$. From the definition of $T = S_2 \circ S_1$ we derive the following formula:

$$T(x) = \sum_{n=1}^{\infty} \chi_n(x) u_n \quad \text{for all } x \in E.$$

It is evident that the formula for its adjoint $T' : E' \to E'$ is

$$T'(\psi) = \sum_{n=1}^{\infty} \psi(u_n) \chi_n \quad \text{for all } \psi \in E'. \qquad (*)$$

Obviously, T is an order weakly compact operator because by, Corollary 3.43 of [1], the operator S_2 is a weakly compact, and so is $T = S_2 \circ S_1$. According to Proposition 2.1(1) of [5], T will be order weakly demicompact operator. But the adjoint operator T' is not order weakly demicompact. In fact, through $(*)$ we have $T'(\chi_n) = \chi_n$ for all n. Now, it can easily be seen that the order bounded sequence (χ_n) in E'_+ satisfying

$$\chi_n \xrightarrow{w} 0 \quad \text{and } \|\chi_n - T'\chi_n\| = 0 \quad \text{, but } \|\chi_n\| \nrightarrow 0 \text{ as } n \to \infty$$

It results from the definition 2.1 of [5] that the adjoint operator T' is not order weakly demicompact, as desired.

$(2) \implies (1)$ Follows from Theorem 2.1 of [5]. □

2.2 Indirect duality (Di)

Theorem 2.2. Let E be an order σ-complete Banach lattice. The following propositions are equivalent:

(i) The class of OWDC(E, E) admits the property of indirect duality (Di).

(ii) E has order continuous norm.

Proof. $(1) \implies (2)$ By way of contradiction, let us assume that the norm of E is not order continuous. We have to construct an operator $T : E \to E$ that satisfies the following two pieces of information:

- T' is order weakly demicompact,
- but its adjoint T is not.

Since the norm of E is not order continuous, they exist some $y \in E^+$ and a disjoint sequence $(x_n) \subset [0, y]$ which $\|x_n\| \nrightarrow 0$. It can be assumed that $\|x_n\| = 1$ for all n. So, by Lemma 3.4 of [3] we can deduce the existence of a disjoint sequence (f_n) of $(E')^+$ with $\|f_n\| \leqslant 1$ such that:

$$f_n(x_m) = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases} \qquad (**)$$

Firstly, let's take the positive operator $R : E \to \ell^\infty$ defined by

$$R(x) = (f_n(x))_{n=1}^\infty \quad \text{for all} \quad x \in E.$$

Secondly, since E is order σ-complete, it results from the proof of Theorem 117.3 of [14] that the operator $S : \ell^\infty \to E$ defined by

$$S\left((t_n)_{n=1}^\infty\right) = (\mathrm{o})\sum_{n=1}^{\infty} t_n x_n \quad \text{for all} \quad \left((t_n)_{n=1}^\infty\right) \in \ell^\infty.$$

is a lattice isomorphism, where (o) $\sum_{n=1}^\infty t_n x_n$ denotes the order limit of the sequence of the partial sums $\sum_{n=1}^m t_n x_n$ for each $(t_n)_{n=1}^\infty \in \ell^\infty$.

Next, we consider the operator $T = S \circ R : E \to E$. From the definition of $T = T = S \circ R$ we derive the following formula:

$$T(x) = (\mathrm{o})\sum_{n=1}^{\infty} f_n(x) x_n \quad \text{for all} \quad x \in E.$$

It is evident that the formula for its adjoint $T' = R' \circ S' : E' \to (\ell^\infty)' \to E'$ is

$$T'(\varphi) = (\mathrm{o})\sum_{n=1}^{\infty} \varphi(x_n) f_n \quad \text{for all} \quad \varphi \in E'.$$

Clearly T' is order weakly demicompact. Indeed, the operator T' is order weakly compact because, the norm of $(\ell^\infty)'$ is order continuous then, and according to Proposition 2.1(1) of [5], the operator T' will be order weakly demicompact. But T is not order weakly demicompact. In fact, according to the relation $(**)$ we have $T(x_n) = x_n$ for all n. Now, it is easy to notice that (x_n) is an order bounded sequence in E_+ verified

$$x_n \xrightarrow{w} 0 \quad \text{and} \quad \|x_n - Tx_n\| = 0 \quad \text{, but} \quad \|x_n\| \nrightarrow 0 \text{ as } n \to \infty$$

It results from Definition 2.1 of [5], that the operator T is not order weakly demicompact, as desired.

$(2) \Longrightarrow (1)$ Follows from Theorem 2.1 of [5]. □

Before defining a new class of operators, we will state the following result which gives a sufficient and necessary condition under which every OWDC operator is order weakly compact.

Proposition 2.1. For a Banach lattice E, the following propositions are equivalent:

(i) Every OWDC operator $T : E \longrightarrow E$ is order weakly compact.

(ii) E has order continuous norm.

Proof. $(1) \Rightarrow (2)$ It is very easy to notice that $-Id_E$ is OWDC operator. So according to our hypothesis $-Id_E$ will be order weakly compact which implies that E has order continuous norm.

$(2) \Rightarrow (1)$ Obvious. □

3 The OW*DC operators class.

We will now pass on the definition of our new operator class.

Definition 3.1. Let E be a Banach lattice. An operator $T : E \to E$ is said to be OW*DC if $\|x'_n\| \to 0$ as $n \to \infty$, for every order bounded sequence (x'_n) in E'_+ such that $x'_n \xrightarrow{w^*} 0$ and $\|x'_n - T'(x'_n)\| \to 0$ as $n \to \infty$.

For the rest, it should be remembered that a Banach lattice E has the positive Grothendieck property if weak* null sequences in E' with the positive terms are weak null.

Remark 3.1. Let E a Banach lattice.

(i) If $T \in \mathrm{OW^*DC}(E, E)$, then its adjoint $T' \in \mathrm{OWDC}(E', E')$.

(ii) If E has the positive Grothendieck property, then an operator $T : E \to E$ is OW*DC if and only if its adjoint T' is OWDC.

Proposition 3.1. Let E be a Banach lattice. The class of OW*DC operators $T : E \to E$ contains, among others, the following operators.

(i) The operators $T : E \to E$ whose adjoints T' is order weakly compact.

(ii) Almost limited operators $T : E \to E$.

(iii) Operators $T : E \to E$ for which $(Id_{E'} - T')^{-1}$ exists and is bounded.

(iv) The operators family $\left(\widetilde{T}_\beta\right)_{\beta \neq 1}$ from $\widetilde{E}$ into $\widetilde{E}$ defined via the matrix:

$$\begin{pmatrix} \beta Id_E & T \\ 0 & 0 \end{pmatrix}$$

where $T : E \to F$ and $\widetilde{E} = E \oplus F$.

For proof, we will follow the same approach of Benkhaled et al. in [5].

Proof. (1) Assume that T' is order weakly compact and let $(x'_n) \subset (E')_+$ be an order bounded sequence satisfying $x'_n \xrightarrow{w^*} 0$ and $\|x'_n - T'(x'_n)\| \to 0$. By Corollary 3.5.6 of [11], the operator T admits a factorization through a Banach lattice F

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "S"'] & & E \\ & F \arrow[ur, "j"'] & \end{tikzcd}$$

such that the norm on F' is order continuous and j is an interval preserving lattice homomorphism. Clearly, j' is positive, and hence the sequence $(j'(x'_n)) \subset (F')_+$ is order bounded and satisfying $j'(x'_n) \xrightarrow{w^*} 0$. So, by Theorem 3.1 of [13] $\|j'(x'_n)\| \to 0$, and hence $\|T'(x'_n)\| = \|S'j'(x'_n)\| \to 0$.

From the following inequality

$$\|x'_n\| \leq \|x'_n - T'(x'_n)\| + \|T'(x'_n)\|,$$

it follows that $\|x'_n\| \to 0$ as $n \to \infty$, as desired.

(2) Follows from (1) because if T almost limited operator then T' is order weakly compact.

(3) Let (x'_n) be an order bounded sequence of E'_+ such that $x'_n \xrightarrow{w^*} 0$ and $\|x'_n - T'(x'_n)\| \to 0$ as $n \to \infty$. Since $(Id_{E'} - T')^{-1}$ exists and is bounded, and from the following inequality

$$\|x'_n\| = \left\|(Id_{E'} - T')^{-1}(Id_{E'} - T')\,x'_n\right\| \leq \left\|(Id_{E'} - T')^{-1}\right\| \|(Id_{E'} - T')\,x'_n\|$$

for each n, we get $\|x'_n\| \to 0$ as $n \to \infty$.

(4) Let $\beta \neq 1$ and $\{\tilde{x}'_n = (x'_n, y'_n)\}$, $x'_n \in E'$, $y'_n \in F'$, be an order bounded sequence of $(\widetilde{E'})_+$ such that $\tilde{x}'_n \xrightarrow{w^*} 0$ and $\left\|\tilde{x}'_n - (\widetilde{T})'\tilde{x}'_n\right\|_{\widetilde{E'}} \to 0$ as $n \to \infty$. We have to show that $\|\tilde{x'_n}\|_{\widetilde{E'}} \to 0$ as $n \to \infty$. Since $\|\tilde{x'_n}\|_{\widetilde{E'}} = \|x'_n\|_{E'} + \|y'_n\|_{F'}$, it suffices to prove that $\|x'_n\|_{E'} \to 0$ and $\|y'_n\|_{F'} \to 0$. Accordingly, for every n, we have

$$\begin{aligned} \left\|\tilde{x}'_n - (\widetilde{T})'\tilde{x}'_n\right\|_{\widetilde{E'}} &= \left\|(x'_n, y'_n) - (\widetilde{T})'(x'_n, y'_n)\right\|_{\widetilde{E'}} = \|(x'_n, y'_n) - (\beta x'_n + T'y'_n, 0)\|_{\widetilde{E'}} \\ &= \|((1-\beta)x'_n - T'y'_n, y'_n)\|_{\widetilde{E'}} = \|(1-\beta)x'_n - T'y'_n\|_{E'} + \|y'_n\|_{F'} \end{aligned}$$

Since $\left\|\tilde{x'}_n - (\widetilde{T})'\tilde{x'}_n\right\|_{\widetilde{E'}} \longrightarrow 0$, then $\|(1-\beta)x'_n - T'y'_n\|_{E'} \longrightarrow 0$ and $\|y'_n\|_{F'} \longrightarrow 0$. On the other hand, from the following inequalities

$$\begin{aligned}|1-\beta|\,\|x'_n\|_{E'} &= \|(1-\beta)x'_n - T'y'_n + T'y'_n\|_{E'}\\ &\le \|(1-\beta)x'_n - T'y'_n\|_{E'} + \|T'y'_n\|_{E'}\end{aligned}$$

and since $\beta \neq 1$, we get $\|x'_n\|_{E'} \longrightarrow 0$. □

Example 3.1. The assumption $\beta \neq 1$ is essential. Indeed, if we consider E = l^1, F = l^∞ and T be an from l^1 into l^∞. Put $\widetilde{E} = l^1 \oplus l^\infty$, let $\widetilde{T}$ be an operator defined as follows:

$$\widetilde{T} = \begin{pmatrix} Id_{l^1} & T \\ 0 & 0 \end{pmatrix}$$

The operator $\widetilde{T}$ is not OW^*DC. In fact, take $\tilde{x'}_n = (e_n, 0)$ for every n, where e_n is the sequence with the nth entry equals to 1 and others are zero. So that $\left(\tilde{x'}_n\right)$ is an order bounded sequence of $(\widetilde{E})'_+$. As (e_n) is weakly null in $c_0, (e_n)$ is also weakly null in $(l^1)' = l^\infty$. In this way, we have $\left(\tilde{x'}_n\right)$ is a weakly null sequence in $(\widetilde{E})'$. Also, $\left\|\tilde{x'}_n - (\widetilde{T})'\tilde{x'}_n\right\|_{(\widetilde{E})'} = \left\|(e_n, 0) - (\widetilde{T})'(e_n, 0)\right\|_{(\widetilde{E})'} = \|(e_n, 0) - (e_n, 0)\|_{(\widetilde{E})'} = 0$. However, $\left\|\tilde{x'}_n\right\|_{(\widetilde{E})'} = \|e_n\|_\infty = 1 \nrightarrow 0$.

Remark 3.2. Note that the adjoint of an OW*DC operator is not necessarily order weakly compact. Indeed, let $Id_{l^1} : l^1 \longrightarrow l^1$ be the identity operator. It is easy to verify that $-Id_{l^1}$ is OW*DC. But since the norm of l^∞ is not order continuous, $(-Id_{l^1})' = -Id_{l^\infty}$ is not order weakly compact.

The next result provides a sufficient and necessary condition under which the adjoint of every OW*DC operator is order weakly compact.

Proposition 3.2. Let E be Banach lattice. The following conditions are equivalent:

(i) The adjoint of every OW*DC operator $T : E \to E$ is order weakly compact.

(ii) E' has order continuous norm.

Proof. (1) $\Rightarrow$ (2) It is easy to notice that the operator $-Id_E$ is OW*DC. So according to our hypothesis $(-Id_E)'$ will be order weakly compact which shows that the norm of E' is order continuous.

(2) $\Rightarrow$ (1) Obvious. □

Remark 3.3. Note that an OW*DC operator is not necessarily almost limited. Indeed, let $Id_{l^1} : l^1 \longrightarrow l^1$ be the identity operator. It is easy to see that $-Id_{l^1}$ is OW*DC, but $-Id_{l^1}$ is not almost limited.

The next result provides a sufficient and necessary condition under which every OW*DC operator is almost limited.

Proposition 3.3. Let E be a Banach lattice, then the following assertions are equivalent:

(i) Every OW*DC operator $T : E \longrightarrow E$ is almost limited.

(ii) E has the dual Schur property.

Proof. (1) $\Rightarrow$ (2) It is clear that the operator $-Id_E$ is OW*DC, so according to our hypothesis $-Id_E$ will be almost limited which implies that E has the dual Schur property.

(2) $\Rightarrow$ (1) Since E has the dual Schur property, $\|f_n\| \to 0$ for every disjoint weak* null sequence $(f_n) \subset E'$. Then $\|T'(f_n)\| \to 0$ for every disjoint weak* null sequence $(f_n) \subset E'$. i.e. $T : E \longrightarrow E$ is almost limited. □

Generally, the sum of two OW*DC operators is not necessarily OW*DC.

Example 3.2. Let S be an operator from l^1 into l^∞. We take $\widetilde{E} = l^1 \oplus l^\infty$ and let the operators T_1 and T_2 define as follows:

$$T_1 = \begin{pmatrix} 2Id_{l^1} & S \\ 0 & 0 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} -Id_{l^1} & S \\ 0 & 0 \end{pmatrix}$$

According to Proposition 3.1, the operators T_1 and T_2 are OW*DC, but the sum $T_1 + T_2$ defined by

$$T_1 + T_2 = \begin{pmatrix} Id_{l^1} & 2S \\ 0 & 0 \end{pmatrix}$$

is not (see Example 3.1).

If $T_1 : E \longrightarrow E$ is OW*DC and the adjoint of $T_2 : E \to E$ is order weakly compact, we have the following:

Proposition 3.4. Let $T_1 : E \longrightarrow E$ be an OW*DC operator. If the adjoint of an operator $T_2 : E \to E$ is order weakly compact, then the operator $T_1 + T_2$ is OW*DC.

Proof. Let (x'_n) be an order bounded sequence of E'_+ such that $x'_n \xrightarrow{w^*} 0$ and $\|x'_n - (T_1 + T_2)' x'_n\| = \|x'_n - (T'_1 + T'_2)x'_n\| \to 0$ as $n \to \infty$. The order weak compactness of T'_2, means that $\|T'_2 x'_n\| \to 0$ as $n \to \infty$. On the other hand, from the following inequalities

$$\|x'_n - T'_1 x'_n\| = \|x'_n - T'_1 x'_n - T'_2 x'_n + T'_2 x'_n\| \leq \|x'_n - (T'_2 + T'_1)x'_n\| + \|T'_2 x'_n\|$$

we get $\|x'_n - T'_1 x'_n\| \to 0$ as $n \to \infty$. Thus, the order weak* demicompactness of T_1 argues that $\|x'_n\| \to 0$ as $n \to \infty$ and therefore $T_1 + T_2$ is OW*DC. □

The next example illustrates that the class of operators OW*DC does not have the structure of a vector space, more precisely does not verify the external product.

Example 3.3. Let S be an operator from l^1 into l^∞. Take $E = l^1 \oplus l^\infty$ and let T be an operator defined by:

$$T = \begin{pmatrix} -Id_{l^1} & S \\ 0 & 0 \end{pmatrix}$$

we can conclude that

$$-T = \begin{pmatrix} Id_{l^1} & -S \\ 0 & 0 \end{pmatrix}$$

From Proposition 3.1 and Example 3.1, the operator T is OW*DC, but $-T$ is not.

The present example shows that the domination problem of class OW*DC operators is not generally verified.

Example 3.4. Let's take $\beta > 1$ and define two operators as follows:
$S, T : l^1 \to l^1$ with $S(x) = x$, and $T(x) = \beta x$. Thus, T is a OW*DC and $0 \leq S \leq T$. On the other hand, it is easy to see that S is not a OW*DC operator.

The next result provides a necessary and sufficient condition under which any operator is of OW*DC.

Theorem 3.1. For a Banach lattice E, the following propositions are equivalent:

(i) Every operator $T : E \longrightarrow E$ is OW*DC.

(ii) The identity operator of E is OW*DC.

(iii) The adjoint of every operator $T : E \to E$ is order weakly demicompact.

(iv) The identity operator of E' is order weakly demicompact.

(v) E' has order continuous norm.

(vi) The adjoint of every operator $T : E \to E$ is order weakly compact.

Proof. $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$ are obvious.

$(2) \Rightarrow (4)$ The identity operator of E is OW*DC, implies that the identity operator $Id : E' \longrightarrow E'$ is order weakly demicompact.

$(4) \Leftrightarrow (5)$ and $(5) \Rightarrow (6)$ and $(4) \Rightarrow (3)$ Follows from Theorem 2.1 of [5].

$(6) \Rightarrow (1)$ Result of the Proposition 3.1 (1).

□

From the above Theorem 3.1, we can deduce the following corollary which gives the characterizations of a Banach lattice whose dual has order continuous norm.

Corollary 3.1. For a Banach lattice E, the following assertions are equivalent:

(i) The norm of E' is order continuous.

(ii) $\|f_n\| \to 0$ as $n \to \infty$ for every order bounded sequence (f_n) in E'_+ satisfying $f_n \stackrel{w}{\to} 0$.

(iii) $\|f_n\| \to 0$ as $n \to \infty$ for every order bounded sequence (f_n) in E'_+ satisfying $f_n \stackrel{w^*}{\to} 0$.

(iv) $f_n(z_n) \to 0$ for every disjoint bounded sequence (z_n) in E_+ and for every order bounded sequence (f_n) in E'_+ satisfying $f_n \stackrel{w^*}{\to} 0$.

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Author information

A. EL ALOUI, MMA, FPL, Abdelmalek Essaadi University, Tetouan,, Morocco.
E-mail: dehmanabdennabi@gmail.com

K. BOURAS, MMA, FPL, Abdelmalek Essaadi University, Tetouan,, Morocco.
E-mail: bouraskhalid@hotmail.com

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