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ON MULTIPLICATIVE (GENERALIZED) SEMIDERIVATIONS OF RINGS

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Abstract The purpose of this paper is to characterize the concept of multiplicative (generalized) semiderivations. Also, the action of these derivations on appropriate subsets of semiprime rings is explored and as a consequence, some significant results regarding the commutativity of prime rings are proved.

1 Introduction

In ring theory, the concept of derivations originated long back in 1957, when Posner [14] proved some important results concerning the commutativity of rings. These results attracted various algebraists to study several additive mappings on prime and semiprime rings (cf. [1]-[4],[6] [8], [11]). It is pertinent to note here that during the study of derivations, the additive property is considered. But the behaviour of derivations can also be investigated in the absence of additive property, as there are many well known results which are untouched in case the map is non additive. In order to accomplish this, authors of present paper study and characterize the concept of multiplicative (generalized) semiderivations which provide a common generalization of derivations, multiplicative derivations and multiplicative semiderivations. Furthermore, the concept of multiplicative derivations was introduced by Daif [7] which was an inspiration of Martindale's work [12]. However, the concept of multiplicative derivations was illustrated by Goldmenn and Semrl [11] and afterwards Daif and Sayiad [9] extended these derivations to multiplicative generalized derivations. In 2013, Dhara and Ali have given a more general definition of a multiplicative (generalized) derivation and it is observed that this definition covers both the concepts (i.e., the multiplicative derivations and multiplicative generalized derivations). Note that every generalized derivation is a multiplicative (generalized) derivation but converse may not be true in general as observed in [10]. In 2020, Öölbaşı and Öğırtıcı defined multiplicative semiderivations and obtained some results regarding these derivations which motivated us to analyse these results by characterizing multiplicative (generalized) semiderivations.

Let R be an associative ring with the center $Z(R)$ and for any $a, b \in R$, the symbol $[a, b]$ (resp. $a \circ b$) represents $ab - ba$ (resp. $ab + ba$). Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and R is semiprime if $aRa = (0)$ implies that $a = 0$. Also, an additive subgroup L is said to be a Lie ideal of R if $[L, R] \subseteq L$ and a Lie ideal L is a 2-Lie ideal of R if $2ab \in L$, for all $a, b \in L$. A mapping $g : R \to R$ is said to be commuting on a subset $A \subseteq R$ if $[g(a), a] = 0$, for all $a \in A$. An additive mapping $D : R \to R$ is called a derivation if $D(ab) = D(a)b + aD(b)$, for all $a, b \in R$. Moreover, an additive mapping $F : R \to R$ is said to be a generalized derivation if $F(ab) = F(a)b + aD(b)$, for all $a, b \in R$, where $D : R \to R$ is a derivation of R. Following [5], an additive mapping $D : R \to R$ is called a semiderivation if there exists a function $\gamma : R \to R$ such that $D(ab) = D(a)\gamma(b) + aD(b) = D(a)b + \gamma(a)D(b)$ and $D(\gamma(a)) = \gamma(D(a))$, for all $a, b \in R$. In case γ is an identity map of R, then semiderivations are ordinary derivations. However, an additive mapping $F : R \to R$ is said to be a generalized semiderivation if $F(ab) = F(a)\gamma(b) + aD(b) = F(a)b + \gamma(a)D(b)$ and $F(\gamma(a)) = \gamma(F(a))$, for all $a, b \in R$, where $D : R \to R$ is a semiderivation associated with a function $\gamma : R \to R$. In this case, F is called a generalized semiderivation associated with a semiderivation D and a function γ of R.

By [7], a mapping $D : R \to R$ (not necessarily additive) is said to be a multiplicative derivation if $D(ab) = D(a)b + aD(b)$, for all $a, b \in R$. Furthermore, a mapping $F : R \to R$ is a multiplicative (generalized) derivation if there exists a map $d : R \to R$ (not necessarily additive) such that $F(ab) = F(a)b + ad(b)$, for all $a, b \in R$. According to [13], a mapping $D : R \to R$ is called a multiplicative semiderivation if there exists a function $\gamma : R \to R$ such that $D(ab) = D(a)\gamma(b) + aD(b) = D(a)b + \gamma(a)D(b)$ and $D(\gamma(a)) = \gamma(D(a))$, for all $a, b \in R$. In addition, a mapping $F : R \to R$ is said to be a multiplicative (generalized) semiderivation if there exists a map $d : R \to R$ (not necessarily additive) and a function $\gamma : R \to R$ such that $F(ab) = F(a)\gamma(b) + ad(b) = F(a)b + \gamma(b)d(b)$ and $F(\gamma(a)) = \gamma(F(a))$, for all $a, b \in R$. On account of this, F is called a multiplicative (generalized) semiderivation associated with a map d and a function γ of R. Therefore, one may observe that the concept of multiplicative (generalized) semiderivations includes the concept of multiplicative semiderivations and multiplicative (generalized) derivations.

In [13], Öölbaşı and Öğırtıcı proved that if R is semiprime ring and I is a nonzero ideal of R and D a multiplicative semiderivation of R associated with an onto endomorphism γ of R such that any one of the following conditions hold: (i) $D([a, b]) = 0$; (ii) $D(ab) \pm ab = 0$; (iii) $D(a)D(b) \pm ab = 0$; (iv) $D(ab) \pm ba = 0$, then $[D(a), a] = 0$, for all $a, b \in I$. Motivated by this, we consider similar situations for multiplicative (generalized) semiderivations on a nonzero ideal of a semiprime ring R. Additionally, we give an example of multiplicative (generalized) semiderivation and list all commutator identities, which will be frequently used in the sequel.

Example 1.1. Let R be a ring and take $R_1 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{bmatrix} : a, b, c \in R \right\}$. Obviously, R_1 is a ring under the usual addition and multiplication of matrices. Define maps $F, d, \gamma : R_1 \to R_1$ by $F\left(\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{bmatrix} \right) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & bc & 0 \end{bmatrix}$, $d\left(\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^2 & 0 \end{bmatrix}$ and $\gamma\left(\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{bmatrix} \right) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$. It is easy to verify that F with maps d and γ is a multiplicative (generalized) semiderivation of R_1, which is not a multiplicative semiderivation.

Lemma 1.2. *If R is a ring and $a, b, c \in R$, then the following statements hold:*
(i) $[a, bc] = b[a, c] + [a, b]c$; (ii) $[ab, c] = a[b, c] + [a, c]b$; (iii) $[a, b + c] = [a, b] + [a, c]$; (iv) $[a + b, c] = [a, c] + [b, c]$; (v) $[ab, a] = a[b, a]$; (vi) $[a, ab] = a[a, b]$; (vii) $[a, ba] = [a, b]a$; $(viii)$ $[ba, a] = [b, a]a$; (ix) $a \circ (bc) = (a \circ b)c - b[a, c] = b(a \circ c) + [a, b]c$; (x) $(ab) \circ c = a(b \circ c) - [a, c]b = (a \circ c)b + a[b, c]$.

Lemma 1.3. *[15, Lemma 2.1] Let R be a semiprime ring, I be a nonzero ideal of R and $a \in R$ such that $aia = 0$, for all $i \in I$, then $a = 0$.*

2 Multiplicative (Generalized) Semiderivations

This section deals with the behaviour of multiplicative (generalized) semiderivations on ideals of prime and semiprime rings. Furthermore, we extend some results of [13] to the case of multiplicative (generalized) semiderivations. We start with the following lemma which is useful to prove the main results of this paper.

Lemma 2.1. *If R is a semiprime ring and I is a nonzero ideal of R such that $[I, I] = (0)$, then R is commutative.*

Proof. Suppose that R is a semiprime ring and I is a nonzero ideal of R such that $[a, b] = 0$, for all $a, b \in I$. Replacing a by $[a, s]r$, we get

$$0 = [[a, s]r, b] = [a, s][r, b] + [[a, s], b]r, \text{ for all } a, b \in I \text{ and } r, s \in R.$$

By using the hypothesis, the above equation infers $[a, s][r, b] = 0$. Now, by replacing s by rs, we obtain $[a, r]s[r, b] = 0$, for all $a, b \in I$ and $r, s \in R$. This gives that $[a, r]R[a, r] = (0)$, for all $a \in I$ and $r \in R$. Since R is semiprime, therefore $[a, r] = 0$, for all $a \in I$ and $r \in R$. Now, putting $[s, r]as$ in place of a, we are left with $[s, r]a[s, r] = 0$, for all $a \in I$ and $r, s \in R$. Moreover, by Lemma 1.3, $[R, R] = (0)$ and hence, R is commutative. □

The upcoming result is an extension of [13, Theorem 2.4].

Theorem 2.2. *Let F be a multiplicative (generalized) semiderivation associated with a map d and an onto endomorphism γ of a 2-torsion free semiprime ring R. If L is a nonzero $2-Lie$ ideal of R such that $F(ab) \pm ab = 0$, for all $a, b \in L$, then $[F(a), a] = (0)$, for all $a \in L$.*

Proof. Assume that
$$F(ab) + ab = 0, \text{ for all } a, b \in L. \tag{2.1}$$
Putting $2bc$ in place of b in (2.1), we find
$$(F(2ab) + 2ab)c + 2\gamma(ab)d(c) = 0, \text{ for all } a, b, c \in L. \tag{2.2}$$
By using (2.1) in (2.2) and the fact that R is 2-torsion free, we obtain
$$\gamma(ab)d(c) = 0. \tag{2.3}$$
Now, replacing a by $[a, r]$, we get
$$\gamma(a)\gamma(r)\gamma(b)d(c) - \gamma(r)\gamma(a)\gamma(b)d(c) = 0, \text{ for all } a, b, c \in L \text{ and } r \in R.$$
On combining (2.3) with the above equation and using the fact that γ is onto, we have $\gamma(a)s\gamma(b)d(c) = 0$, for all $a, b, c \in L$ and $s \in R$. By taking $s = d(c)s$, the last relation infers that $\gamma(a)d(c)R\gamma(b)d(c) = (0)$, for all $a, b, c \in L$. So $\gamma(b)d(c)R\gamma(b)d(c) = (0)$, for all $b, c \in L$. Since R is semiprime, therefore
$$\gamma(b)d(c) = 0, \text{ for all } b, c \in L. \tag{2.4}$$
By using (2.4), equation (2.1) infers that $(F(a) + a)b = 0$, for all $a, b \in L$. Further, putting $[b, r]$ in place of b, we have $(F(a) + a)rb = 0$, for all $a, b \in L$ and $r \in R$. By taking $b = [F(a), a]$, this concludes that
$$(F(a) + a)r[F(a), a] = 0. \tag{2.5}$$
By considering $r = ar$, equation (2.5) leads to
$$(F(a) + a)ar[F(a), a] = 0, \text{ for all } a \in L \text{ and } r \in R. \tag{2.6}$$
Also, (2.5) infers that $a(F(a) + a)r[F(a), a] = 0$ and subtracting this from (2.6), we obtain $[F(a), a]R[F(a), a] = (0)$, for all $a \in L$. The semiprimeness of R infers that $[F(a), a] = 0$, for all $a \in L$.
By using the same technique, $F(ab) - ab = 0$ gives that $[F(a), a] = (0)$, for all $a \in L$. □

Now, we list two direct consequences of the above theorem as follows:

Corollary 2.3. *Let D be a multiplicative semiderivation associated with an onto endomorphism γ of a 2-torsion free semiprime ring R. If L is a nonzero $2-Lie$ ideal of R such that $D(ab) \pm ab = 0$, for all $a, b \in L$, then $[D(a), a] = (0)$, for all $a \in L$.*

Corollary 2.4. *Let F be a multiplicative (generalized) derivation associated with a map d of a 2-torsion free semiprime ring R. If L is a nonzero $2-Lie$ ideal of R such that $F(ab) \pm ab = 0$, for all $a, b \in L$, then $[F(a), a] = (0)$, for all $a \in L$.*

Proposition 2.5. *Let R be a prime ring and F a multiplicative (generalized) semiderivation associated with a map d and an automorphism γ of R. If I is a nonzero ideal of R and $0 \neq x \in R$ such that $x(F(a)F(b) \pm ab) = 0$, for all $a, b \in I$, then either R is commutative or $d = 0$.*

Proof. Let us assume that

$$x(F(a)F(b)+ab)=0, \text{ for all } a,b\in I. \tag{2.7}$$

Replacing b by br in equation (2.7), we have

$$x(F(a)F(b)+ab)r+xF(a)\gamma(b)d(r)=0, \text{ for all } a,b\in I \text{ and } r\in R.$$

Further, by using (2.7), the last relation leads to

$$xF(a)\gamma(b)d(r)=0, \text{ for all } a,b\in I \text{ and } r\in R.$$

Since R is prime and γ is an automorphism of R, therefore the above equation infers that either $xF(I)=(0)$ or $d=0$. In view of the former case, equation (2.7) concludes that $xab=0$, for all $a,b\in I$. This implies that

$$xrab=0, \text{ for all } a,b\in I \text{ and } r\in R. \tag{2.8}$$

The above equation also gives $xrba=0$. On subtracting the previous equation from (2.8), we find $xR[a,b]=(0)$, for all $a,b\in I$. The primeness of R gives that $[I,I]=(0)$, as $x\neq 0$. Moreover, by Lemma 2.1, R is commutative. Similarly, from $x(F(a)F(b)-ab)=0$, we get that either R is commutative or $d=0$. □

As a consequence of the above result, we have

Corollary 2.6. *Let R be a prime ring and F a multiplicative (generalized) derivation associated with a map d of R. If I is a nonzero ideal of R and $0\neq x\in R$ such that $x(F(a)F(b)\pm ab)=0$, for all $a,b\in I$, then either R is commutative or $d=0$.*

In the upcoming results, R represents a semiprime ring and I is a nonzero ideal of R unless stated otherwise. Also, Theorem 2.7 generalizes [13, Theorem 2.5].

Theorem 2.7. *Let F be a multiplicative (generalized) semiderivation associated with a map d and an endomorphism γ of R. If any one of the following statements hold:*
(i) $F(ab)+ba=0$;
(ii) $F(ab)-ba=0$;
for all $a,b\in I$, then R is commutative.

Proof. (i) Assume that $F(ab)+ba=0$, for all $a,b\in I$. Now, putting ai in place of a, we obtain that

$$F(ai)\gamma(b)+bai+aid(b)=0, \text{ for all } a,b,i\in I. \tag{2.9}$$

From hypothesis, $F(ai)=-ia$ and using this in equation (2.9), we find

$$-ia\gamma(b)+bai+aid(b)=0, \text{ for all } a,b,i\in I. \tag{2.10}$$

Further, replacing a by ia, we get

$$i(-ia\gamma(b)+aid(b))+biai=0. \tag{2.11}$$

From (2.10), $-ia\gamma(b)+aid(b)=-bai$, so (2.11) implies that

$$0=-ibai+biai=[b,i]ai, \tag{2.12}$$

for all $a,b,i\in I$. By taking $a=ab$, we have

$$[b,i]abi=0. \tag{2.13}$$

Also, $[b,i]ai=0$ gives that $[b,i]aib=0$ and subtracting this from equation (2.13), we find $[b,i]I[b,i]=(0)$, for all $b,i\in I$. By Lemma 1.3, $[I,I]=(0)$ and hence Lemma 2.1 concludes that R is commutative. Similarly, (ii) can be proved. □

A direct consequence of the above result is

Corollary 2.8. *If D is a multiplicative semiderivation associated with an endomorphism γ of R such that $D(ab) \pm ba = 0$, for all $a, b \in I$, then R is commutative.*

Theorem 2.9. *Let F be a generalized semiderivation associated with a semiderivation d and an endomorphism γ of R such that $F([a, b]) = 0$, for all $a, b \in I$. Then d maps R into $Z(R)$.*

Proof. By hypothesis, we have $F([a, b]) = 0$, for all $a, b \in I$. Now, taking br in place of b, we get

$$\begin{aligned} 0 &= F([a, b]r + b[a, r]) \\ &= F([a, b])\gamma(r) + [a, b]d(r) + F(b)\gamma([b, r]) + bd([a, r]), \end{aligned}$$

for all $a, b \in I$ and $r \in R$. By using the hypothesis, the above equation leads to $[a, b]d(r) + F(b)\gamma([b, r]) + bd([a, r]) = 0$. Further, by replacing b by $[b, i]$ and then again using the hypothesis, we have

$$[a, [b, i]]d(r) + [b, i]d([a, r]) = 0, \text{ for all } a, b, i \in I \text{ and } r \in R. \tag{2.14}$$

Putting bi in place of i in (2.14), we find

$$[a, b][b, i]d(r) = 0, \text{ for all } a, b, i \in I \text{ and } r \in R. \tag{2.15}$$

Replacing a by $id(r)a$, equation (2.15) infers that

$$0 = i[d(r)a, b][b, i]d(r) + [i, b]d(r)a[b, i]d(r).$$

By using (2.15), the above equation gives that

$$0 = [i, b]d(r)a[b, i]d(r) = -[b, i]d(r)a[b, i]d(r).$$

Therefore, we have

$$[b, i]d(r)I[b, i]d(r) = (0), \text{ for all } b, i \in I \text{ and } r \in R.$$

Moreover, by Lemma 1.3, $[b, i]d(r) = 0$ and by taking b by $d(r)b$, we have

$$[d(r), i]bd(r) = 0, \text{ for all } b, i \in I \text{ and } r \in R. \tag{2.16}$$

By taking bi in place of b,

$$[d(r), i]bid(r) = 0. \tag{2.17}$$

On multiplying (2.16) from the right hand side by i, we get $[d(r), i]bd(r)i = 0$. Subtracting (2.17) from the previous equation, we conclude that

$$[d(r), i]I[d(r), i] = (0), \text{ for all } i \in I \text{ and } r \in R.$$

Thus, by Lemma 1.3, $[d(r), i] = 0$ and by replacing i by $si[d(r), s]$, we obtain

$$\begin{aligned} 0 &= [d(r), si[d(r), s]] \\ &= s[d(r), i[d(r), s]] + [d(r), s]i[d(r), s], \text{ for all } i \in I \text{ and } r, s \in R. \end{aligned}$$

As $[d(r), I] = (0)$, so the preceding relation gives that $[d(r), s]I[d(r), s] = (0)$, for all $r, s \in R$. Therefore, by Lemma 1.3, $[d(R), R] = (0)$. Hence d maps R into $Z(R)$. □

Corollary 2.10. *Let F be a generalized semiderivation associated with a semiderivation d and an endomorphism γ of a prime ring R such that $F([a, b]) = 0$, for all $a, b \in I$. Then either R is commutative or $d = 0$.*

Proof. In view of Theorem 2.9, we have $[b, i]d(r) = 0$, for all $b, i \in I$ and $r \in R$. On replacing b by ab, this infers that $[a, i]bd(r) = 0$, for all $a, b, i \in I$ and $r \in R$. Therefore, $[a, i]Id(r) = (0)$, for all $a, i \in I$ and $r \in R$. As R is prime and I is a nonzero ideal of R, so the preceding equation infers that, either $[I, I] = (0)$ or $d = 0$. By Lemma 2.1, the former case gives that R is commutative. □

In view of Theorem 2.9, we immediately have the next corollary.

Corollary 2.11. *Let d be a semiderivation associated with an endomorphism γ of R such that $d([a,b]) = 0$, for all $a, b \in I$. Then d maps R into $Z(R)$.*

Now, we introduce a more general theorem which is an extension of [13, Theorem 2.8].

Theorem 2.12. *Let F be a multiplicative (generalized) semiderivation associated with a map d and an endomorphism γ of R such that $F(ab) + F(a)F(b) = 0$, for all $a, b \in I$. Then F is commuting on I.*

Proof. By hypothesis, we have

$$F(ab) + F(a)F(b) = 0, \text{ for all } a, b \in I. \tag{2.18}$$

Replacing b by bi in (2.18), we get $0 = (F(ab) + F(a)F(b))\gamma(i) + abd(i) + F(a)bd(i)$, for all $a, b, i \in I$. By hypothesis, the previous equation gives that

$$abd(i) + F(a)bd(i) = 0, \text{ for all } a, b, i \in I. \tag{2.19}$$

Further, putting ac in place a, we find $a(cb)d(i) + F(a)(cb)d(i) + \gamma(a)d(c)bd(i) = 0$, for all $a, b, c, i \in I$. Now, using (2.19), the previous equation gives

$$\gamma(a)d(c)bd(i) = 0, \text{ for all } a, b, c, i \in I.$$

By taking $b = b\gamma(a)$ and $c = i$, we conclude that $\gamma(a)d(i)I\gamma(a)d(i) = (0)$, for all $a, i \in I$. In view of Lemma 1.3,

$$\gamma(a)d(i) = 0, \text{ for all } a, i \in I. \tag{2.20}$$

By using (2.20), equation(2.18) leads to $F(a)(b + F(b)) = 0$ and by putting ba in place of a, we have

$$F(b)a(b + F(b)) = 0, \text{ for all } a, b \in I. \tag{2.21}$$

Further, by taking ba in place of a, we have

$$F(b)ba(b + F(b)) = 0$$

and by multiplying (2.21) from the left hand side by b, then subtracting the resulting equation from the above equation, we have $[F(b), b]a(b + F(b)) = 0$, for all $a, b \in I$. This gives

$$[F(b), b]a(b + F(b))b = 0 \tag{2.22}$$

and

$$[F(b), b]ab(b + F(b)) = 0.$$

Now, subtracting the above equation from (2.22), we conclude that

$$[F(b), b]I[F(b), b] = 0, \text{ for all } b \in I.$$

Hence, by Lemma 1.3, $[F(b), b] = 0$, for all $b \in I$ i.e., F is commuting on I. □

The next two results are the direct consequences of the above theorem.

Corollary 2.13. *Let D be a multiplicative semiderivation associated with an endomorphism γ of R such that $D(ab) + D(a)D(b) = 0$, for all $a, b \in I$. Then D is commuting on I.*

Corollary 2.14. *Let F be a multiplicative (generalized) derivation associated with a map d of R such that $F(ab) + F(a)F(b) = 0$, for all $a, b \in I$. Then F is commuting on I.*

Theorem 2.15. *Let F, G be two multiplicative (generalized) semiderivations associated with (d, γ), (g, γ) of a prime ring R respectively, where d, g be two maps and γ be an endomorphism of R. If any one of the following statements hold:*
(i) $F(ab) + G(a)G(b) + [a, b] = 0$;
(ii) $F(ab) + G(a)G(b) + a \circ b = 0$;
for all $a, b \in I$, then either $[G(a), a] = 0$, for all $a \in I$ or $g = 0$.

Proof. (i) Let us assume that

$$F(ab)+G(a)G(b)+[a,b]=0,\ \text{for all } a,b\in I. \tag{2.23}$$

Replacing b by br, we get

$$(F(ab)+G(a)G(b))\gamma(r)+abd(r)+G(a)bg(r)+[a,b]r+b[a,r]=0 \tag{2.24}$$

for all $a,b\in I$ and $r\in R$. From (2.23), $F(ab)+G(a)G(b)=-[a,b]$ and using this in (2.24), we conclude that

$$-[a,b]\gamma(r)+abd(r)+G(a)bg(r)+[a,b]r+b[a,r]=0. \tag{2.25}$$

Further, putting ab in place of b, we find that

$$a(-[a,b]\gamma(r)+abd(r)+[a,b]r+b[a,r])+G(a)abg(r)=0,\ \text{for all } a,b\in I \text{ and } r\in R.$$

In view of (2.25), $-[a,b]\gamma(r)+abd(r)+[a,b]r+b[a,r]=-G(a)bg(r)$ and combining this with the last equation, we have

$$[G(a),a]Ig(r)=(0),\ \text{for all } a,b\in I \text{ and } r\in R.$$

Since R is prime and I is a nonzero ideal of R, therefore either, $[G(a),a]=0$, for all $a\in I$ or $g=0$.
By the same technique with suitable variations, we can prove that the same conclusion holds for (ii). □

Theorem 2.16. *Let F, G be two multiplicative (generalized) semiderivations associated with (d,γ), (g,γ) respectively, where d,g be two maps and γ be an automorphism of R such that $F(a)b-bG(a)=0$, for all $a,b\in I$. Then d is commuting on R.*

Proof. By hypothesis, we have

$$F(a)b-bG(a)=0,\ \text{for all } a,b\in I. \tag{2.26}$$

Replacing b by rb, we find

$$F(a)rb-rbG(a)=0,\ \text{for all } a,b\in I \text{ and } r\in R. \tag{2.27}$$

From (2.26), we get $bG(a)=F(a)b$ and combining this with (2.27), $[F(a),r]b=0$. This infers that

$$[F(a),r]I[F(a),r]=(0),\ \text{for all } a\in I \text{ and } r\in R.$$

Further, by Lemma 1.3, $[F(a),r]=0$, for all $a\in I$ and $r\in R$. Now, taking as in place of a, $0=[F(a)s,r]+[\gamma(a)d(s),r]$, for all $a\in I$ and $r,s\in R$. By using $[F(a),r]=0$, the last relation gives

$$F(a)[s,r]+[\gamma(a)d(s),r]=0. \tag{2.28}$$

Putting rs in place r, (2.28) concludes that $r[\gamma(a)d(s),s]=0$. So

$$[\gamma(a)d(s),s]R[\gamma(a)d(s),s]=(0),\ \text{for all } a\in I \text{ and } s\in R.$$

Since R is a semiprime ring, therefore

$$\begin{aligned} 0 &= [\gamma(a)d(s),s] \\ &= \gamma(a)[d(s),s]+[\gamma(a),s]d(s). \end{aligned}$$

Replacing a by ra, we have $[\gamma(r),s]\gamma(a)d(s)=0$, for all $a\in I$ and $r,s\in R$. As γ is an automorphism, so

$$[d(s),s]\gamma(a)d(s)=0,\ \text{for all } a\in I \text{ and } s\in R. \tag{2.29}$$

Replacing a by ar, $[d(s),s]\gamma(a)\gamma(r)d(s)=0$, for all $a\in I$ and $r,s\in R$. Since γ is an automorphism, therefore

$$[d(s),s]\gamma(a)sd(s)=0. \tag{2.30}$$

On multiplying (2.29) from right hand side by s, we have $[d(s), s]\gamma(a)d(s)s = 0$, for all $a \in I$ and $s \in R$. By subtracting (2.30) from the previous equation, we conclude that

$$[d(s), s]\gamma(a)[d(s), s] = 0, \text{ for all } a \in I \text{ and } s \in R.$$

Since γ is an automorphism and I is a nonzero ideal, so $\gamma(I)$ is a nonzero ideal and by Lemma 1.3, the last relation implies that $[d(s), s] = 0$, for all $s \in R$. Hence, d is commuting on R. □

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