Palestine Journal of Mathematics

Vol. 12(3)(2023) , 26–33 © Palestine Polytechnic University-PPU 2023

The numerical solution of hyperbolic differential equations using Wavelet lifting scheme

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Communicated by Suheil Khoury

MSC 2020 Classifications: 65T60, 97N40, 35L04.

Keywords and phrases: Orthogonal and biorthogonal wavelets; Lifting scheme; Hyperbolic differential equations.

Authors are thankful to KLECET, Chikodi and KLE's G. I. Bagewadi College, Nipani for their support to research.

Abstract Recently Wavelet theory is an advanced tool in science and engineering research; mainly wavelets are applicable in fast algorithms for easy implementation. In this paper, we present an innovative wavelet technique that is lifting scheme for the numerical solutions of hyperbolic differential equations come across in mathematical physics. Here we used orthogonal and biorthogonal wavelets to expose the efficiency and effectiveness of the proposed scheme. The proposed scheme speeds up the convergence in less computational time, which is illustrated through the numerical examples.

1 Introduction

Simulation of physical phenomena has important effects on applied mathematics, physics and engineering field. Many such physical phenomena are modeled in terms of partial differential equations (PDEs). The importance of obtaining the analytical or numerical solutions of PDEs in physics and mathematics is still a significant problem that needs new techniques to discover. The hyperbolic PDE is derived from the relativistic energy formula is one of the most important mathematical models in quantum field theory. It is used to describe dispersive wave phenomena in relativistic physics. It has attracted much attention in studying solitons and condensed matter physics, in investigating the interaction of solitons in collisionless plasma and in examining the linear and nonlinear wave equations. The study of numerical solutions of such equations has been investigated considerably since from the last few years.
Several techniques including finite difference, collocation, finite element methods have been used to handle such equations. To find solution to the PDEs using these techniques, it is necessary to employ discretization methods to reduce the PDEs to systems of algebraic equations. For large systems, most of these methods have some the drawbacks like divergent results and huge computational work. Multigrid methods [1-5] are well known error minimizing techniques for solving boundary-value problems. Since from 35 years, multi-grid methods have earned a reputation as an efficient and versatile approach for solving PDEs. However, when problems with discontinuous or highly oscillatory coefficients, multigrid procedure converge slowly with larger computational time or may break down. To overcome this difficulty, wavelet plays a very important role.

Wavelet analyses have significant applications in signal and image processing during the 1980s. The smooth orthonormal basis obtained by the translation and dilation of a single function in a hierarchical fashion proved very useful to develop compression algorithms for signals and images upto a chosen threshold of relevant amplitudes. While the existence of the Haar type of wavelet functions has been known for a long time, the study of wavelets acquired the present growth after the mathematical analysis of wavelets [6-8].

The multiresolution approximation of Mallat [9] and Meyer [10] led to Daubechies [11] orthogonal family of wavelets. Recently wavelets have been applied in a wide range of engineering disciplines; particularly, wavelets are successfully used in signal analysis, time-frequency analysis and fast algorithms for easy implementation. Wavelet based numerical methods are used for solving the system of equations with better convergence in less computational cost. Some of the earlier works on wavelet based methods can be found in Dahmen et al. [12]. A collection of the discrete wavelet transforms (DWT) and the wavelet based full-approximation scheme (WFAS) were introduced recently in [13-16]. Shiralashetti et al. [17] had proposed the new wavelet based full-approximation scheme for the numerical solution of non-linear elliptic partial differential equations.

Similarly, the biorthogonal wavelet method is applied for the solution of elliptic partial differential equations [18]. The method can be either used as an iterative solver or as a preconditioning technique, offering in many cases a better performance than some of the most innovative and existing FAS algorithms. Due to the efficiency and potentiality of WFAS, researches further have been carried out for its enrichment. In order to realize this task, the new work build i.e. lifting scheme using orthogonal / biorthogonal wavelet filters [19]. Wavelet based lifting technique is introduced by Sweldens [20], which permits some improvements on the properties of existing wavelet transforms. The technique has many numerical benefits such as a reduced number of operations, which are fundamental in the context of the iterative solvers. In addition to this, Shiralashetti et al. [21, 22] applied the lifting scheme for the numerical solution of Elasto-hydrodynamic lubrication problems and non-linear partial differential equations. Extension to this, the present paper illustrates the lifting technique for the solution of hyperbolic differential equations.

The paper is divided as follows: Preliminaries of wavelets are given in section 2. Lifting technique is presented in section 3. Section 4 highlights the method of solution. Numerical solutions of the test problems are presented in section 5. Finally, concluding remarks of the paper are discussed in section 6.

2 Preliminaries of wavelets

Orthogonal and biorthogonal wavelet filters based on the orthogonality and smoothness conditions that must be satisfied by scaling and wavelet functions. These conditions impose restrictions on the value of filter coefficients through dilation equations. Fortunately we have two distinct functions called scaling functions and wavelet functions with coefficients $\{h_k\}$ and $\{g_k\}$ that define the refinement relation. These coefficients decide shape of the scaling and wavelet functions and act as signal filters, in which the application where we can use the particular wavelet.

2.1 Wavelet filters

The most important classes of filters are those of finite impulse response (FIR). The main characteristics of these filters are the convenient time-localization properties. These filters are initiated from wavelets with compact support and are such that,

$$h_n = 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad n > L$$

in which L is the length of the filter.

The minimum requirements for these compact FIR filters are:

(i) The length of the scaling filter h_n must be even.

(ii) $\sum_n h_n = \sqrt{2}$

(iii) $\sum_n (h_n \cdot h_{n-2k}) = \delta(k)$,

in which $\delta(k)$ is the Kronecker delta, such that, it is equal to 1 for $k = 0$ or 0 for $k = 1$.

2.2 Haar wavelet filter coefficients

We know that low pass filter coefficients $h = [h_0,\ h_1]^T = \left[\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right]^T$ and high pass filter coefficients $g = [g_0,\ g_1]^T = \left[\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right]^T$ play an important role in the decomposition. Thus it is natural to wonder that, it is possible to model the decomposition in terms of linear transformations. Moreover, since the digital signals and images are composed of discrete data, then we need a discrete analog of the decomposition algorithm so that we can process the signal and image data.

2.3 Daubechies wavelet filter coefficients

Daubechies introduced scaling functions that satisfy the above requirements and distinguished by having the shortest possible support. The scaling function ϕ has support$[0,\ L-1]$, while the corresponding wavelet function ψ has support in the interval $[1-L/2,\ L/2]$. We have filter coefficients [23], $h = [h_0,\ h_1,\ h_2,\ h_3]^T = \left[\frac{1+\sqrt{3}}{4\sqrt{2}},\ \frac{3+\sqrt{3}}{4\sqrt{2}},\ \frac{3-\sqrt{3}}{4\sqrt{2}},\ \frac{1-\sqrt{3}}{4\sqrt{2}}\right]^T$ are low pass filter coefficients and $g = [g_0,\ g_1,\ g_2,\ g_3]^T = \left[\frac{1-\sqrt{3}}{4\sqrt{2}},\ -\frac{3-\sqrt{3}}{4\sqrt{2}},\ \frac{3+\sqrt{3}}{4\sqrt{2}},\ -\frac{1+\sqrt{3}}{4\sqrt{2}}\right]^T$ are the high pass filter coefficients.

2.4 Biorthogonal (CDF (2, 2)) wavelet filter coefficients

In many filtering applications, we need filter coefficients having symmetry to get a better accuracy. None of the orthogonal wavelet systems except Haar are having symmetrical coefficients. But Haar is too insufficient for many applications in science and engineering. Biorthogonal wavelet system can be constructed to have this feature. This is the motivation for designing such wavelet systems. The following are the biorthogonal (CDF (2, 2)) wavelet filter coefficients [24, 25] are,
low pass filters: $h = [h_0,\ h_1,\ h_2] = \left[\frac{1}{2\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{1}{2\sqrt{2}}\right]$ and

$$\tilde{h} = [\tilde{h}_0,\ \tilde{h}_1,\ \tilde{h}_2,\ \tilde{h}_3,\ \tilde{h}_4] = \left[\frac{-\sqrt{2}}{8},\ \frac{\sqrt{2}}{4},\ \frac{3\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{-\sqrt{2}}{8}\right].$$

Similarly, high pass filters: $g_k = (-1)^k \tilde{h}_{4-k}$ and $\tilde{g}_k = (-1)^{k+1} h_{2-k}$.

3 Lifting Technique

The wavelets transform customs averages and differences, which brings us to the definition of the lifting procedure. The operations, average and difference, can be observed as distinct cases of more general operations. The concluding viewpoint is based on the fact that the pair-wise average values containing the overall structure of the data, but with only half the number of data. The lifting procedure has three steps i.e. split, prediction and update. The detailed procedure about this is explained in [21].

4 Method of solution

Consider the hyperbolic partial differential equation of the form,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.1}$$

subjected to boundary conditions.
By applying the finite difference scheme to Eq. (1), which gives the system of algebraic equations,

$$Au = B \tag{4.2}$$

By solving Eq. (2), we get approximate solution u. Approximate solution containing some error, therefore required solution equals to sum of approximate solution and error. There are many methods to minimize such error to get the accurate solution. Some of them are multigrid, wavelet multigrid and bi-orthogonal wavelet multigrid etc. Now, we are using the advanced technique based on different wavelets called as lifting scheme. Recently, lifting schemes are very useful in the signal analysis and image processing in the science and engineering field. But nowadays, extends to approximations in the numerical analysis. We discussed the algorithm [19] of the lifting scheme as same as explained by Shiralashetti et al. [21].

5 Numerical examples

Here, we present some of the test problems to demonstrate the validity and applicability of the Haar wavelet lifting scheme (HWLS), Daubechies wavelet lifting scheme (DWLS) and Biorthogonal wavelet lifting scheme (BWLS).

Example-1. First, we consider the linear ODE,

$$x\frac{d^2u}{dx^2}+\frac{du}{dx}+xu=0 \tag{5.1}$$

subject to boundary conditions $u(0)=1,\;\; u(1)=0.7652$. The exact solution of which is known as the Bessel function of the zero order [14],

$$u(x)=J_0(x)=\sum_{q=0}^{N}\frac{(-1)^q}{(q!)^2}\left(\frac{x}{2}\right)^{2q}.$$

The wavelet based numerical solutions of Eqn. (3) are obtained as per the procedure explained in section 4 and are presented in comparison with exact solution in figure 1 and the error analysis and CPU time versus grid points is given in table 1.

The maximum error, $E_{\max}=\max|u_e-u_a|$, where $u_e\&u_a$ are exact and approximate solutions respectively.

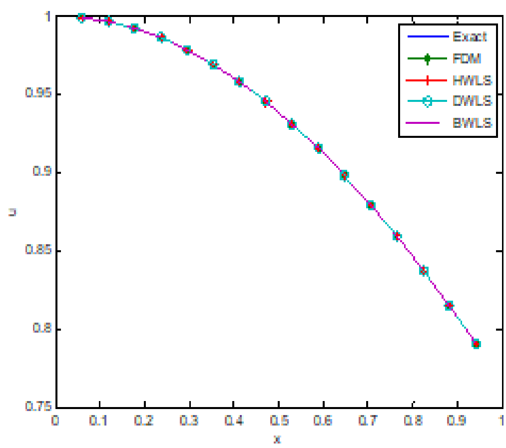

Fig. 1. Comparison of numerical solutions with exact solution for $N=16$ of the Example-1.

Table 1. The maximum error with CPU time (in seconds) versus grid points of the Example-1.

N	Method	$E_{\max}$	Setup time	Running time	Total time
8	FDM	1.9646e-04	6.9296e+00	3.0721e-01	7.2368e+00
	HWLS	1.9646e-04	3.9889e-04	1.0013e-03	1.4002e-03
	DWLS	1.9646e-04	3.4894e-04	5.6040e-03	5.9529e-03
	BWLS	1.9646e-04	2.8634e-04	1.1491e-03	1.4355e-03
16	FDM	5.9936e-05	1.6227e+01	5.2815e-02	1.6280e+01
	HWLS	5.9936e-05	3.8658e-04	9.6200e-04	1.3486e-03
	DWLS	5.9936e-05	3.3150e-04	4.7118e-03	5.0433e-03
	BWLS	5.9936e-05	2.8053e-04	2.7160e-03	2.9965e-03
32	FDM	1.8362e-05	4.1082e+00	6.6163e-02	4.1743e+00
	HWLS	1.8362e-05	4.1394e-04	9.7636e-04	1.3903e-03
	DWLS	1.8362e-05	3.2979e-04	4.4713e-03	4.8011e-03
	BWLS	1.8362e-05	2.7608e-04	2.5206e-03	2.7967e-03
64	FDM	6.6190e-06	5.8301e+00	9.2253e-02	5.9223e+00
	HWLS	6.6190e-06	4.0129e-04	9.8560e-04	1.3869e-03
	DWLS	6.6190e-06	3.3800e-04	4.6037e-03	4.9417e-03
	BWLS	6.6190e-06	2.8155e-04	2.5285e-03	2.8100e-03
128	FDM	3.5117e-06	5.4662e+00	1.4743e-01	5.6136e+00
	HWLS	3.5117e-06	4.0231e-04	9.8389e-04	1.3862e-03
	DWLS	3.5117e-06	3.3218e-04	4.5842e-03	4.9163e-03
	BWLS	3.5117e-06	2.8531e-04	2.5254e-03	2.8107e-03

Example-2. Now, we consider the nonlinear ODE,

$$\frac{\partial^2 u}{\partial x^2} - \left(\frac{8}{1+2x}\right) u^2 = 0 \tag{5.2}$$

with respect to boundary conditions $u(0) = 1,\ u(1) = 1/3$. The exact solution of the problem is $u(x) = \left(\frac{1}{1+2x}\right)$. As in the previous example, we obtained the numerical results of Eqn. (4) and are presented in figure 2. The error analysis and CPU time versus grid points is given in table 2.

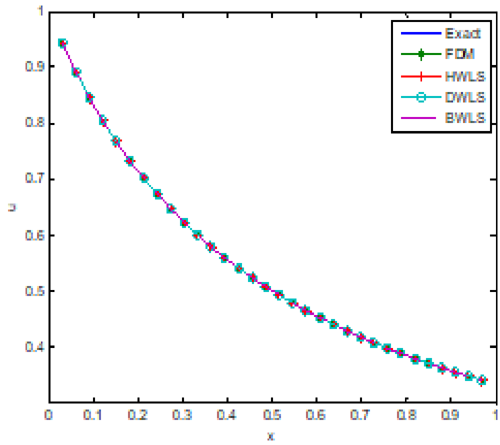

Fig. 2. Comparison of numerical solutions with exact solution for $N = 32$ of the Example-2.

Table 2. The maximum error with CPU time (in seconds) versus grid points of the Example-2.

N	Method	$E_{\max}$	Setup time	Running time	Total time
8	FDM	2.5194e-03	1.0251e+01	3.9489e-03	1.0255e+01
	HWLS	2.5194e-03	1.2692e-03	1.0680e-03	2.3372e-03
	DWLS	2.5194e-03	6.3597e-04	7.5806e-03	8.2166e-03
	BWLS	2.5194e-03	5.4018e-04	2.4973e-03	3.0375e-03
16	FDM	7.2752e-04	7.2048e+00	3.2277e-03	7.2080e+00
	HWLS	7.2752e-04	9.8628e-04	1.1854e-03	2.1717e-03
	DWLS	7.2752e-04	7.2902e-04	8.7670e-03	9.4961e-03
	BWLS	7.2752e-04	6.0860e-04	2.7707e-03	3.3793e-03
32	FDM	1.9133e-04	8.4371e+00	1.0800e-02	8.4479e+00
	HWLS	1.9133e-04	7.2252e-04	9.7739e-04	1.6999e-03
	DWLS	1.9133e-04	5.8910e-04	7.5971e-03	8.1862e-03
	BWLS	1.9133e-04	7.2936e-04	2.5688e-03	3.2982e-03
64	FDM	4.6112e-05	2.1881e+01	3.4267e-02	2.1915e+01
	HWLS	4.6112e-05	7.4476e-04	9.7533e-04	1.7201e-03
	DWLS	4.6112e-05	5.9149e-04	7.8027e-03	8.3942e-03
	BWLS	4.6112e-05	5.6378e-04	2.5175e-03	3.0813e-03
128	FDM	3.2735e-05	1.8036e+01	1.5862e-01	1.8195e+01
	HWLS	3.2735e-05	7.4510e-04	1.0373e-03	1.7824e-03
	DWLS	3.2735e-05	6.8865e-04	7.7178e-03	8.4065e-03
	BWLS	3.2735e-05	5.5557e-04	2.4960e-03	3.0515e-03

Example-3. Finally, we consider the hyperbolic differential equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin(\pi x) \tag{5.3}$$

subjected to boundary conditions. The exact solution of the problem is given by $u(x,t) = \left(\frac{1}{\pi^2}\right)(1 - \cos \pi t)\sin(\pi x)$. As in the previous examples, we obtained the numerical solutions of Eqn. (5) and are presented in figure 3. The error analysis and CPU time versus grid points is given in table 3.

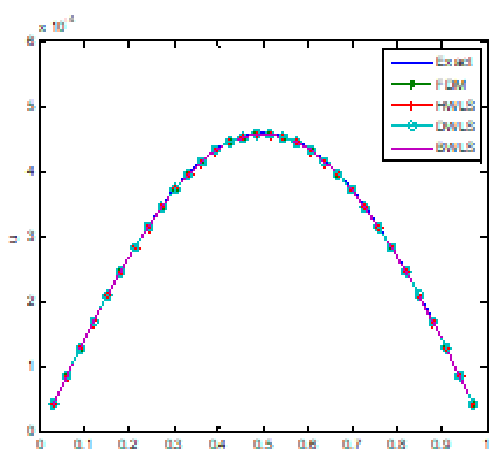

Fig. 3. Comparison of numerical solutions with exact solution for $N = 32$ of the Example-3.

Table 3. The maximum error with CPU time (in seconds) versus grid points of the Example-3.

N	Method	$E_{\max}$	Setup time	Running time	Total time
8	FDM	2.8428e-04	3.1531e+00	7.6103e-02	3.2292e+00
	HWLS	2.8428e-04	8.3558e-04	1.7611e-03	2.5967e-03
	DWLS	2.8428e-04	4.8386e-04	8.8291e-03	9.3129e-03
	BWLS	2.8428e-04	5.8946e-04	7.6756e-03	8.2650e-03
16	FDM	2.3944e-05	4.4419e+00	7.4926e-02	4.5168e+00
	HWLS	2.3944e-05	6.9730e-04	1.7351e-03	2.4324e-03
	DWLS	2.3944e-05	4.7771e-04	8.6346e-03	9.1123e-03
	BWLS	2.3944e-05	5.9225e-04	7.1621e-03	7.7543e-03
32	FDM	1.7210e-06	3.6995e+00	7.5644e-02	3.7752e+00
	HWLS	1.7210e-06	6.9590e-04	1.7357e-03	2.4316e-03
	DWLS	1.7210e-06	4.8414e-04	8.6371e-03	9.1213e-03
	BWLS	1.7210e-06	6.0175e-04	7.1861e-03	7.7879e-03
64	FDM	1.1497e-07	3.0844e+00	7.4721e-02	3.1591e+00
	HWLS	1.1497e-07	6.8864e-04	1.7290e-03	2.4176e-03
	DWLS	1.1497e-07	4.8135e-04	8.6005e-03	9.0819e-03
	BWLS	1.1497e-07	5.9114e-04	7.1626e-03	7.7538e-03
128	FDM	7.4215e-09	4.9891e+00	7.5477e-02	5.0646e+00
	HWLS	7.4215e-09	6.9422e-04	1.7323e-03	2.4266e-03
	DWLS	7.4215e-09	4.8246e-04	8.6282e-03	9.1107e-03
	BWLS	7.4215e-09	5.9058e-04	7.1573e-03	7.7479e-03

6 Conclusions

Here, we developed an efficient numerical method that is wavelet based lifting technique for the numerical solution of hyperbolic differential equations. The proposed schemes are very convenient and effective by referring the figures and tables. However the computational (CPU) time of the proposed scheme shows the super convergence. Thus, the proposed method has wide range of applications in science and engineering field.

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Received: 2020-12-10
Accepted: 2023-03-02