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CHARATERISTIC POLYNOMILAL AND RANK OF SPECIAL TYPES OF EVEN ORDER MATRICES

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Abstract In this paper we introduce a different types of matrices. We determine the characteristic polynomial theoretically and by using software. In some cases we draw conclusions from it about the eigenvalues of the matrices. The proofs are sometimes constructive in the sense that we construct a basis of the nullspace.

1 Introduction

In [7] Nordgren proved that QR magic squares have signed pairs of eigenvalues just as do regular magic squares according to a well-known theorem of Mattingly. This leads to the fact that odd powers of QR magic squares are magic squares which also can be established directly from the QR condition. Since all pandiagonal magic squares of order 4 are MP, they are QR. Also, the researcher showed that all pandiagonal magic squares of order 5 are QR but higher order ones may or may not be. Similarly, in [8] Stephens studied magic squares of order 4 and 5 and detailed the results concerning the eigenvalues. A n by n matrix $A = (A_{ij})$ is called a pentadiagonal matrix if $A_{ij} = 0$ whenever $|i - j| > 2$. Tridiagonal and pentadiagonal matrices appear in several areas of mathematics and engineering, specially involving linear systems of differential equations. Since every 3 by 3 matrix is a pentadiagonal one, it is clear that not every pentadiagonal matrix is similar to a symmetric matrix. In [3] Alvarez et. al. consider two classes of pentadiagonal matrices and obtain recursive formulas for the characteristic polynomials and explicit formulas for the eigenvalues of these classes of pentadiagonal matrices. They show that if A is a pentadiagonal matrix, then under certain conditions on the sign of the product of the entries the matrix A is similar to a symmetric pentadiagonal matrix. Thus, all the eigenvalues of A are real and A is diagonalizable.

We are going to consider different types of symbolic matrices. For each type we determine the rank or the characteristic polynomial.

2 Squares with Four Blocks Property

In the literature we encounter the concept of a Franklin square (see [1] and [2]). Usually its a 8 by 8 matrix satisfying many conditions. One of these conditions is the four blocks property; i.e.

$$a_{ij} + a_{(i+1)(j+1)} + a_{i(j+1)} + a_{(i+1)j} = s$$

for all $1 \leq i, j \leq 7$, where s is a constant. We are going to study the class of matrices, which satisfy this condition and call it squares with the four blocks property. A simple example of a 6 by 6 square of this type is

$$\begin{bmatrix} p & b & j & k & l & m \\ c & s-p-b-c & p+c-j & s-k-p-c & p+c-l & s-p-c-m \\ d & p+b-d & j-p+d & p-d+k & d+l-p & p+m-d \\ f & s-p-b-f & f+p-j & s-p-f-k & f-l+p & s-p-f-m \\ g & p+b-g & g+j-p & p-g+k & g+l-p & p-g+m \\ h & s-p-b-h & p-j+h & s-h-k-p & p-l+h & s-h-m-p \end{bmatrix}$$

Its nullspace has as a basis the following set of vectors

$$\begin{bmatrix} b-k \\ j-p \\ k-b \\ p-j \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} j-l \\ 0 \\ l-p \\ 0 \\ p-j \\ 0 \end{bmatrix}, \begin{bmatrix} b-m \\ j-p \\ m-b \\ 0 \\ 0 \\ p-j \end{bmatrix}.$$

We will prove that all matrices of this type of an even order have rank 3. We consider the following notation of this type, which we illustrate by this 4 by 4 square:

$$\begin{bmatrix} p & c_1 & c_2 & c_3 \\ r_1 & s-p-r_1-c_1 & p+r_1-c_2 & s-p-r_1-c_3 \\ r_2 & p-r_2+c_1 & -p+r_2+c_2 & p-r_2+c_3 \\ r_3 & s-p-r_3-c_1 & p+r_3-c_2 & s-p-r_3-c_3 \end{bmatrix}$$

Definition 2.1. A matrix (m_{ij}) is a square with four blocks property if and only if $m_{11} = p$ and for $2 \leq i, j \leq n$ the following hold

$m_{1j} = c_{j-1}, m_{i1} = r_{i-1}$, $m_{ij} = s - p - r_{j-1} - c_{j-1}$ when i, j are even,
$m_{ij} = r_{j-1} + c_{j-1} - p$ when i, j are odd,
$m_{ij} = p + (-1)^{i-1} c_{j-1} + (-1)^{j-1} r_{j-1}$ when i, j have different parity.

Proposition 2.2. *Assume that* $s\,(p - c_2)\,(p - r_2) \neq 0$. *For all* $n \geq 3$ *a matrix* (m_{ij}) *is a square with four blocks property of order* $2n$ *by* $2n$ *has rank three.*

Proof. If we calculate the determinant of the 3 by 3 upper left corner of this matrix then we obtain

$$-s\,(p - c_2)\,(p - r_2)\,.$$

Thus the rank is at least 3. We give now the explicit form of $2n - 3$ independent vectors in the nullspace, which can be checked by forward computation. The first set of vectors has the structure for $2 \leq i \leq n - 1$:

$$(c_2 - c_{2i}, -p + c_{2i}, 0, ..., 0, p - c_2, 0, ..., 0)^t$$

where $p - c_2$ will be located at the position $2i + 1$.

The second set of vectors has the structure for $2 \leq i \leq n - 1$:

$$(c_1 - c_{2i-1}, -p + c_2, c_{2i-1} - c_1, 0, ..., 0, p - c_2, 0, ..., 0)^t$$

where $p - c_2$ will be located at the position $2i + 2$. □

Proposition 2.3. *Assume that* $s = 0, (p + c_1)\,(p + r_1) \neq 0$. *For all* $n \geq 2$ *a matrix* (m_{ij}) *is a square with four blocks property of order* $2n$ *by* $2n$ *has rank two.*

Proof. If we calculate the determinant of the 2 by 2 upper left corner of this matrix then we obtain $-(p + c_1)\,(p + r_1)$. Thus, the rank is at least 3. We give now the explicit form of $2n - 2$

independent vectors in the nullspace, which can be checked by forward computation: The first set of vectors has the structure for $1 \le i \le n-1$:

$$(-c_1 - c_{2i}, p - c_{2i}, 0, ..., 0, p + c_1, 0, ..., 0)^t$$

where $p + c_1$ will be located at the position $2i + 1$. The second set of vectors has the structure for $1 \le i \le n-1$:

$$(c_1 - c_{2i+1}, -p - c_{2i+1}, 0, ..., 0, p + c_1, 0, ..., 0)^t$$

where $p + c_1$ will be located at the position $2i + 2$. □

Remark 2.4. If we add the vector

$$(-c_2 - c_3 - ...0 - c_{2n-1}, -c_2 - c_3 - ...0 - c_{2n-1}, p + c_1, p + c_1, ..., p + c_1)^t$$

to the set of independent vectors in the nullspace, which were described in Proposition 2.2, then we obtain a basis for the nullspace in case $s = 0$.

The following 8 by 8 square of the structure

$$\begin{bmatrix} \Gamma & \Theta \\ \Lambda & \Psi \end{bmatrix}$$

where

$$\Gamma = \begin{bmatrix} F & m-p-q-F & f+k-J & m-k-x-f \\ f+s-F & -j+2b+x-s & J+s-k & -s+k+x \\ G & f-b+k & L & -j+f+b \\ m-f-s-G & s+b-k & m-f-s-L & j+s-b \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} b-q+x & -b-p-x+m & p+x-U & q-x-U \\ -b+p+q & b & U & -b+j+k \\ H & j-2b+k+p & q & p \\ m-p-x-H & -j+2b-k+x & J & x \end{bmatrix}$$

$$\Theta = \begin{bmatrix} f+s-M & m-b-x-f & R & m+D-f \\ M & -s+b+x & f+s-R & -D-s \\ -f-k-a+m & f-j+k & -f+j+a & f \\ k-s+a & j-k+s & -j-s+m-a & s \end{bmatrix}$$

$$\Psi = \begin{bmatrix} -a+p+x & m-j-x-p & B & -k-p-x+m \\ a & j & -a-j-k+m & k \\ q-a-U & p-b+k & b-p+a & -b+p+j \\ N & x+b-k & m-b-a-x & -j+b+x \end{bmatrix}$$

$$D = j - b - k - x, B = p + a + x - m + j + k, F = -j + f + 2b - q + x - p,$$

$$G = -f + b - k - q - p + m, H = -j + 2b - k - 2p - q + m, J = m - q - x - p,$$

$$L = j + q + p - f - b, M = a + j + s - b - x, N = j + a - x - b + k,$$

$$U = b + m - j - k - p - q, R = b + a + x + k + f - m.$$

is a traditional Franklin square 8 by 8. We know that Γ, Θ, Λ, Ψ are all semi magic squares. If we require the condition that $x = m - \frac{1}{2}b$ then we obtain also a traditional magic square since Franklin squares do not strictly qualify as "magic squares" according to the common definition of the term that includes the diagonal sums. Its characteristic polynomial is according to computer calculations $X^5 (X - 2m) (X^2 - 2(b + m) X - 4A)$, where

$$A = 2\left(b-j\right)^2 - 4k^2 + b(m+3a+4(f-s)) + k(5b-2f) + 2(j+k)(s-a)-$$
$$(4f+b)(p+q) - 4p(a+k) + 2m(p-j) + 2(m-j)(f+k) + 2q(j-k).$$

We consider another class of 8 by 8 squares, namely

$$\begin{bmatrix} c+d+x & -x & -c & -d & b & l & a & -b-l-a \\ h+k+m & -h & -k & -m & n & o & p & -n-o-p \\ r & -r & 0 & 0 & 0 & 0 & u & -u \\ w & -w & 0 & 0 & 0 & 0 & y & -y \\ y & -y & 0 & 0 & 0 & 0 & w & -w \\ u & -u & 0 & 0 & 0 & 0 & r & -r \\ n+o+p & -p & -o & -n & m & k & h & -(h+k+m) \\ b+l+a & -a & -l & -b & d & c & x & -c-d-x \end{bmatrix}$$

We notice that it consists of four semi magic squares (here the sum is zero 0) like a Franklin square. It is apparent that the two following independent vectors belong to the nullspace:

$$\left(1,1,1,1,0,0,0,0\right)^t, \left(0,0,0,0,1,1,1,1\right)^t.$$

After tedious computations with the aid of computer we found that its characteristic polynomial is

$$X^2(X^6 + \alpha X^4 + \beta X^2)$$

where

$$\beta = (r+u-w-y)(r-u-w+y)$$
$$((k+o)(b+d) - (m+n)(c+l))((k-o)(b-d) + (m-n)(c-l)).$$

So, in general the rank is 6.

3 S-Squares

Another interesting structure in the case of matrices with even order is the following 6 by 6 matrix

$$S = \begin{bmatrix} h & d & -c & c & -d & -h \\ -k & f & -u & u & -f & k \\ j & t & g & -y & t & j \\ -j & -t & y & -g & -t & -j \\ -k & f & -u & u & -f & k \\ h & d & -c & c & -d & -h \end{bmatrix}$$

It has in general rank four, and a basis of the nullspace basis is

$$\begin{bmatrix} g-y \\ 0 \\ -2j \\ -2j \\ 0 \\ g-y \end{bmatrix}, \begin{bmatrix} 0 \\ g-y \\ -2t \\ -2t \\ g-y \\ 0 \end{bmatrix}$$

Its minimum polynomial is $X^4 + \left(y^2 - g^2 + 4cj + 4tu\right) X^2$ while its characteristic polynomial is $X^6 + \left(y^2 - g^2 + 4cj + 4tu\right) X^4$. We generalize for matrices of order $2n+2$. We denote by E the matrix order n, which has on ones on the opposite diagonal, i.e. $E = (e_{ij})$ with $e_{i(n-i)} = 1$ for $i = 1, ..., n$ and 0 else. It is well known that left/right multiplication of a suitable matrix K by E will reverse the order of the rows/columns of K and, hence, $E^2 = I$.

Definition 3.1. A matrix $S \in M_{2n+2}(R)$ is called a S-matrix iff it has the form

$$\begin{bmatrix} A & C & -C & -AE \\ R & g & -y & RE \\ -R & y & -g & -RE \\ EA & -EC & EC & -EAE \end{bmatrix}$$

where $A \in M_{2n}(R), R \in M_{1n}(R), C \in M_{n1}(R)$ and $g, y \in R$.

We notice that S^2 can be calculated by employing block matrix multiplications, and yields

$$\begin{bmatrix} 2CR & (g-y)C & -C & 2CRE \\ (g+y)R+2RA & (g^2-y^2)+2RC & -2RC & (g+y)RE-2RAE \\ (g+y)R-2RA & -2RC & (g^2-y^2)+2RC & (g+y)RE+2RAE \\ 2ECR & (g-y)EC & EC & 2ECRE \end{bmatrix}$$

We obtain also

$$S^4 = (g^2 - y^2 + 2RC) + S^2 \quad \textbf{(1)}$$

For example the entries in $n+1, n+2$ rows and columns of S^2 are calculated by

$$\begin{bmatrix} R \\ -R \end{bmatrix} \begin{bmatrix} C & -C \end{bmatrix} + \begin{bmatrix} g & -y \\ y & -g \end{bmatrix} \begin{bmatrix} g & -y \\ y & -g \end{bmatrix} + \begin{bmatrix} RE \\ -RE \end{bmatrix} \begin{bmatrix} EC & -EC \end{bmatrix} =$$

$$\begin{bmatrix} (g^2-y^2)+2RC & -2RC \\ -2RC & (g^2-y^2)+2RC \end{bmatrix}$$

Lemma 3.2. *The number* $(g^2 - y^2) + 4RC$ *is an eigenvalue of* S^2.

Proof. By direct computation we verify that

$$(0, ..., 0, -1, 1, 0, ..., 0)^t, (C, g+y, 0, EC)^t$$

are corresponding eigenvalues. We notice that these two eigenvectors are linearly independent. □

Proposition 3.3. *The rank of* S^2 *is two and the nullity is* $2n$.

Proof. Without loss of generality we assume that for the first element in C, $c_1 \neq 0$. Clearly, S^2 is now equivalent to

$$\begin{bmatrix} 2CR & (g-y)C & (g-y)C & 2CRE \\ (g+y)R+2RA & (g^2-y^2)+2RC & -2RC & (g+y)RE-2RAE \\ (g+y)R-2RA & -2RC & (g^2-y^2)+2RC & (g+y)RE+2RAE \\ [0]_{n\times n} & \vec{0} & \vec{0} & [0]_{n\times n} \end{bmatrix}$$

We consider two cases:

i) For $g+y \neq 0$, apply the row operations sequence

$$R_i \to R_i - \frac{c_i}{c_1} R_1, 2 \leq i \leq n, R_{n+1} \to R_{n+1} + R_{n+2},$$

$$R_{n+1} \to \frac{1}{g+y} R_{n+1}, R_{n+1} \to R_{n+1} - \frac{1}{c_1} R_1$$

to obtain the form

$$\begin{bmatrix} 2c_1R & (g-y)c_1 & (g-y)c_1 & 2c_1RE \\ [0]_{n\times n} & \vec{0} & \vec{0} & [0]_{n\times n} \\ (g+y)R-2RA & -2RC & (g^2-y^2)+2RC & (g+y)RE+2RAE \\ [0] & \vec{0} & \vec{0} & [0] \end{bmatrix}$$

Therefore, rank of S^2 is 2 and its nullity is $2n$.
ii) For $g + y = 0$ we proceed similarly, observing that $R_{n+1} = -R_{n+2}$. □

Theorem 3.4. *The matrix S has minimal polynomial*

$$q(x) = x^2(x^2 - ((g^2 - y^2) + 4RC))$$

and characteristic polynomial

$$\Phi_S(x) = x^{2n}(x^2 - (g^2 - y^2 + 4RC))$$

Proof. From the relation (1) we conclude that is a root of the polynomial

$$x(x^2 - ((g^2 - y^2) + 4RC)).$$

Thus, $p(x)$ is the minimal polynomial of S^2, since none of its factors is annihilated by S^2. Furthermore, since nullity of S^2 is $2n$, the algebraic multiplicity of the zero eigenvalue is greater than or equal $2n$ and the algebraic multiplicity of the eigenvalue $(g^2 - y^2) + 4RC$ is greater than or equal two. Putting everything together, we conclude that the characteristic polynomial of S^2 is

$$x^{2n}(x - (g^2 - y^2 + 4RC))^2.$$

Now, we proceed with matrix S. From relation (1) again, we find that S is a root of the polynomial $q(x)$. By direct calculation, we verify that

$$S(S^2 - (g^2 - y^2 + 4RC)I) \neq 0.$$

Hence, $q(x)$ is the minimal polynomial.

It is well known that the algebraic multiplicities of the variable x in the characteristic polynomial of S^2 as well S are the same. Furthermore, every factor of has to be prsentd in the characteristic polynomial of S. Therefore, the characteristic polynomial of S is given by

$$x^{2n}(x^2 - (g^2 - y^2 + 4RC)).\square$$

Remark 3.5. (i) Let $\overrightarrow{0}_j$ repersent the zero j-column vector and r_j the entries of R. We verify that for the n linearly independent vectors for $0 \leq k \leq n-1$:

$$v_k = (\overrightarrow{0}_{n-k-1}, g-y, \overrightarrow{0}_k, -2r_{k+1}, -2r_{k+1}, \overrightarrow{0}_k, g-y, \overrightarrow{0}_{n-k-1})^t$$

that $Sv_k = \overrightarrow{0}_n$ holds. Hence, $v_1, v_2, ..., v_n$ form a bais for the nullspace of S. This forces the rank of S to be equal to $n + 2$.
(ii) We avoided the trivial cases like that R, C are zero in this section. Also, we discard the case that the matrix is one and the same entry in the squares with four blocks property.

4 Self-Complementary Squares of Even Order

So far we saw many squares like S-squares, where the coefficients of odd powers of X in the characteristic polynomial vanish. There are other cases of squares, which pocess this property. They are self-complementary squares with a condition on the summation of the rows or columns. We give a similar definition to the one in [6].

Definition 4.1. A $2n$ by $2n$ magic square $A = (A_{ij})$ is called self-complementary square iff $A_{ij} + A_{(2n+1-i)(2n+1-j)} = \frac{\mu}{n}$ whenever $1 \leq i \leq 2n$, $1 \leq j \leq n$, μ is the magic constant.

We start with a self-complementary 4 by 4 square with the magic constant zero:

$$\begin{bmatrix} -j-l-n & n & j & l \\ j+k+2l+n & -k-l-n & -j-k-l & k \\ -k & j+k+l & k+l+n & -j-k-2l-n \\ -l & -j & -n & j+l+n \end{bmatrix}$$

Its characteristic polynomial is $X^2(X^2 + \big(4jk - 4jn - 4kn - 8ln - 4n^2\big))$, while its minimum polynomial is

$$X(X^2 + \big(4jk - 4jn - 4kn - 8ln - 4n^2\big)).$$

Its nullspace basis is

$$\left\{(j-n, -\left(l+n\right), j+l, 0)^t, (l+n, j+2l+n, 0, j+l)^t\right\}.$$

When we add the same value for all entries we obtain the following self-complementary 4 by 4 square with magic constant $4s$

$$\begin{bmatrix} s-j-l-n & n+s & j+s & l+s \\ j+k+2l+n+s & s-k-l-n & s-j-k-l & k+s \\ s-k & j+k+l+s & k+l+n+s & -j-k-2l-n+s \\ s-l & s-j & s-n & j+l+n+s \end{bmatrix}$$

Its characteristic polynomial is

$$X\left(X-4s\right)\left(X^2 + 4\left(jk - jn - kn - 2ln - n^2\right)\right).$$

The nullspace basis is spanned by the vector

$$(n-j, j+2l+n, -\left(j+2l+n\right), -(n-j))^t.$$

This can be explained by the theorem, which is proved in [4]: For any A is a semi-magic square with eigenvalues μ (magic constant) $m_2, ..., m_n$ in the field of complex numbers and p a complex number, then $A+pE$, where E is the all 1's matrix, has the same eigenvalues $m_2, ..., m_n$ except that μ is replaced with $\mu + pn$.

On the other side a general self-complementary square, which is not necessarily magic like

$$\begin{bmatrix} m & n & j & l \\ p & q & h & k \\ s-k & s-h & s-q & s-p \\ s-l & s-j & s-n & s-m \end{bmatrix}$$

has rank four and a characteristic polynomial in general of the form

$$X^4 - 2sX^3 - AX^2 - BX + C.$$

We see that zero is now not necessarily an eigenvalue. The following matrix

$$\begin{bmatrix} m & n & j & 2s-m-n-j \\ p & q & h & 2s-p-q-h \\ h+p+q-s & s-h & s-q & s-p \\ j+m+n-s & s-j & s-n & s-m \end{bmatrix}$$

has zero as an eigenvalues, and the characteristic polynomial is

$$X(X-2s)\left(X^2 + 2V + h^2 + j^2 + n^2 - q^2\right),$$

$$V = j(m-h+n-p-q) + s(p+q-m-n) + mn - np.$$

It is similar to a self-complementary square. We explain the need for the row or column sum condition as follows: In general for a self-complementary square we look for a vector in nullpace as follows: $(-x, -y, y.x)^t$. The multiplication of the matrix with this vector yields just the following two different entries $xD - Ay + By - ax$ and $Kx - dy - fx + gy$. Since the entries are equal this vector is zero if and only if

$$x(D-a) + (B-A)y = 0,$$
$$(K-f)x + (g-d)y = 0.$$

It has a nontrivial solution if one equation is a multiple of the other. The conditions

$$B - A = d - g, D - a = f - K$$

will be enough. The requirement that the sum in all columns is $2s$ (to be columns magic) means that

$$A + d + s - g + s - B = 2s, a + f + s - K + s - D = 2s.$$

This leads to the previous condition. Also when we encounter

$$\begin{bmatrix} m & n & j & 2s-m-n-j \\ p & q & h & 2s-p-q-h \\ h+p+q-s & s-h & s-q & s-p \\ j+m+n-s & s-j & s-n & s-m \end{bmatrix}$$

We transpose the matrix

$$\begin{bmatrix} m & p & h+p+q-s & j+m+n-s \\ n & q & s-h & s-j \\ j & h & s-q & s-n \\ 2s-m-n-j & 2s-p-q-h & s-p & s-m \end{bmatrix}$$

Its determinant is zero since it satisfies the column sum condition. Hence, the original matrix has a nontrivial nullspace. So any self-complementary square of even order, which satisfies the row sum condition, will have zero as an eigenvalue. We consider the 6 by 6 squares. A general self-complementary square 6 by 6 with row sum $3s$ is the matrix

$$\begin{bmatrix} a & b & c & d & e & 3s-a-b-c-d-e \\ f & g & h & i & j & 3s-f-g-h-i-j \\ k & l & m & n & o & 3s-k-l-m-n-o \\ k+l+m+n+o-2s & s-o & s-n & s-m & s-l & s-k \\ f+g+h+i+j-2s & s-j & s-i & s-h & s-g & s-f \\ a+b+c+d+e-2s & s-e & s-d & s-c & s-b & s-a \end{bmatrix}$$

By using computer we find that its characteristic polynomial is

$$X(X-3s)(X^4 + GX^2 + H),$$

where
$G = 2(a+b+c+e-k-l-m-n-o)d + 2e(a+b+c-f-g-h-i-j) + 2(io+ab+ac+bc-bf-ck-hl) - 4s(a+b+c) + b^2 + c^2 + d^2 + j^2 + n^2 + e^2 - g^2 - m^2$.

We consider a subset of such self-complementary square 6 by 6 with row sum $3s$, where the squares have a zero summation of the middle columns

$$S_1 = \begin{bmatrix} a & -c-d-e & c & d & e & 3s-a \\ f & -h-j-p & h & p & j & 3s-f \\ k & -m-n-o & m & n & o & 3s-k \\ k-2s & s-o & s-n & s-m & m+n+o+s & s-k \\ f-2s & s-j & s-p & s-h & h+j+p+s & s-f \\ a-2s & s-e & s-d & s-c & c+d+e+s & s-a \end{bmatrix}$$

Proposition 4.2. *If the condition*

$$s = \frac{2(m+n-p)h + 2(f-k)(c+d) + 2(h+p)(o-j) - (h^2+m^2+p^2-n^2)}{4(a-d-e) - 2(f+k-j-n-o-p)}$$

holds, then the eigenvalues of S_1 are the four complex roots of a real number besides 0 *and* $2s$.

Proof. The characteristic polynomial of S_1 is

$$X(X-2s)(X^4+KX^2+L)$$

where

$$K=2\,(f+k-j-n-o-p)\,s+2\,(m+n-p)\,h+2\,(f-k)\,(c+d)+$$

$$2(h+p)(o-j)-4s\,(a-d-e)-(h^2+m^2+p^2-n^2)$$

When the condition holds then the characteristic polynomial becomes

$$X(X-2s)(X^4+L).$$

and we are done. □

In case of 8 by 8 we have an interesting case of pandiagonal self-complementary square. Actually, the structure is 8 by matrix, where the first four columns are shown below (the last four columns are obtained by the self-complementary property)

$$\begin{bmatrix} e & A+g+n & G & x \\ f & L & H & m \\ t & p & q & f+\alpha+E \\ v & w & g & \gamma \\ M & -\alpha & -s & -\delta \\ A & -u & -\beta & R \\ R-G+w & -\lambda & Q & -n \\ C & t-s-m & E & F \end{bmatrix}$$

where

$$\begin{gathered} A=2f+g+m-n+p+q+s+t+u+2v+w+\alpha+\lambda+\gamma+e, \\ C=-2f-2s-2t-2u-2v-2\alpha-\beta-\lambda-\delta-2e, \\ E=-2f-g-m-p-q-s-t-u-v-w-x-2\alpha-\beta-\gamma-e, \\ F=4(f+v)+3(e+g+s+w+\alpha)+2(m+p+q+t+u+\gamma)+x+\beta+\lambda+\delta, \\ G=w-m-2s-t-2u-v-2f-x-2\alpha-\beta-\lambda-2e, \\ H=2f+p+2s+2t+2u+v+x+2\alpha+2\beta+2e, \\ M=g+s+v+w+\alpha+\gamma+\delta, \\ L=-2f-2g-2p-q-2t-2v-2w-\gamma-e, \\ Q=2f+2m+2s+u+v+x+2\alpha+\beta+\lambda+\gamma+e, \\ R=n-2g-2m-3f-p-q-2s-t-u-3v-2w-x-2\alpha-\lambda-2\gamma-2e. \end{gathered}$$

We already explained that 0 is a double eigenvalue. We observed that the other eigenvalues are always pairs in the form $\pm t$ or $\pm yi$ for some real numbers t, y.

In closing, we conjecture that this is the situation for all squares of this type, and conjecture that the characteristic polynomial of a self-complementary square is

$$X^8+a_1X^6+a_2X^4+a_3X^2$$

for some real numbers a_1, a_2, a_3. We think also that a theoretical mathematical computation of the characteristic polynomial instead of a calculation by using computer is a better approach for studying a wider class of matrices.

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