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A note on generalized relative index-pair (α, β) of entire function with respect to another entire function

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Abstract . In this paper we wish to introduce the concept of generalized relative index-pair (α, β) of an entire function with respect to another entire function and then prove some results relating to the growth rates of composition of two entire functions with their corresponding left and right factors.

1 Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max\limits_{|z|=r} |f(z)|$. We use the standard notations and definitions of the theory of entire functions which are available in [11] and [12], and therefore we do not explain those in details.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1, β_2 and γ_1 always denote the functions belonging to L^0.

Considering this, the value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

is called [10] generalized order (α,β) of an entire function f. For details about generalized order (α,β) one may see [10]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different directions. For the purpose of further applications, Biswas et al. [3] rewrite the definition of the generalized order (α,β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [10]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [6]:

Definition 1.1. [3] The generalized order (α, β) and generalized lower order (α, β) of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

The function f is said to be of regular generalized growth (α, β) when generalized order (α, β) and generalized lower order (α, β) of f are the same. Functions which are not of regular generalized growth (α, β) are said to be of irregular generalized growth (α, β).

In this connection Biswas et al. [4] give the following definition:

Definition 1.2. [4] An entire function f is said to have generalized index-pair (α,β) if $b<\rho_{(\alpha,\beta)}[f]<+\infty$ and $\rho_{(\exp\alpha,\exp\beta)}[f]$ is not a non-zero finite number, where $b=1$ if $\alpha=\beta$ and $b=0$ for otherwise. Moreover if $0<\rho_{(\alpha,\beta)}[f]<+\infty$, then for any $\gamma_1\in L$ and $\gamma_1(r)\neq r$

$$\begin{cases} \rho_{(\gamma_1(\alpha),\beta)}[f]=+\infty \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho\beta(r))}{\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f],\\ \rho_{(\gamma_1(\alpha),\beta)}[f]=0 \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho_1\beta(r))}{\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f],\\ \rho_{(\alpha,\gamma_1(\beta))}[f]=+\infty \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\rho\beta(r)}{\gamma_1(\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f],\\ \rho_{(\alpha,\gamma_1(\beta))}[f]=0 \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\rho_1\beta(r)}{\gamma_1(\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f],\\ \rho_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]=1 \quad \text{when } \gamma_1\in L^0. \end{cases}$$

Similarly for $0<\lambda_{(\alpha,\beta)}[f]<+\infty$, one can easily verify that

$$\begin{cases} \lambda_{(\gamma_1(\alpha),\beta)}[f]=+\infty \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\lambda\beta(r))}{\beta(r))}=+\infty \text{ for any } \lambda<\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\gamma_1(\alpha),\beta)}[f]=0 \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\lambda_1\beta(r))}{\beta(r))}=0 \text{ for any } \lambda_1>\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\alpha,\gamma_1(\beta))}[f]=+\infty \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\lambda\beta(r)}{\gamma_1(\beta(r))}=+\infty \text{ for any } \lambda<\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\alpha,\gamma_1(\beta))}[f]=0 \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\lambda_1\beta(r)}{\gamma_1(\beta(r))}=0 \text{ for any } \lambda_1>\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]=1 \quad \text{when } \gamma_1\in L^0. \end{cases}$$

Definition 1.2 extend the definition of index-pair (p,q) of entire function f introduced by Juneja et al. [9]. For details about index-pair (p,q) of entire function f, one may see [9].

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function with respect to another entire function in the following way:

Definition 1.3. The generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function f with respect to an entire function g denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\rho_{(\alpha,\beta)}[f]_g=\limsup_{r\to+\infty}\frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g=\liminf_{r\to+\infty}\frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

Now we introduce the following definition which will be needed in the sequel:

Definition 1.4. An entire function f is said to have generalized relative index-pair (α,β) with respect to an entire function g if $b<\rho_{(\alpha,\beta)}[f]_g<+\infty$ and $\rho_{(\exp\alpha,\exp\beta)}[f]_g$ is not a non-zero finite number, where $b=1$ if $\alpha=\beta$ and $b=0$ for otherwise. Moreover if $0<\rho_{(\alpha,\beta)}[f]_g<+\infty$, then for any $\gamma_1\in L$ and $\gamma_1(r)\neq r$

$$\begin{cases} \rho_{(\gamma_1(\alpha),\beta)}[f]_g=+\infty \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho\beta(r))}{\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f]_g,\\ \rho_{(\gamma_1(\alpha),\beta)}[f]_g=0 \quad \text{when } \gamma_1(\alpha)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho_1\beta(r))}{\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f]_g,\\ \rho_{(\alpha,\gamma_1(\beta))}[f]_g=+\infty \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\rho\beta(r)}{\gamma_1(\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f]_g,\\ \rho_{(\alpha,\gamma_1(\beta))}[f]_g=0 \quad \text{when } \gamma_1(\beta)\in L^0 \text{ and } \lim\limits_{r\to+\infty}\frac{\rho_1\beta(r)}{\gamma_1(\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f]_g,\\ \rho_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]_g=1 \quad \text{when } \gamma_1\in L^0. \end{cases}$$

Similarly for $0 < \lambda_{(\alpha,\beta)}[f]_g < +\infty$, one can easily verify that

$$\begin{cases} \lambda_{(\gamma_1(\alpha),\beta)}[f]_g = +\infty \quad \text{when } \gamma_1(\alpha) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\gamma_1(\lambda\beta(r))}{\beta(r))} = +\infty \text{ for any } \lambda < \lambda_{(\alpha,\beta)}[f]_g, \\ \lambda_{(\gamma_1(\alpha),\beta)}[f]_g = 0 \quad \text{when } \gamma_1(\alpha) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\gamma_1(\lambda_1\beta(r))}{\beta(r))} = 0 \text{ for any } \lambda_1 > \lambda_{(\alpha,\beta)}[f]_g, \\ \lambda_{(\alpha,\gamma_1(\beta))}[f]_g = +\infty \quad \text{when } \gamma_1(\beta) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\lambda\beta(r)}{\gamma_1(\beta(r))} = +\infty \text{ for any } \lambda < \lambda_{(\alpha,\beta)}[f]_g, \\ \lambda_{(\alpha,\gamma_1(\beta))}[f]_g = 0 \quad \text{when } \gamma_1(\beta) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\lambda_1\beta(r)}{\gamma_1(\beta(r))} = 0 \text{ for any } \lambda_1 > \lambda_{(\alpha,\beta)}[f]_g, \\ \lambda_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]_g = 1 \quad \text{when } \gamma_1 \in L^0. \end{cases}$$

In this paper we wish to prove some results relating to the growth rates of composition of two entire functions with their corresponding left and right factors on the basis of their generalized relative order (α, β) and generalized relative lower order (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

2 Lemma

In this section, we present a lemma which will be needed in the sequel.

Lemma 2.1. *[5] Let f and g are any two entire functions with $g(0) = 0$. Also let B satisfy $0 < B < 1$ and $c(B) = \frac{(1-B)^2}{4B}$. Then for all sufficiently large values of r,*

$$M_f\left(c(B)\, M_g(Br)\right) \le M_{f\circ g}(r) \le M_f\left(M_g(r)\right).$$

In addition if $B = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \ge M_f\Big(\frac{1}{8} M_g\Big(\frac{r}{2}\Big)\Big).$$

3 Main results

In this section, we present the main results of the paper.

Theorem 3.1. *Let f be any entire function with generalized relative index-pairs (α_1,β_1) with respect to another entire function h. Also let g be an entire function with generalized index-pairs (α_2,β_2). Then*
(i) *the generalized relative index-pair of $f \circ g$ is (α_1,β_2) when $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

(a) $\lambda_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_1,\beta_2)}[f\circ g]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ *and*

(b) $\lambda_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_1,\beta_2)}[f\circ g]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$;

(ii) *the generalized relative index-pair of $f \circ g$ is $(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))$ when $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

(a) $\lambda_{(\alpha_1,\beta_1)}[f]_h \le \rho_{(\alpha_1,\, \beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ *and*

(b) $\rho_{(\alpha_1,\, \beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]_h = \rho_{(\alpha_1,\beta_1)}[f]_h$ *if* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$;

(iii) *the generalized relative index-pair of $f \circ g$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

(a) $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\, \beta_2)}[f\circ g]_h = \rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ *and*

(b) $\lambda_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\, \beta_2)}[f\circ g]_h \leqslant \rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$.

Proof. In view of the first part of Lemma 2.1, it follows for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1)) \left(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right) \beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big) \tag{3.1}$$

and also for a sequence of values of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1)) \left(\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right)\beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big). \tag{3.2}$$

Similarly, in view of the second part of Lemma 2.1, we have for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right) \beta_1\left(M_g(r)\right). \tag{3.3}$$

Now the following two cases may arise:
Case I. Let $\beta_1(r) = \alpha_2(r)$.
Now we have from (3.3) for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right) \left(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon\right) \beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\beta_2(r)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.4}$$

Also from (3.1), we obtain for a sequence of values of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1)) \left(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right) \left(\rho_{(\alpha_2,\beta_2)}[g] - \varepsilon\right) \beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\beta_2(r)} \geq \lambda_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.5}$$

Moreover, we have from (3.2) for a sequence of values of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1)) \left(\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right) \left(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon\right) \beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\beta_2(r)} \geq \rho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.6}$$

Therefore from (3.4) and (3.5), we get for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\lambda_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty} \frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]$$

$$i.e.,\ \lambda_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_1,\beta_2)}\left[f\circ g\right]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.7}$$

Likewise, from (3.4) and (3.6), we obtain for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\rho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty} \frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]$$

$$i.e.,\ \rho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_1,\beta_2)}\left[f\circ g\right]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h \rho_{(\alpha_2,\beta_2)}[g]. \tag{3.8}$$

Also from (3.7) and (3.8) one can easily verify that

(i) $\rho_{(\alpha_1(\gamma_1^{-1}),\, \beta_2)}[f\circ g]_h = \infty$ when $\alpha_1(\gamma_1^{-1}) \in L^0$ and $\lim_{r\to+\infty} \frac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)} = +\infty$,
(ii) $\rho_{(\alpha_1,\, \beta_2(\gamma_1^{-1}))}[f\circ g]_h = 0$ when $\beta_2\left(\gamma_1^{-1}\right) \in L^0$ and $\lim_{r\to+\infty} \frac{\beta_2(\gamma_1^{-1}(r))}{\beta_2(r)} = +\infty$ and
(iii) $\rho_{(\alpha_1(\gamma_1),\, \beta_2(\gamma_1))}[f\circ g]_h = 1$ when $\lim_{r\to+\infty} \frac{\alpha_1(\gamma_1(r))}{\alpha_1(r)} = 0$ and $\lim_{r\to+\infty} \frac{\beta_2(\gamma_1(r))}{\beta_2(r)} = 0$.

Therefore we obtain that the generalized relative index-pair of $f \circ g$ is (α_1, β_2) when $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the first part of the theorem is established.

Case II. Let $\beta_1(\alpha_2^{-1}(r)) \in L^0$.

Now we obtain from (3.3) for all sufficiently large values of r that

$$\begin{aligned}
\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) &\leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\alpha_2\left(M_g(r)\right)\right)\right)\\
i.e.,\ \alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) &\leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2(r)\right)\right)\\
i.e.,\ \alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) &\leqslant (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)
\end{aligned}$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h. \tag{3.9}$$

Also from (3.1), we have for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) \geq (1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2\left(\frac{r}{2}\right)\right)\right)$$

$$i.e.,\ \alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) \geq (1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \geq \lambda_{(\alpha_1,\beta_1)}[f]_h. \tag{3.10}$$

Further, we get from (3.2) for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) \geq (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\left(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2\left(\frac{r}{2}\right)\right)\right)$$

$$\begin{aligned}
i.e.,\ \alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right) &\geq (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)\\
i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} &\geq \rho_{(\alpha_1,\beta_1)}[f]_h. \qquad (3.11)
\end{aligned}$$

Therefore from (3.9) and (3.10), we get for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\begin{aligned}
\lambda_{(\alpha_1,\beta_1)}[f]_h &\leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h\\
i.e.,\ \lambda_{(\alpha_1,\beta_1)}[f]_h &\leq \rho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}[f\circ g]_h \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h. \qquad (3.12)
\end{aligned}$$

Likewise, from (3.9) and (3.11) we get for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\rho_{(\alpha_1,\beta_1)}[f]_h \leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]_h$$

$$i.e.,\ \rho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}[f\circ g]_h = \rho_{(\alpha_1,\beta_1)}[f]_h. \tag{3.13}$$

Further from (3.12) and (3.13) one can easily verify that

(i) $\rho_{(\alpha_1(\gamma_1^{-1}),\, \beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]_h = \infty$ when $\alpha_1(\gamma_1^{-1}) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)} = +\infty$,

(ii) $\rho_{(\alpha_1,\, \beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}))))}[f\circ g]_h = 0$ when $\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}))) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))} = +\infty$ and

(iii) $\rho_{(\alpha_1(\gamma_1),\, \beta_1(\alpha_2^{-1}(\beta_2(\gamma_1))))}[f\circ g]_h = 1$ when $\lim\limits_{r\to+\infty} \frac{\alpha_1(\gamma_1(r))}{\alpha_1(r)} = 0$ and $\lim\limits_{r\to+\infty} \frac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))} = 0.$

Therefore we get that the generalized index-pair of $f\circ g$ is $\left(\alpha_1, \beta_1\left(\alpha_2^{-1}\left(\beta_2\right)\right)\right)$ when $\beta_1\left(\alpha_2^{-1}(r)\right) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the second part of the theorem follows.

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$

Then we obtain from (3.3) for all sufficiently large values of r that

$$\begin{aligned}
\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\le \alpha_2\left(\beta_1^{-1}\left(\left(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(M_g(r)\right)\right)\right)\\
i.e.,\ \alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\leqslant (1+o(1))\,\alpha_2\left(M_g(r)\right)\\
i.e.,\ \alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\leqslant (1+o(1))\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2(r)\\
i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right)}{\beta_2(r)} &\leqslant \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.14)
\end{aligned}$$

Also from (3.1) we have for a sequence of values of r tending to infinity that

$$\begin{aligned}
\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\ge (1+o(1))\,\alpha_2\left(M_g\left(\frac{r}{2}\right)\right)\\
i.e.,\ \alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\ge (1+o(1))\left(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2(r)\\
\limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right)}{\beta_2(r)} &\ge \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.15)
\end{aligned}$$

Similarly, we get from (3.2) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right) &\ge (1+o(1))\left(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2(r)\\
\limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right)}{\beta_2(r)} &\ge \lambda_{(\alpha_2,\beta_2)}[g]. \qquad (3.16)
\end{aligned}$$

Therefore from (3.14) and (3.15), we obtain for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\rho_{(\alpha_2,\beta_2)}[g] \le \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_2,\beta_2)}[g]$$

$$i.e.,\ \rho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}[f\circ g]_h = \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.17)$$

Similarly, from (3.14) and (3.16) we get for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\begin{aligned}
\lambda_{(\alpha_2,\beta_2)}[g] &\le \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_h^{-1}(M_{f\circ g}(r))\right)\right)\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_2,\beta_2)}[g]\\
i.e.,\ \lambda_{(\alpha_2,\beta_2)}[g] &\le \rho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}[f\circ g]_h \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.18)
\end{aligned}$$

So from (3.17) and (3.18) one can easily verify that

(i) $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}))),\,\beta_2)}[f\circ g]_h=\infty$ when $\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1})))\in L^0$ and $\lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))}=+\infty$,

(ii) $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\,\beta_2(\gamma_1^{-1}))}[f\circ g]_h=0$ when $\beta_2\left(\gamma_1^{-1}\right)\in L^0$ and $\lim_{r\to+\infty}\frac{\beta_2\left(\gamma_1^{-1}(r)\right)}{\beta_2(r)}=+\infty$ and

(iii) $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1))),\,\beta_2(\gamma_1))}[f\circ g]_h=1$ when $\lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))}=0$ and $\lim_{r\to+\infty}\frac{\beta_2(\gamma_1(r))}{\beta_2(r)}=0$.

So we obtain that the generalized index-pair of $f\circ g$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r))\in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and thus the third part of the theorem is established. □

Theorem 3.2. *Let f be any entire function with generalized relative index-pairs (α_1,β_1) with respect to another entire function h. Also let g be an entire function with generalized index-pairs (α_2,β_2).*

(i) *If $\beta_1(r)=\alpha_2(r)$, $\lambda_{(\alpha_1,\beta_1)}[f]_h>0$ and $\lambda_{(\alpha_2,\beta_2)}[g]>0$, then*

$$\lambda_{(\alpha_1,\beta_1)}[f]_h\lambda_{(\alpha_2,\beta_2)}[g]\le\lambda_{(\alpha_1,\beta_2)}[f\circ g]_h\le\min\left\{\rho_{(\alpha_1,\beta_1)}[f]_h\lambda_{(\alpha_2,\beta_2)}[g],\ \lambda_{(\alpha_1,\beta_1)}[f]_h\rho_{(\alpha_2,\beta_2)}[g]\right\}.$$

(ii) *If $\beta_1\left(\alpha_2^{-1}(r)\right)\in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f]_h>0$ and $\lambda_{(\alpha_2,\beta_2)}[g]>0$, then*

$$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]_h=\lambda_{(\alpha_1,\beta_1)}[f]_h.$$

(iii) *If $\alpha_2\left(\beta_1^{-1}(r)\right)\in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f]_h>0$ and $\lambda_{(\alpha_2,\beta_2)}[g]>0$, then*

$$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]_h=\lambda_{(\alpha_2,\beta_2)}[g].$$

In the line of Theorem 3.1 one can easily deduce the conclusion of Theorem 3.2 and so its proof is omitted.

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