Palestine Journal of Mathematics

Vol. 12(3)(2023) , 294–300

Graded Almost Prime Ideals over Non-Commutative Graded Rings

Jenan Shtayat, Rashid Abu-Dawwas and Ghadeer Bani Issa

Communicated by Jawad Abuhlail

MSC 2010 Classifications: Primary 13A02; Secondary 16W50.

Keywords and phrases: Graded prime ideals, graded almost prime ideals, graded weakly prime ideals.

Abstract The purpose of this article is to define and examine graded almost prime ideals over a non-commutative graded ring, and consider some cases where all graded right ideals of a non-commutative graded ring are graded almost prime.

1 Introduction

Let G be a group. Then a ring R is called a G-graded ring if $R = \bigoplus_{g\in G} R_g$ with the property $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. The set of all homogeneous elements of R is $\bigcup_{g\in G} R_g$ and is denoted by $h(R)$. The component R_e is a subring of R, and if R has unity 1, then $1 \in R_e$. For more terminology, see [7]. Let R be a G-graded and P be a right ideal of R. Then P is said to be a graded right ideal if $P = \bigoplus_{g\in G}(P \bigcap R_g)$, i.e, for $x \in P$, $x_g \in P$ for all $g \in G$. A right deal of a graded ring is not necessary to be graded right ideal, see ([3], Example 1).

A proper ideal P of a commutative ring R is said to be prime if $xy \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in R$. The concept of graded prime ideals over commutative graded rings was introduced in [8]. A proper graded ideal P of a commutative graded ring R is said to be graded prime if $xy \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in h(R)$. Assume that P is a prime ideal of a commutative ring R. If R is graded and P is a graded ideal of R, then it will be easy to see that P is a graded prime ideal of R. On the other hand, if P is a graded prime ideal of R, then P is not necessary to be prime ideal of R, see the following example:

Example 1.1. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the graded ideal $P = 2R$ of R. We show that P is a graded prime ideal of R. Let $xy \in P$ for some $x, y \in h(R)$.*

Case (1): Assume that $x, y \in R_0$. In this case, $x, y \in \mathbb{Z}$ such that 2 divides xy, and then either 2 divides x or 2 divides y as 2 is a prime number, which implies that $x \in P$ or $y \in P$.

Case (2): Assume that $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that 2 divides $xy = -ab$, and then 2 divides a or 2 divides b in $\mathbb{Z}$, which implies that 2 divides $x = ia$ or 2 divides $y = ib$ in R. Then we have that $x \in P$ or $y \in P$.

Case (3): Assume that $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that 2 divides $xy = ixb$ in R, that is $ixb = 2(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$. Then we obtain $xb = 2\beta$, that is 2 divides xb in $\mathbb{Z}$, and again 2 divides x or 2 divides b, which implies that 2 divides x or 2 divides $y = ib$ in R. Thus, $x \in P$ or $y \in P$.

One can similarly show that $x \in P$ or $y \in P$ in other cases. So, P is a graded prime ideal of R. On the other hand, P is not a prime ideal of R since $(1 - i)(1 + i) \in P$, $(1 - i) \notin P$ and $(1 + i) \notin P$.

Almost prime ideals over commutative rings have been appeared for the first time in [5]. A proper ideal P of a commutative ring R is said to be almost prime if $xy \in P - P^2$ implies $x \in P$

or $y \in P$ for all $x, y \in R$. The concept of graded almost prime ideals over commutative graded rings was introduced in [6]. A proper graded ideal P of a commutative graded ring R is said to be graded almost prime if $xy \in P - P^2$ implies $x \in P$ or $y \in P$ for all $x, y \in h(R)$. Assume that P is an almost prime ideal of a commutative ring R. If R is graded and P is a graded ideal of R, then it will be easy to see that P is a graded almost prime ideal of R. On the other hand, if P is a graded almost prime ideal of R, then P is not necessary to be almost prime ideal of R, see the following example:

Example 1.2. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the graded ideal $P = 2R$ of R. Clearly, P is a graded almost prime ideal of R since it is a graded prime ideal by Example 1.1. Indeed, P is not an almost prime ideal of R since $(1-i)(1+i) \in P - P^2$, $(1-i) \notin P$ and $(1+i) \notin P$.*

Clearly, every graded prime ideal over a commutative graded ring is graded almost prime. However, the next example shows that converse is not true in general:

Example 1.3. *Consider $R = \mathbb{Z}_{12}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_{12}$ and $R_1 = i\mathbb{Z}_{12}$. Consider the graded ideal $P = 4R$ of R. Clearly, P is a graded almost prime ideal of R since $P^2 = P$. Indeed, P is not a graded prime ideal of R since $2.2 \in P$ and $2 \notin P$.*

Graded prime ideals over non-commutative graded rings have been defined and examined in [3]. A proper graded right ideal P of R is called graded prime if $XY \subseteq P$ implies $X \subseteq P$ or $Y \subseteq P$ for all graded right ideals X, Y of R. If R has unity, then the previous definition is equivalent to the following: a graded right ideal P of R is called graded prime if $xRy \subseteq P$ implies $x \in P$ or $y \in P$ for all $x, y \in h(R)$. Graded weakly prime ideals over non-commutative graded rings have been introduced and investigated in [4]. A proper graded right ideal P of R is called graded weakly prime if $0 \neq XY \subseteq P$ implies $X \subseteq P$ or $Y \subseteq P$, for all graded right ideals X, Y of R. The purpose of this article is following [1] to define and examine graded almost prime ideals over a non-commutative graded ring. We propose the following definition: a proper graded right ideal P of R is called graded almost prime if $XY \subseteq P$ and $XY \nsubseteq P^2$ imply $X \subseteq P$ or $Y \subseteq P$ for all graded right ideals X, Y of R. A speedy attention provides that our definition and the notion of graded almost prime ideals over commutative graded rings with unity are equivalent. Nevertheless, the definitions disagree over non-commutative graded rings. We show in Theorem 2.6 that I is a graded almost prime ideal in a non-commutative graded ring R with unity if and only if $xRy \subseteq I$ and $xRy \nsubseteq I^2$ imply either $x \in I$ or $y \in I$ for all $x, y \in h(R)$. Among several results, we prove that if R is a graded ring with unity and P is a graded ideal of R, then P is a graded almost prime right ideal of R if and only if P is a graded almost prime ideal of R (Proposition 2.5). We show that if R is a graded ring with unity and I is a graded right ideal of R with $(I^2 : I) \subseteq I$, then I is a graded prime right ideal of R if and only if I is a graded almost prime right ideal of R (Theorem 2.7). We prove that if I is a graded ideal of R, then I is a graded almost prime right ideal of R if and only if I/I^2 is a graded weakly prime right ideal of R/I^2 (Theorem 2.11). In Theorem 2.12, Corollary 2.13, Theorem 2.14 and Corollary 2.15, we study graded almost prime right ideals over graded homomorphism. In Theorem 2.16, we examine graded almost prime right ideals over graded quotient rings. Finally, we consider some cases where all graded right ideals of a non-commutative graded ring are graded almost prime. Throughout this article all rings are associative, non-commutative, and without unity unless stated otherwise, and by ideal we mean a proper two sided ideal.

2 Graded Almost Prime Right Ideals

In this section, we introduce and study the concept of graded almost prime ideals over non-commutative graded rings.

Definition 2.1. *Let R be a graded ring and P be a proper graded right ideal of R. Then P is said to be graded almost prime if $XY \subseteq P$ and $XY \nsubseteq P^2$ imply $X \subseteq P$ or $Y \subseteq P$ for all graded right ideals X, Y of R.*

Clearly, every graded prime right ideal is graded almost prime. Indeed, $P = \{0\}$ is a graded almost prime right ideal in any graded ring R that is not necessary to be graded prime. Also, the next examples show that the converse is not true in general:

Example 2.2. *Consider the non-commutative ring $R = \{0, x, y, z\}$ under the operations:*

$$\begin{array}{ccccc} + & 0 & x & y & z \\ 0 & 0 & x & y & z \\ x & x & 0 & z & y \\ y & y & z & 0 & x \\ z & z & y & x & 0 \end{array} \qquad \begin{array}{ccccc} . & 0 & x & y & z \\ 0 & 0 & 0 & 0 & 0 \\ x & 0 & x & x & 0 \\ y & 0 & y & y & 0 \\ z & 0 & z & z & 0 \end{array}$$

The only additive subgroups of R are $\{0\}$, $\{0, x\}$, $\{0, y\}$ and $\{0, z\}$. Let R be G-graded for some group G and assume that $g \in G - \{e\}$. Then $R_g = \{0\}$, $\{0, x\}$, $\{0, y\}$ or $\{0, z\}$. If $R_g = \{0, x\}$, then $x = x.x \in R_gR_g \subseteq R_{g^2}$, and then $0 \neq x \in R_g \bigcap R_{g^2}$, which implies that $g = g^2$, and hence $g = e$, a contradiction. Similarly, if $R_g = \{0, y\}$, then $g = e$, a contradiction. Hence, for $g \neq e$, $R_g = \{0\}$ or $\{0, z\}$, and $R_e = \{0\}$, $\{0, x\}$ or $\{0, y\}$. So, if we take $G = \mathbb{Z}_3$, then we can choose R to be G-graded by $R_0 = \{0, x\}$, $R_1 = \{0, z\}$ and $R_2 = \{0\}$. The only right ideals of R are $\{0\}$, $P = \{0, x\}$, $I = \{0, y\}$ and $J = \{0, z\}$. Note that, I is not a graded right ideal of R since $y \in I$ and $y = y_0 + y_1$ with $y_0 = x, y_1 = z \notin I$. So, $\{0\}$, P and J are the only graded right ideals of R. Since $P^2 = P$, P is a graded almost prime right ideal of R. On the other hand, P is not a graded prime right ideal of R since $J.J = \{0\} \subseteq P$ and $J \nsubseteq P$.

As an application on Example 2.2, we introduce the following example:

Example 2.3. *Consider the subring of $M_2(\mathbb{Z}_2)$:*

$$R = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), A = \left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right), B = \left(\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right), C = \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \right\}$$

and $G = \mathbb{Z}_3$. Depending on the discussion in Example 2.2, R is G-graded by

$$R_0 = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), A \right\}, \ R_1 = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), C \right\} \text{ and } R_2 = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\}$$

with $P = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), A \right\}$ is a graded almost prime right ideal of R which is not graded prime.

Example 2.4. *Consider the ring with unity $R = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) : a, b, c \in \mathbb{R} \right\}$ and $G = \mathbb{Z}_4$. Then R is G-graded by*

$$R_0 = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & c \end{array} \right) : a, c \in \mathbb{R} \right\}, \ R_2 = \left\{ \left(\begin{array}{cc} 0 & b \\ 0 & 0 \end{array} \right) : b \in \mathbb{R} \right\} \text{ and } R_1 = R_3 = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\}.$$

Indeed, $P = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & c \end{array} \right) : c \in \mathbb{R} \right\}$ is a graded almost prime right ideal of R which is not graded prime since $A = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), B = \left(\begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array} \right) \in h(R)$ with $ARB = \left\{ \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right\} \subseteq P$ and $A, B \notin P$.

Proposition 2.5. *Let R be a graded ring with unity and P be a graded ideal of R. Then P is a graded almost prime right ideal of R if and only if P is a graded almost prime ideal of R.*

Proof. Suppose that P is a graded almost prime ideal of R. Let X, Y be two graded right ideals of R such that $XY \subseteq P$, and $XY \nsubseteq P^2$. Since R has unity, $XR = X$, and then $(RX)(RY) = RXY \subseteq RP = P$, where RX and RY are graded ideals of R. Further, if $(RX)(RY) \subseteq P^2$, then $XY \subseteq RXY = (RX)(RY) \subseteq P^2$ which is a contradiction. So, $(RX)(RY) \nsubseteq P^2$. Hence, since P is graded almost prime, we have either $X \subseteq RX \subseteq P$ or $Y \subseteq RY \subseteq P$. Thus, P is a graded almost prime right ideal of R. The converse is clear. □

Let R be a ring and $P, K \subseteq R$. Define $(P : K) = \{x \in R : Kx \subseteq P\}$ and $(P :_* K) = \{x \in R : xK \subseteq P\}$. Suppose that R is a graded ring and P, K are graded right ideals of R. Then using a similar proof to ([2], Lemma 2.19), one can prove that $(P : K)$ is a graded right ideal of R. Also similarly, If P and K are graded left ideals of R, then $(P :_* K)$ is a graded left ideal of R. Note that, if P and K are graded ideals of R, then so are $(P : K)$ and $(P :_* K)$.

Theorem 2.6. *Let R be a graded ring with unity and P be a graded ideal of R. Then the following assertions are equivalent:*

(i) P is a graded almost prime ideal of R.

(ii) If $x, y \in h(R)$ such that $\langle x\rangle\langle y\rangle \subseteq P$ and $\langle x\rangle\langle y\rangle \nsubseteq P^2$, then either $x \in P$ or $y \in P$.

(iii) If $x, y \in h(R)$ such that $xRy \subseteq P$ and $xRy \nsubseteq P^2$, then either $x \in P$ or $y \in P$.

(iv) $(P : \langle x\rangle) = P \bigcup (P^2 : \langle x\rangle)$, and $(P :_* \langle x\rangle) = P \bigcup (P^2 :_* \langle x\rangle)$ for all $x \in h(R) - P$.

(v) Either $(P : \langle x\rangle) = P$ or $(P : \langle x\rangle) = (P^2 : \langle x\rangle)$, and either $(P :_* \langle x\rangle) = P$ or $(P :_* \langle x\rangle) = (P^2 :_* \langle x\rangle)$ for all $x \in h(R) - P$.

Proof. (1) $\Rightarrow$ (2): Suppose that $x, y \in h(R)$ such that $\langle x\rangle\langle y\rangle \subseteq P$ and $\langle x\rangle\langle y\rangle \nsubseteq P^2$. Then $(RxR)(RyR) = R\langle x\rangle R\langle y\rangle = R\langle x\rangle\langle y\rangle \subseteq RP = P$. If $(RxR)(RyR) \subseteq P^2$, then $\langle x\rangle\langle y\rangle \subseteq R\langle x\rangle\langle y\rangle = (RxR)(RyR) \subseteq P^2$ which is a contradiction. Hence, $(RxR)(RyR) \nsubseteq P^2$. Thus, either $RxR \subseteq P$ or $RyR \subseteq P$ by (1), which implies that either $x \in P$ or $y \in P$.

(2) $\Rightarrow$ (3): Assume that $x, y \in h(R)$ such that $xRy \subseteq P$ and $xRy \nsubseteq P^2$. Then $\langle x\rangle\langle y\rangle = xRyR \subseteq PR = P$. If $\langle x\rangle\langle y\rangle \subseteq P^2$, then we get $xRy \subseteq xRyR = \langle x\rangle\langle y\rangle \subseteq P^2$, a contradiction. So, $\langle x\rangle\langle y\rangle \nsubseteq P^2$, and then either $x \in P$ or $y \in P$ by (2).

(3) $\Rightarrow$ (4): Let $x \in h(R) - P$ and $y \in (P : \langle x\rangle)$. Then $y_g \in (P : \langle x\rangle)$ for all $g \in G$, and then for $g \in G$, $xRy_g \subseteq \langle x\rangle y_g \subseteq P$. If $xRy_g \nsubseteq P^2$, then by (3), we have $y_g \in P$ for all $g \in G$ since $x \notin P$, and then $y \in P$. If $xRy_g \subseteq P^2$, then $\langle x\rangle y_g = RxRy_g \subseteq RP^2 = P^2$. Hence, $y_g \in (P^2 : \langle x\rangle)$ for all $g \in G$, which implies that $y \in (P^2 : \langle x\rangle)$. Therefore, $(P : \langle x\rangle) \subseteq P \bigcup (P^2 : \langle x\rangle)$. Let $y \in P \bigcup (P^2 : \langle x\rangle)$. If $y \in P$, then $y_g \in P$ for all $g \in G$, and then for $g \in G$, $\langle x\rangle y_g \subseteq \langle x\rangle P \subseteq P$, and thus $y_g \in (P : \langle x\rangle)$ for all $g \in G$, that gives $y \in (P : \langle x\rangle)$. If $y \in (P^2 : \langle x\rangle)$, then $y_g \in (P^2 : \langle x\rangle)$ for all $g \in G$, and then for $g \in G$, $\langle x\rangle y_g \subseteq P^2 \subseteq P$, which implies that $y_g \in (P : \langle x\rangle)$ for all $g \in G$, which yields that $y \in (P : \langle x\rangle)$. Thus, $P \bigcup (P^2 : \langle x\rangle) \subseteq (P : \langle x\rangle)$, and consequently, $(P : \langle x\rangle) = P \bigcup (P^2 : \langle x\rangle)$. Similarly, one can prove that $(P :_* \langle x\rangle) = P \bigcup (P^2 :_* \langle x\rangle)$.

(4) $\Rightarrow$ (5): It is understandable.

(5) $\Rightarrow$ (1): Let X and Y be two graded ideals of R such that $XY \subseteq P$. Suppose that $X \nsubseteq P$ and $Y \nsubseteq P$. We show that $XY \subseteq P^2$. Firstly, we show that $(X - P)Y \subseteq P^2$. Let $x \in X - P$. Then we have $\langle x\rangle Y \subseteq XY \subseteq P$, which implies that $Y \subseteq (P : \langle x\rangle)$, and then by assumption, we have $Y \subseteq (P^2 : \langle x\rangle)$ since $Y \nsubseteq P$. Therefore, $xY \subseteq \langle x\rangle Y \subseteq P^2$. Consequently, $(X - P)Y \subseteq P^2$. Secondly, we show that $X(Y - P) \subseteq P^2$. Let $y \in Y - P$. Then, $X\langle y\rangle \subseteq XY \subseteq P$, and so $X \subseteq (P :_* \langle y\rangle)$, and then by assumption we get $X \subseteq (P^2 :_* \langle y\rangle)$ since $X \nsubseteq P$. Thus, $Xy \subseteq X\langle y\rangle \subseteq P^2$, consequently $X(Y - P) \subseteq P^2$. The final step that completes the proof is observing that

$$XY = (X - P)Y + (X \bigcap P)(Y - P) + (X \bigcap P)(Y \bigcap P) \subseteq$$
$$(X - P)Y + X(Y - P) + (X \bigcap P)(Y \bigcap P) \subseteq P^2.$$

□

Examples 2.2, 2.3 and 2.4 show that a graded almost prime right ideal does not have to be a graded prime right ideal. In the next result, we consider a case such that the concepts above are corresponding.

Theorem 2.7. *Let R be a graded ring with unity and I be a graded right ideal of R with $(I^2 : I) \subseteq I$. Then I is a graded prime right ideal of R if and only if I is a graded almost prime right ideal of R.*

Proof. Suppose that I is a graded almost prime right ideal of R which is not graded prime. Then there exist graded right ideals X, Y of R such that $XY \subseteq I$ with $X \nsubseteq I$ and $Y \nsubseteq I$, and then $XY \subseteq I^2$. Let $x \in X - I$ and $y \in Y - I$. Then there exist $g, h \in G$ such that $x_g \in X - I$ and $y_h \in Y - I$, and then $(x_gR + I)y_hR = x_gRy_hR + Iy_hR \subseteq XY + Iy_hR \subseteq I$.

If $(x_gR + I)y_hR \subseteq I^2$, then $Iy_hR \subseteq I^2$. This implies that $Iy_h \subseteq I^2$, and thus $y_h \in (I^2 : I) \subseteq I$, a contradiction. Hence, either $x_gR + I \subseteq I$ or $y_hR \subseteq I$, which implies that $x_g \in I$ or $y_h \in I$, a contradiction. The converse is obvious. □

Theorem 2.8. *Let R be graded ring and I be a graded right ideal of R such that $I^2 = \{0\}$. Then I is a graded weakly prime right ideal of R if and only if I is a graded almost prime right ideal of R.*

Proof. It is understandable. □

Remark 2.9. *In Example 2.2, $J = \{0, z\}$ is a graded almost prime right ideal of R and $J^2 = \{0\}$. Hence, J is a graded weakly prime right ideal of R by Theorem 2.8.*

Corollary 2.10. *Let R be graded ring such that $R^2 = \{0\}$ and I be a graded right ideal of R. Then I is a graded weakly prime right ideal of R if and only if I is a graded almost prime right ideal of R.*

Proof. The result is a consequence of Theorem 2.8. □

Let R be a G-graded ring and P be a graded ideal of R. Then R/P is a G-graded ring by $(R/P)_g = (R_g + P)/P$ for all $g \in G$. It has been proved in ([6], Theorem 3) that if R is a commutative graded ring, then I is a graded almost prime ideal of R if and only if I/I^2 is a graded weakly prime ideal of R/I^2. In the next result, we prove that the same holds for graded almost prime right ideals and graded weakly prime right ideals in a non-commutative graded ring.

Theorem 2.11. *Let R be a graded ring and I be a graded ideal of R. Then I is a graded almost prime right ideal of R if and only if I/I^2 is a graded weakly prime right ideal of R/I^2.*

Proof. Suppose that I is a graded almost prime right ideal of R. Assume that $\overline{X}$ and $\overline{Y}$ are graded right ideals of R/I^2 such that $\overline{0} \neq \overline{XY} \subseteq \overline{I} = I/I^2$. Then there exist graded right ideals $X \supseteq I^2$ and $Y \supseteq I^2$ of R such that $\overline{X} = X/I^2$ and $\overline{Y} = Y/I^2$. Therefore, $I^2/I^2 \neq (XY + I^2)/I^2 \subseteq I/I^2$, hence $I^2 \neq XY \subseteq I$. So, we have that either $X \subseteq I$ or $Y \subseteq I$ since $XY \nsubseteq I^2$. This gives that either $\overline{X} \subseteq \overline{I}$ or $\overline{Y} \subseteq \overline{I}$. Conversely, suppose that X and Y are graded right ideals of R such that $XY \subseteq I$ and $XY \nsubseteq I^2$. Then $\overline{X} = (X + I^2)/I^2$ and $\overline{Y} = (Y + I^2)/I^2$ are graded right ideals of R/I^2. Moreover, $\overline{X}\,\overline{Y} = (XY + XI^2 + I^2Y + I^4 + I^2)/I^2 \subseteq I/I^2 = \overline{I}$, and $\overline{X}\,\overline{Y} \nsubseteq \overline{I^2}$. Thus, $\overline{0} \neq \overline{X}\,\overline{Y} \subseteq \overline{I}$ and hence either $\overline{X} \subseteq \overline{I}$ or $\overline{Y} \subseteq \overline{I}$. Consequently, $X \subseteq I$ or $Y \subseteq I$. □

Let R and T be two G-graded rings. Then a ring homomorphism $f : R \to T$ is said to be a graded homomorphism if $f(R_g) \subseteq T_g$ for all $g \in G$ [7].

Theorem 2.12. *Let $f : R \to T$ be a graded ring epimorphism and I be a graded almost prime right ideal of R such that $Ker(f) \subseteq I$. Then $f(I)$ is a graded almost prime right ideal of T.*

Proof. Suppose that $CD \subseteq f(I)$ and $CD \nsubseteq (f(I))^2$ for graded right ideals C and D of T. Then $Ker(f) \subseteq f^{-1}(C) = X$ and $Ker(f) \subseteq f^{-1}(D) = Y$ are graded right ideals of R. Hence, $f(X) = C$ and $f(Y) = D$ since f is an epimorphism, and then we have that $f(XY) = CD \subseteq f(I)$, and $f(XY) \nsubseteq (f(I))^2 = f(I^2)$. Thus $XY \subseteq f^{-1}(f(XY)) \subseteq f^{-1}(f(I)) = I$ and $XY \nsubseteq I^2$. So, either $X \subseteq I$ or $Y \subseteq I$, and so either $C \subseteq f(I)$ or $D \subseteq f(I)$. □

Corollary 2.13. *Let $f : R \to T$ be a graded ring epimorphism and J be a graded right ideal of T such that $f^{-1}(J)$ is a graded almost prime right ideal of R. Then J is a graded almost prime right ideal of T.*

Proof. Since $f^{-1}(J)$ is a graded right ideal of R and $Ker(f) \subseteq f^{-1}(J)$, the result follows by Theorem 2.12. □

Theorem 2.14. *Let $f : R \to T$ be a graded ring epimorphism and I be a graded right ideal of R such that $Ker(f) \subseteq I^2$. If $f(I)$ is a graded almost prime right ideal of T, then I is a graded almost prime right ideal of R.*

Proof. Suppose that $XY \subseteq I$ and $XY \nsubseteq I^2$ for graded right ideals X and Y of R. Then $f(X)f(Y) = f(XY) \subseteq f(I)$. If $f(XY) \subseteq f(I^2)$, then $XY \subseteq f^{-1}(f(XY)) \subseteq f^{-1}(f(I^2)) = I^2$, a contradiction. Hence, $f(X)f(Y) = f(XY) \nsubseteq (f(I))^2$. Since $f(I)$ is a graded almost prime right ideal of T, either $f(X) \subseteq f(I)$ or $f(Y) \subseteq f(I)$. Therefore, either $X \subseteq f^{-1}(f(X)) \subseteq f^{-1}(f(I)) = I$ or $Y \subseteq I$. □

Corollary 2.15. *Let $f : R \to T$ be a graded ring epimorphism and J be a graded almost prime right ideal of T such that $Ker(f) \subseteq (f^{-1}(J))^2$. Then $f^{-1}(J)$ is a graded almost prime right ideal of R.*

Proof. Assume that $I = f^{-1}(J)$. Then I is a graded almost prime right ideal of R by Theorem 2.14 since $Ker(f) \subseteq I^2$ and $f(I) = f(f^{-1}(J)) = J$ is a graded almost prime right ideal of T. □

Theorem 2.16. *Let R be a graded ring and K be a graded ideal of R. Suppose that P is a graded right ideal of R such that $K \subseteq P$. If P is a graded almost prime right ideal of R, then P/K is a graded almost prime right ideal of R/K.*

Proof. Suppose that $\overline{X}\,\overline{Y} \subseteq \overline{P} = P/K$ and $\overline{X}\,\overline{Y} \nsubseteq \overline{P}^2$ for graded right ideals $\overline{X}$ and $\overline{Y}$ of R/K. Assume that $\overline{X} = X/K$ and $\overline{Y} = Y/K$ for some graded right ideals $X \supseteq K$ and $Y \supseteq K$. Then $(XY + K)/K \subseteq P/K$ and $(XY + K)/K \nsubseteq (P^2 + K)/K$ which implies that $XY \subseteq P$ and $XY \nsubseteq P^2$. So, either $X \subseteq P$ or $Y \subseteq P$, and hence $\overline{X} \subseteq \overline{P}$ or $\overline{Y} \subseteq \overline{P}$. □

Remark 2.17. *One can choose a graded ring R and a graded ideal K of R that is not a graded almost prime right ideal of R. Indeed, $\overline{0} = K/K$ is a graded almost prime right ideal of R/K. So, the converse of Theorem 2.16 is not true in general.*

In Example 2.2, every graded right ideal of R is graded almost prime. In the rest of our article, we consider some cases where all graded right ideals of a non-commutative graded ring are graded almost prime. The next result is a repercussion of Theorem 2.8.

Corollary 2.18. *Let R be a graded ring such that $I^2 = \{0\}$ for every graded right ideal I of R. Then every graded right ideal of R is graded almost prime if and only if every graded right ideal of R is graded weakly prime.*

The next result is a consequence of Corollary 2.10.

Corollary 2.19. *Let R be a graded ring such that $R^2 = \{0\}$. Then every graded right ideal of R is graded almost prime if and only if every graded right ideal of R is graded weakly prime.*

Theorem 2.20. *Let $f : R \to T$ be a graded ring epimorphism. If every graded right ideal of R is graded almost prime, then so is T.*

Proof. Let I be a graded right ideal of T. Then $f^{-1}(I) \supseteq Ker(f)$ is a graded almost prime right ideal of R, and then by Theorem 2.12 we get that $f(f^{-1}(I)) = I$ is a graded almost prime right ideal of T. □

Theorem 2.21. *Let $f : R \to T$ be a graded ring epimorphism such that $Ker(f) \subseteq P^2$ for any graded right ideal P of R. If every graded right ideal of T is graded almost prime, then so is R.*

Proof. Let I be a graded right ideal of R. Then $f(I)$ is a graded almost prime right ideal of T, and then by Theorem 2.14 we get that I is a graded almost prime right ideal of R. □

Theorem 2.22. *Let R be a graded ring and P be a graded ideal of R. If every graded right ideal of R is graded almost prime, then so is R/P.*

Proof. Suppose that $\overline{I}$ is a graded right ideal of R/P. Then there exists a graded right ideal $I \supseteq P$ of R such that $\overline{I} = I/P$. Clearly, I is a graded almost prime right ideal of R. Hence, by Theorem 2.16 $\overline{I}$ is a graded almost prime right ideal of R/P. □

References

[1] A. Abouhalaka and S. Findik, Almost prime ideals in non-commutative rings, (2022), arXiv:2201.09758v1.

[2] R. Abu-Dawwas, On graded strongly 1-absorbing primary ideals, Khayyam Journal of Mathematics, 8 (1) (2022), 42-52.

[3] R. Abu-Dawwas, M. Bataineh and M. Al-Muanger, Graded prime submodules over non-commutative rings, Vietnam Journal of Mathematics, 46 (3) (2018), 681-692.

[4] A. S. Alshehry and R. Abu-Dawwas, Graded weakly prime ideals of non-commutative rings, Communications in Algebra, 49 (11) (2021), 4712-4723.

[5] M. S. Bhatwadekar, P. K. Sharma, Unique factorization and brith of almost prime, Communications in Algebra, 33 (1) (2005), 43-49.

[6] A. Jaber, M. Bataineh and H. Khashan, Almost graded prime ideals, Journal of Mathematics and Statistics, 4 (4) (2008), 232-236.

[7] C. Nastasescu and F. Oystaeyen, Methods of graded rings, 1836, Berlin, Springer-Verlag, 2004.

[8] M. Refai, M. Hailat and S. Obiedat, Graded radicals and graded prime spectra, Far East Journal of Mathematical Sciences, Special Vol. (2000), Part I, 59-73.

Author information

Jenan Shtayat, Department of Mathematics, Yarmouk University, 21163 Irbid, Jordan.
E-mail: `jenan@yu.edu.jo`

Rashid Abu-Dawwas, Department of Mathematics, Yarmouk University, 21163 Irbid, Jordan.
E-mail: `rrashid@yu.edu.jo`

Ghadeer Bani Issa, Department of Mathematics, Yarmouk University, 21163 Irbid, Jordan.
E-mail: `gadeerbanyissa5@gmail.com`

Received: 2022-04-20
Accepted: 2023-01-08