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The semiconformal curvature tensor on Relativistic spacetimes

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Abstract. The semiconformal curvature tensor has been studied for the spacetime of general relativity. It is shown that the energy-momentum tensor with divergence-free semiconformal curvature tensor is of Codazzi type, as well as the energy-momentum tensor of a spacetime having semi-symmetric semiconformal curvature tensor is also semi-symmetric. The semiconformal curvature tensor has also been expressed in terms of different tensors already known in the literature, and the relationship between their divergences has been established.

1 Introduction

The study of curvature properties play a significant role in differential geometry and General Relativity. G. P. Pokhariyal and others ([6]-[9]) have studied some curvature tensors and, their geometrical and, physical properties. Furthermore, some other authors ([2], [3], [4], [15], [16], [21]) studied these tensors in different ambient spaces. Recently, J. Kim ([11], [12]) introduces a new curvature tensor which is invariant under conharmonic transformation ([19]) and, this curvature-like tensor is called semiconformal curvature tensor denoted by $\mathcal{P}$. This $(0,4)$ type tensor on a Riemannian manifold is defined by

$$\mathcal{P}_{lbcd} = -(n-2)\alpha\mathcal{C}_{lbcd} + [\alpha + (n-2)\beta]\mathcal{Z}_{lbcd}, \tag{1.1}$$

where α and β are constants (not simultaneously zero). The Weyl conformal tensor denoted by $\mathcal{C}$ and conharmonic curvature tensors $\mathcal{Z}$ are respectively defined by

$$\mathcal{W}_{lbcd} = \mathcal{R}_{lbcd} - \frac{1}{n-2}(g_{ld}\mathcal{R}_{bc} - g_{lc}\mathcal{R}_{bd} + g_{bc}\mathcal{R}_{ld} - g_{bd}\mathcal{R}_{lc}) + \frac{\mathcal{R}}{(n-1)(n-2)}(g_{ld}g_{bc} - g_{lc}g_{bd}), \tag{1.2}$$

and

$$\mathcal{Z}_{lbcd} = \mathcal{R}_{lbcd} - \frac{1}{n-2}(g_{ld}\mathcal{R}_{bc} - g_{lc}\mathcal{R}_{bd} + g_{bc}\mathcal{R}_{ld} - g_{bd}\mathcal{R}_{lc}), \tag{1.3}$$

Making use of (1.2) and (1.3), Equation (1.1) may be expressed as (for $n = 4$).

$$\mathcal{P}_{lbcd} = \alpha\big[\mathcal{R}_{lbcd} - \frac{1}{n-2}(g_{ld}\mathcal{R}_{bc} - g_{lc}\mathcal{R}_{bd} + g_{bc}\mathcal{R}_{ld} - g_{bd}\mathcal{R}_{lc})\big] - \frac{\beta\mathcal{R}}{3}(g_{ld}g_{bc} - g_{lc}g_{bd}). \tag{1.4}$$

It is clear that from the Equation (1.4), we can obtain conformal and conharmonic curvature tensors for certain values of α and β, with some conditions on them. It is evident from Equation (1.4), that the semiconformal curvature tesnor satisfies the following properties

$$\begin{cases} \mathcal{P}_{lbcd} = -\mathcal{P}_{blcd} \\ \mathcal{P}_{lbcd} = -\mathcal{P}_{lbdc} \\ \mathcal{P}_{lbcd} = \mathcal{P}_{cdlb} \\ \mathcal{P}_{lbcd} + \mathcal{P}_{bcld} + \mathcal{P}_{clbd} = 0 \end{cases} \tag{1.5}$$

A Riemannian manifold M is semi-symmetric ([1], [17], [23]), if

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{R}_{lbcd} = 0, \tag{1.6}$$

and Ricci semi-symmetric, if

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{R}ic = 0. \tag{1.7}$$

A semi-symmetric manifold implies a Ricci semi-symmetric but converse need not to be true in general. Hassan Abu-Donia et al. ([10]) studied the W^*-curvature tensor on relativistic space-times. They investigated that the energy-momentum tensor of a spacetime with divergence-free W^*-curvature tensor is of Codazzi type whereas the energy-momentum tensor of a spacetime having a semi-symmetric W^*-curvature tensor is semi-symmetric. Motivated by the above, we have defined a semi-symmetric semiconformal curvature tensor as

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{P}_{lbcd} = 0. \tag{1.8}$$

This paper is organised as follows: Section 2 divided in, two subsections 2.1 and 2.2. In 2.1, we study semiconformally semi-symmetric spacetime and in 2.2, we study semiconformally symmetric spacetime. It is found that the divergence of the semiconformal curvature tensor and other curvature tensor are proportional or identical under certain conditions even though the algebraic properties of these tensors (projective, conformal, concircular and conharmonic curvature tensors) are different, this study we have arranged in Section 3 with four subsections.

2 Semiconformally semi-symmetric and symmetric spacetimes

2.1 Semiconformally semi-symmetric spacetimes

A four dimensional spacetime manifold M is said to admit a semi-symmetric semiconformal curvature tensor if

$$\mathcal{R}(X,Y).\mathcal{P} = 0, \tag{2.1}$$

where $\mathcal{R}(X,Y)$ act as a derivation on the tensor P. In local coordinate system we may get

$$\begin{aligned}(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{P}_{lbcd} &= \alpha(\nabla_i\nabla_j - \nabla_j\nabla_i)\big[\mathcal{R}_{lbcd} - \frac{1}{n-2}(g_{ld}\mathcal{R}_{bc} - g_{lc}\mathcal{R}_{bd} \\ &+ g_{bc}\mathcal{R}_{ld} - g_{bd}\mathcal{R}_{lc})\big].\end{aligned} \tag{2.2}$$

Here, $\mathcal{R} = 0$ for semi-symmetric spacetime.
Contracting both sides with g^{ld} in Equation (2.2), we get

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{P}_{bc} = 2\alpha(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{R}_{bc}, \tag{2.3}$$

where $\mathcal{P}_{bc} = g^{ld}\mathcal{P}_{lbcd}$.
Thus, we may state

Theorem 2.1. *A spacetime manifold M is Ricci semi-symmetric if and only if $\mathcal{P}_{bc} = g^{ld}\mathcal{P}_{lbcd}$ is semi-symmetric.*

The following result is direct consequence of this theorem.

Corollary 2.2. *A spacetime manifold M is Ricci semi-symmetric if $\mathcal{P}$-curvature is semi-symmetric.*

A spacetime manifold is conformally and conharmonically semi-symmetric if the conformal and conharmonic curvature tensors are semi-symmetric respectively, that is,

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{W}_{lbcd} = 0, \tag{2.4}$$

and

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{Z}_{lbcd} = 0 \tag{2.5}$$

In view of Equations (1.1), (2.4) and (2.5), we may state the following

Theorem 2.3. *Assume that a spacetime manifold M is admitting a semi-symmetric $\mathcal{P}_{bc} = g^{ld}\mathcal{P}_{lbcd}$. Then M is conharmonically semi-symmetric if and only if M is conformally semi-symmetric, provided, $\alpha + 2\beta \neq 0$.*

The Einstein field equation with cosmological terms

$$\mathcal{R}_{bc} - \frac{1}{2} g_{bc}\mathcal{R} + g_{bc}\wedge = k\mathcal{T}_{bc}, \tag{2.6}$$

where $\mathcal{R}$, k and, $\wedge$ are the scalar curvature, the gravitational constant and, cosmological constant respectively. Then

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{R}_{bc} = k(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{T}_{bc} \tag{2.7}$$

From the Equations (2.3) and, (2.7), we may establish the following

Theorem 2.4. *The energy-momentum tensor is semi-symmetric if and only if* $\mathcal{P}_{bc} = g^{ld}\mathcal{P}_{lbcd}$ *is semi-symmetric.*

De and Velimirovic ([18]) studied a spacetime M with semi-symmetric energy-momentum tensor. It is clear that $\nabla_i\mathcal{P}_{lbcd} = 0$ implies $(\nabla_i\nabla_j - \nabla_j\nabla_i)\mathcal{P}_{lbcd} = 0$. Thus we may state the following result

Corollary 2.5. *Let* M *be a spacetime with covaraintly constant* $\mathcal{P}$ *– curvature tensor. Then* M *is semiconformally semi-symmetric and the energy-momentum tensor is semi-symmetric.*

2.2 Semiconformally symmetric spacetime

A spacetime manifold M admits $\mathcal{P}$-symmetric, if the covariant derivative of semiconformal curvature tensor is zero, i.e.,

$$\nabla_e\mathcal{P}_{lbcd} = 0. \tag{2.8}$$

Now, the covariant form of Equation (1.4), may take the following form

$$\begin{aligned} \nabla_e\mathcal{P}_{lbcd} &= \alpha\big[\nabla_e\mathcal{R}_{lbcd} - \frac{1}{2}(g_{ld}\nabla_e\mathcal{R}_{bc} - g_{lc}\nabla_e\mathcal{R}_{bd} + g_{bc}\nabla_e\mathcal{R}_{ld} \\ &- g_{bd}\nabla_e\mathcal{R}_{lc})\big] - \frac{\beta\nabla_e\mathcal{R}}{3}(g_{ld}g_{bc} - g_{lc}g_{bd}). \end{aligned} \tag{2.9}$$

If the spacetime manifold M is $\mathcal{P}$-symmetric, then

$$\begin{aligned} -\alpha\nabla_e\mathcal{R}_{lbcd} &= -\frac{\alpha}{2}(g_{ld}\nabla_e\mathcal{R}_{bc} - g_{lc}\nabla_e\mathcal{R}_{bd} + g_{bc}\nabla_e\mathcal{R}_{ld} - g_{bd}\nabla_e\mathcal{R}_{lc}) \\ &- \frac{\beta\nabla_e\mathcal{R}}{3}(g_{ld}g_{bc} - g_{lc}g_{bd}). \end{aligned} \tag{2.10}$$

Multiplying by g^{ld} to the above equation, one gets

$$2\alpha\nabla_e\mathcal{R}_{bc} = -\left(\frac{\alpha + 2\beta}{2}\right) g_{bc}\nabla_e\mathcal{R}. \tag{2.11}$$

Thus, we may establishe the following theorem.

Theorem 2.6. *The spacetime manifold* M *is Ricci symmetric if the scalar curvature tensor is covariantly constant for a* $\mathcal{P}$*-symmetric spacetime, provided* $(\alpha + \beta) \neq 0$.

3 Semiconformal curvature tensor and other cuvature tensors

In this section, we will represent semiconformal cuvature tensor with regard to projective, conformal, conharmonic and concircular curvature tensors, and obtain the relationships between the divergences of semiconformal curvature tensor with these curvature tensors. The semiconformal curvature tensor is defined in the section 1 [Equation (1.1)] for the spacetime of general relativity.

3.1 Projective curvature tensor

For a Riemannian space V_4, the projective curvature tensor $\mathcal{W}^h_{bcd}$, defined as ([20])

$$\mathcal{W}^h_{bcd} = \mathcal{R}^h_{bcd} - \frac{1}{3}(\delta^h_d \mathcal{R}_{bc} - \delta^h_c \mathcal{R}_{bd}). \tag{3.1}$$

The projective curvature tensor vanishes, if we take its contraction over h and d. The covariant form of the projective curvature tensor is

$$\mathcal{W}_{abcd} = \mathcal{R}_{abcd} - \frac{1}{3}(\mathcal{R}_{bc} g_{ad} - \mathcal{R}_{bd} g_{ac}). \tag{3.2}$$

It's covariant derivative is

$$\nabla_e \mathcal{W}^h_{bcd} = \nabla_e \mathcal{R}^h_{bcd} - \frac{1}{3}(\delta^h_d \nabla_e \mathcal{R}_{bc} - \delta^h_c \nabla_e \mathcal{R}_{bd}), \tag{3.3}$$

and the divergence of this tensor is ([5])

$$\nabla_h \mathcal{W}^h_{bcd} = \nabla_h \mathcal{R}^h_{bcd} - \frac{1}{3}(\nabla_d \mathcal{R}_{bc} - \nabla_c \mathcal{R}_{bd}). \tag{3.4}$$

Using the Equations (2.9) and (3.4), we have

$$\begin{aligned} \nabla_h \mathcal{P}^h_{bcd} &= \alpha[\nabla_h \mathcal{W}^h_{bcd} + \frac{1}{6}(\nabla_c \mathcal{R}_{bd} - \nabla_d \mathcal{R}_{bc}) + \frac{1}{2}(g_{bd}\nabla_c \mathcal{R} - g_{bc}\nabla_d \mathcal{R})] \\ &- \frac{\beta}{3}(g_{bc}\nabla_d \mathcal{R} - g_{bd}\nabla_c \mathcal{R}). \end{aligned} \tag{3.5}$$

Put $\mathcal{R}_{bc} = \frac{\mathcal{R}}{4} g_{bc}$, (for Einstein spaces the scalar curvature is constant i.e. $\nabla_d \mathcal{R} = 0$).

Thus we state

Theorem 3.1. *For Einstein spaces, the divergences of semiconformal curvature tensor and projective curvature tensor are related through* $\nabla_h \mathcal{P}^h_{bcd} = \alpha \mathcal{W}^h_{bcd}$.

Corollary 3.2. *For Einstein spaces, the divergences of semiconformal and projective curvature tensors are identical if* $\alpha = 1$.

Writing Equation (3.5) as

$$\nabla_h \mathcal{P}^h_{bcd} = \alpha \nabla_h \mathcal{W}^h_{bcd} + \frac{\alpha}{6}(\nabla_c \mathcal{R}_{bd} - \nabla_d \mathcal{R}_{bc}) + \left(\frac{3\alpha + 2\beta}{6}\right)(g_{bd}\nabla_h \mathcal{R}^h_c - g_{bc}\nabla_h \mathcal{R}^h_d) \tag{3.6}$$

and we thus have

Theorem 3.3. *For a Riemannian space, the divergences of semiconformal and projective curvature tensors are proportional if the Ricci curvature tensor is divergence-free and is of Codazzi type.*

Corollary 3.4. *For a Riemannian space with* α*=1, the divergences of semiconformal and projective curvature tensors are identical, if the Ricci curvature tensor is divergence-free and is of Codazzi type.*

3.2 Conformal curvature tensor

For a Riemannian space, the conformal curvature tensor is defined in section 2 (equation (2.2)). In Einstein space from equation (2.1), the semiconformal curvature tensor in terms of conformal curvature tensor can be expressed as

$$\mathcal{P}^h_{bcd} = -2\beta \mathcal{C}^h_{bcd} + (\alpha + 2\beta)[\mathcal{R}^h_{bcd} + \frac{1}{4}(\delta^h_c \mathcal{R} g_{bd} - \delta^h_d \mathcal{R} g_{bc})], \tag{3.7}$$

so that the divergence of the semiconformal curvature tensor, equation (3.7) takes the form

$$\nabla_h \mathcal{P}^h_{bcd} = -2\beta\nabla_h \mathcal{C}^h_{bcd} + (\alpha+2\beta)[\nabla_h \mathcal{R}^h_{bcd} + \frac{1}{4}(g_{bd}\nabla_c \mathcal{R} - g_{bc}\nabla_d \mathcal{R})]. \tag{3.8}$$

We know that the Bianchi identity is given by

$$\nabla_h \mathcal{R}^h_{bcd} + \nabla_c \mathcal{R}^h_{bdh} + \nabla_d \mathcal{R}^h_{bhc} = 0,$$

or

$$\nabla_h \mathcal{R}^h_{bcd} = \nabla_d \mathcal{R}_{bc} - \nabla_c \mathcal{R}_{bd}, \tag{3.9}$$

Using Equation (3.9) and conditions for Einstein space, Equation (3.8) leads to

$$\nabla_h \mathcal{P}^h_{bcd} = -2\beta\nabla_h \mathcal{C}^h_{bcd} \tag{3.10}$$

Thus we have

Theorem 3.5. *In an Einstein space, the divergences of semiconformal and conformal curvature tensors are proportional.*

Corollary 3.6. *For an Einstein space, the divergences of two tensors are identical if* $\beta = -\frac{1}{2}$.

Now using the equations (3.8) and (3.9), we have

$$\nabla_h \mathcal{P}^h_{bcd} = -2\beta\nabla_h \mathcal{C}^h_{bcd} + (\alpha+2\beta)[(\nabla_d \mathcal{R}_{bc} - \nabla_c \mathcal{R}_{bd}) + \frac{1}{4}(g_{bd}\nabla_h \mathcal{R}^h_c - g_{bc}\nabla_h \mathcal{R}^h_d)], \tag{3.11}$$

However, for a semi-Riemannian space it is seen that ([20], [22])

$$2\nabla_h \mathcal{C}^h_{bcd} = \nabla_c \mathcal{R}_{bd} - \nabla_d \mathcal{R}_{bc}. \tag{3.12}$$

Now from the equations (3.11) and (3.12), we get

$$\nabla_h \mathcal{P}^h_{bcd} = 2(\alpha+\beta)\nabla_h \mathcal{C}^h_{bcd} + \left(\frac{\alpha+2\beta}{4}\right)(g_{bd}\nabla_h \mathcal{R}^h_c - g_{bc}\nabla_h \mathcal{R}^h_d) \tag{3.13}$$

From Equation (3.13), we may state

Theorem 3.7. *In a semi-Riemannian space, the divergences of conformal curvature tensor and semiconformal curvature tensors are connected through the following relation*

$$\nabla_h \mathcal{P}^h_{bcd} = 2(\alpha+\beta)\nabla_h \mathcal{C}^h_{bcd},$$

in case, the Ricci tensor is divergence-free.

Corollary 3.8. *For a semi-Riemannian space, the divergences of two tensors semiconformal and conformal curvature tensor are identical if* $\alpha+\beta = \frac{1}{2}$*, and Ricci tensor is divergence-free.*

Equation (3.11) will also lead to the result

Theorem 3.9. *For a Riemannian space, the divergences of semiconformal and conformal curvature tensor are proportional if Ricci tensor is divergence-free and, is of a Codazzi type.*

Corollary 3.10. *For a Riemannian space with* $\beta = -\frac{1}{2}$*, the divergences of semiconformal and conformal curvature tensor are identical if Ricci tensor is divergence-free and, is of a Codazzi type*

3.3 Conharmonic curvature tensor

We have discussed about the conharmonic curvature tensor in section 1.
Now from equations (1.1) and (1.2), we have

$$\begin{aligned}\mathcal{P}^h_{bcd} &= (\alpha+2\beta)\mathcal{Z}^h_{bcd} - 2\beta\big[\mathcal{R}^h_{bcd} + \frac{1}{2}(\delta^h_c\mathcal{R}_{bd} - \delta^h_d\mathcal{R}_{bc} + g_{bd}\mathcal{R}^h_c - g_{bc}\mathcal{R}^h_d) \\ &+ \frac{\mathcal{R}}{6}(\delta^h_d g_{bc} - \delta^h_c g_{bd})\big]. \end{aligned} \tag{3.14}$$

Operating the covariant derivative on Equation (3.14), we have

$$\begin{aligned}\nabla_e\mathcal{P}^h_{bcd} &= (\alpha+2\beta)\nabla_e\mathcal{Z}^h_{bcd} - 2\beta\big[\nabla_e\mathcal{R}^h_{bcd} + \frac{1}{2}(\delta^h_c\nabla_e\mathcal{R}_{bd} - \delta^h_d\nabla_e\mathcal{R}_{bc} \\ &+ g_{bd}\nabla_e\mathcal{R}^h_c - g_{bc}\nabla_e\mathcal{R}^h_d) + \frac{\nabla_e\mathcal{R}}{6}(\delta^h_d g_{bc} - \delta^h_c g_{bd})\big], \end{aligned} \tag{3.15}$$

on contraction over h and e, equation (3.15) leads to

$$\begin{aligned}\nabla_h\mathcal{P}^h_{bcd} &= (\alpha+2\beta)\nabla_h\mathcal{Z}^h_{bcd} - 2\beta\big[\nabla_h\mathcal{R}^h_{bcd} + \frac{1}{2}(\nabla_c\mathcal{R}_{bd} - \nabla_d\mathcal{R}_{bc}) \\ &+ \frac{1}{3}(g_{bd}\nabla_h\mathcal{R}^h_c - g_{bc}\nabla_h\mathcal{R}^h_d)\big]. \end{aligned} \tag{3.16}$$

Using the Equation (3.9), Equation (3.16) takes the form

$$\nabla_h\mathcal{P}^h_{bcd} = (\alpha+2\beta)\nabla_h\mathcal{Z}^h_{bcd} + \beta(\nabla_c\mathcal{R}_{bd} - \nabla_d\mathcal{R}_{bc}) + \frac{2\beta}{3}(g_{bc}\nabla_h\mathcal{R}^h_d - g_{bd}\nabla_h\mathcal{R}^h_c). \tag{3.17}$$

Thus we have

Theorem 3.11. *In a spacetime, the divergences of semiconformal and conharmonic curvature tensors are connected through the following relation*

$$\nabla_h\mathcal{P}^h_{bcd} = (\alpha+2\beta)\nabla_h\mathcal{Z}^h_{bcd}$$

if the Ricci tensor is divergence-free and, is of a Codazzi type.

Corollary 3.12. *In a spacetime V_4 with $\alpha+2\beta=1$, the divergences of semiconformal and conharmonic curvature tensors are identical, if the Ricci tensor is divergence-free and, is of a Codazzi type.*

Also from equation (3.17), we have

Theorem 3.13. *For Einstein spaces, the divergences of semiconformal and conharmonic curvature tensors are proportional.*

Corollary 3.14. *For Einstein spaces with $\alpha+2\beta=1$, the divergences of semiconformal and conharmonic curvature tensors are identical.*

3.4 Concircular curvature tensor

Generally, we cannot transform a geodesic circle into a geodesic circle by the conformal transformation

$$\tilde{g_{ab}} = \Psi^2 g_{ab}, \tag{3.18}$$

where g_{ab} is the first fundamental tensor. K. Yano ([13], [14]) introduced a transformation that preserves the geodesic circles. The conformal transformation (3.18) satisfying the equation

$$\Psi_{;a;b} = \zeta g_{ab} \tag{3.19}$$

transforms a geodesic circle into a geodesic circle and corresponding transformation is concircular transformation.

A $(0, 4)$ type tensor M_{abcd} that remains invariant under such transformation, for a Riemannian space V_4, is given by ([14], [20])

$$\mathcal{M}_{abcd} = \mathcal{R}_{abcd} - \frac{\mathcal{R}}{12}(g_{bc}g_{ad} - g_{bd}g_{ac}) \tag{3.20}$$

Also

$$\mathcal{M}^h_{bcd} = \mathcal{R}^h_{bcd} - \frac{\mathcal{R}}{12}(\delta^h_d g_{bc} - \delta^h_c g_{bd}). \tag{3.21}$$

The tensor $\mathcal{M}_{abcd}$ or $\mathcal{M}^h_{bcd}$ defined by means of Equation (3.20) or (3.21) is known as concircular curvature tensor.
Contraction over h and d, Equation (3.21) takes the form

$$\mathcal{M}_{bc} = \mathcal{R}_{bc} - \frac{\mathcal{R}}{4}g_{bc}, \tag{3.22}$$

which is also invariant under conharmonic transformation and it may be noted that $g^{bc}\mathcal{M}_{bc} = 0$.

From equations (1.4) and (3.21), we have

$$\mathcal{P}^h_{bcd} = \alpha\mathcal{M}^h_{bcd} + \left(\frac{\alpha - 4\beta}{12}\right)(\delta^h_d\mathcal{R}g_{bc} - \delta^h_c\mathcal{R}g_{bd}) + \frac{\alpha}{2}(\delta^h_c\mathcal{R}_{bd} - \delta^h_d\mathcal{R}_{bc} + g_{bd}\mathcal{R}^h_c - g_{bc}\mathcal{R}^h_d). \tag{3.23}$$

On applying the covariant derivative, Equation (3.23) leads to

$$\begin{aligned}\mathcal{P}^h_{bcd;e} = \alpha\mathcal{M}^h_{bcd;e} + \left(\frac{\alpha - 4\beta}{12}\right)(\delta^h_d\mathcal{R}_{;e}g_{bc} - \delta^h_c\mathcal{R}_{;e}g_{bd}) + \tfrac{\alpha}{2}(\delta^h_c\mathcal{R}_{bd;e} - \delta^h_d\mathcal{R}_{bc;e} \\ + g_{bd}\mathcal{R}^h_{c;e} - g_{bc}\mathcal{R}^h_{d;e}),\end{aligned} \tag{3.24}$$

contraction over h and e, caries the Equation (3.24) in the form

$$\mathcal{R}^h_{bcd;h} = \alpha\mathcal{M}^h_{bcd;h} - \left(\frac{5\alpha + 4\beta}{12}\right)(g_{bc}\mathcal{R}_{,d} - g_{bd}\mathcal{R}_{,c}) + \frac{\alpha}{2}(\mathcal{R}_{bd;c} - \mathcal{R}_{bc;d}). \tag{3.25}$$

We thus have

Theorem 3.15. *For an Einstein space, the divergences of semiconformal and concircular curvature tensor are proportional.*

Corollary 3.16. *For an Einstein space with $\alpha = 1$, the divergences of two tensors $\mathcal{P}^h_{bcd}$ and $\mathcal{M}^h_{bcd}$ are identical.*

Also from Equation (3.25), we have

$$\mathcal{P}^h_{bcd;h} = \alpha\mathcal{M}^h_{bcd;h} - \left(\frac{5\alpha + 4\beta}{12}\right)(g_{bc}\mathcal{R}^h_{d;h} - g_{bd}\mathcal{R}^h_{c;h}) + \frac{\alpha}{2}(\mathcal{R}_{bd;c} - \mathcal{R}_{bc;d}). \tag{3.26}$$

Thus we have

Theorem 3.17. *The divergences of semiconformal and concircular curvature tensors are proportional if the Ricci Tensor is divergence-free and is of a Codazzi type.*

Corollary 3.18. *If $\alpha = 1$ and when the Ricci tensor is divergence-free as well as Codazzi type the divergences of semiconformal and concircular curvature tensors become identical.*

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