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A new generalization of Hopfian modules

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Abstract In this paper, we introduce the notion of μ-weakly Hopfian modules which is a new generalization of Hopfian modules. It is shown that over a ring R, every quasi-projective (projective, free) R-module is μ-weakly Hopfian if and only if R has no nonzero semisimple injective R-module. Some basic characterizations of projective μ-weakly Hopfian modules are proved. We demonstrate that if the ACC holds on μ-small submodules of an R-module M, then M is μ-weakly Hopfian. Other properties of μ-weakly Hopfian modules are also obtained with examples.

1 Introduction

Throughout this paper all rings have identity and all modules are unital right modules. We will use the notations $\leq$, $\ll$, $\ll_\mu$ and $\leq^\oplus$ to denote submodule, small submodule, μ-small submodule and direct summand, respectively, and $\text{Rad}(M)$, $Z^*(M)$, $E(M)$, $\text{End}_R(M)$ will denote the radical, the cosingular submodule, the injective hull, and the ring of endomorphisms of an R-module M.

Recall that a submodule K of an R-module M is said to be small in M, if for each $L \leq M$ such that $K + L = M$ implies $L = M$. A submodule P of an R-module M is said to be δ-small in M ($P \ll_\delta M$), if for every submodule N of M such that $P + N = M$ with M/N singular implies $N = M$ (see [12]). For a right R-module M , Ozcan [8], defined the submodule $Z^*(M) = \{m \in M : mR \ll E(M)\}$ as a dual of singular submodule. A module M is called cosingular, (resp, noncosingular) if $Z^*(M) = M$ $(resp, Z^*(M) = 0)$. It is clear that $\text{Rad}(M) \leq Z^*(M)$. A submodule K of an R-module M is said to be μ-small in M, if for all $L \leq M$ such that $K + L = M$ and M/L cosingular implies $M = L$ ([11]). It is clear that if A is a small submodule of M, then A is a μ-small submodule of M, but the converse is not true in general.

In [6], Hiremath introduced the concept of Hopfian modules. An R-module M is said to be Hopfian if any surjective endomorphism of M is an automorphism. In [9], Varadarajan investigated and studied the notion of co-Hopfian modules. An R-module M is said to be co-Hopfian if every injective endomorphism of M is an automorphism. In [5], Ghorbani and Haghany introduced the concept of generalized Hopfian modules. A right R-module M is called generalized Hopfian if every surjective endomorphism of M has a small kernel. In [10], Wang studied the notion of weakly Hopfian modules. A right R-module M is called weakly Hopfian if any small surjective endomorphism of M is an automorphism. In [3], we studied the concept of μ-Hopfian modules. A right R-module M is said to be μ-Hopfian if every surjective endomorphism of M has a μ-small kernel. In [2], the concept of δ-weakly Hopfian modules was introduced and studied. A right R-module M is called δ-weakly Hopfian if every δ-small surjective endomorphism of M is an automorphism. Such modules and other generalizations have been examined by many authors ([2, 3, 4, 5, 6, 9, 10]).

By works mentioned we are motivated to introduce in this paper the notion of μ-weakly Hopfian modules which is a proper generalizations of that of Hopfian modules. We call a module μ-weakly Hopfian if every its μ-small surjective endomorphism is an automorphism, which implies that a right R-module M is Hopfian if and only if M is both μ-Hopfian and μ-weakly

Hopfian.

The paper is organized as follows:

In Section 2, we show that if M is a quasi-projective cosingular module then it is μ-weakly Hopfian (Proposition 2.3). A submodule N of an R-module M is said to be fully invariant if $f(N) \subseteq N$ for every endomorphism f of M. We obtain that if M is a quasi-projective cosingular module and if N is a fully invariant μ-small submodule of M, then M/N is μ-weakly Hopfian (Corollary 2.7).

In Section 3, some basic characterizations of projective μ-weakly Hopfian modules are proved in (Theorem 3.4). It is proved that a projective module M is μ-weakly Hopfian if and only if whenever $f \in \text{End}_R(M)$ has a right inverse and $Ker(f)$ is semisimple, then f has a left inverse in $\text{End}_R(M)$. Moreover, we show that every quasi-projective (projective, free) R-module is μ-weakly Hopfian if and only if R has no nonzero semisimple injective R-module (Theorem 3.5).

At the end of the paper, some open problems are given.

We list some properties of cosingular modules that will be used in the paper.

Lemma 1.1. *[8]*

(i) For any ring R, the class of cosingular R-modules is closed under submodules, homomorphic images and direct sums but not (in general) under essential extensions or extensions.

(ii) Let R be a right cosingular ring. Then any (right) R-module is cosingular.

Now we list some properties of μ-small submodules that will be utilized in the paper.

Lemma 1.2. *[3] Let M be an R-module and $N \leq M$. The following are equivalent.*

(i) $N \ll_\mu M$.

(ii) If $X + N = M$, then $X \oplus L = M$ for a semisimple injective submodule L of M.

Lemma 1.3. *[11]. Let $M = M_1 \oplus M_2$ be an R-module and let $A_1 \leq M_1$ and $A_2 \leq M_2$, then $A_1 \oplus A_2 \ll_\mu M_1 \oplus M_2$ if and only if $A_1 \ll_\mu M_1$ and $A_2 \ll_\mu M_2$.*

Definition 1.4. [7] Let M be an R-module. We say that M is duo module provided every submodule of M is fully invariant.

Definition 1.5. [1]. A module M is called semi Hopfian if any surjective endomorphism of M has a direct summand kernel, i.e. any surjective endomorphism of M splits.

Example 1.6. .

(i) Every semisimple R-module is semi Hopfian. [1]

(ii) Every quasi-projective R-module is semi Hopfian. [1]

(iii) By [6, Theorem 16(ii)], a vector space V over a field K is Hopfian if and only if it is finite dimensional. Hence an infinite-dimensional vector space over a field is semi Hopfian, but it is not Hopfian. [4]

(iv) Every module with D2 is semi Hopfian. (Recall that a module M has D2 if any submodule N of M such that M/N is isomorphic to a direct summand of M is a direct summand of M). [1]

(v) Every semi Hopfian indecomposable R-module is Hopfian. [4]

(vi) Every semi Hopfian co-Hopfian R-module is Hopfian. [4]

2 μ-weakly Hopfian

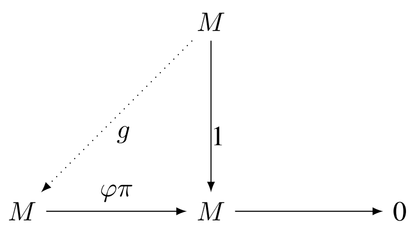
Definition 2.1. Let M be an R-module. We say that M is μ-weakly Hopfian if any μ-small surjective endomorphism of M is an automorphism.

The following example introduces a module that is not μ-weakly Hopfian.

Example 2.2. Let $G = Z_{p^\infty}$. Since in G every proper subgroup is μ-small, hence every its surjective endomorphism has a μ-small kernel. But the multiplication by p induces an epimorphism of G which is not an isomorphism.

Proposition 2.3. *Let M be a quasi-projective module. If M is cosingular, then it is μ-weakly Hopfian.*

Proof. $\diamond$ Let M be a quasi-projective cosingular module. Suppose $M \cong M/K$ for some $K \ll_\mu M$. Let $\varphi : M/K \to M$ be an isomorphism. The map $\varphi\pi : M \to M$, where $\pi : M \to M/K$ is a canonical epimorphism with kernel K i.e. $Ker(\varphi\pi) = K$. Since M is quasi-projective, there is $g : M \to M$ which makes the following diagram commutative.



Thus, $M = Ker(\varphi\pi) \oplus Im(g)$. Since M is cosingular, by lemma 1.1 $M/Im(g)$ is cosingular. Now since $Ker(\varphi\pi) = K \ll_\mu M$, $K = 0$. Therefore M is μ-weakly Hopfian by lemma 3.1. $\square$

$\square$

Proposition 2.4. *Let M be an R-module and N a nonzero μ-small submodule of M. If M/N is μ-weakly Hopfian, then M is μ-weakly Hopfian.*

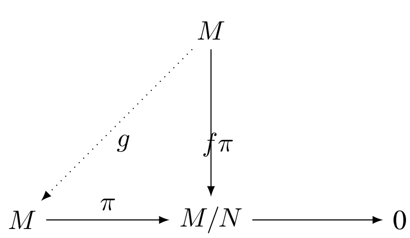
Proof. If M is not μ-weakly Hopfian. Then there exists a μ-small surjection f of M which is not an isomorphism, and f induces an isomorphism $g : M/Kerf \to M$. If $\pi : M \to M/Kerf$ denotes the canonical quotient morphism, then $\pi g : M/Kerf \to M/Kerf$ is a μ-small surjection which is not an isomorphism. This is a contradiction. $\square$

$\square$

Example 2.5. Let P be a set of all primes and $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$. If $\bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$ is a μ-weakly Hofian $\mathbb{Z}$-module, hence $\mathbb{Z}_{p^\infty}$ is μ-weakly Hofian by Proposition 2.11, contradiction with example 2.2. Then $\mathbb{Q}/\mathbb{Z}$ is not μ-weakly Hopfian, but $\mathbb{Q}$ is a μ-weakly Hofian $\mathbb{Z}$-module.

Theorem 2.6. *Let M be a quasi-projective cosingular module. If N is a fully invariant μ-small submodule of M, then M/N is Hopfian.*

Proof. Let M be a quasi-projective cosingular module and N a fully invariant μ-small submodule of M. If $f : M/N \to M/N$ is an epimorphism, then by the canonical epimorphism $\pi : M \to M/N$, we have $f\pi : M \to M/N$ is an epimorphism. Since M is quasi-projective, there exists an endomorphism g of M which makes the following diagram commutative.



i.e., $\pi g = f\pi$, then $N + Img = M$. Therefore g is onto, hence g is an isomorphism. We have $f(x + N) = f\pi(x) = \pi g(x) = g(x) + N$, and $Kerf = K/N$ where $N \subset K = \{x \in M; g(x) \in N\} = g^{-1}(N) \subset M$. Since N is a fully invariant submodule of M, $g^{-1}(N) \subset N$. Hence $Kerf = g^{-1}(N)/N = 0$. Therefore M/N is Hopfian.

$\square$

□

Corollary 2.7. *Let M be a quasi-projective cosingular module. If N is a fully invariant μ-small submodule of M, then M/N is μ-weakly Hopfian.*

Corollary 2.8. *Let M be a finitely generated quasi-projective module. If M is cosingular, then $M/Rad(M)$ is μ-weakly Hopfian.*

Proof. $Rad(M)$ is a fully invariant submodule of M. Since M is finitely generated, $Rad(M)$ is small in M, then it is μ-small. Thus $M/Rad(M)$ is μ-weakly Hopfian, by corollary 2.7. □

Proposition 2.9. *Let M be a semi Hopfian R-module. If M is co-Hopfian, then it is μ-weakly Hopfian.*

Proof. Let $f : M \to M$ be a μ-small surjective endomorphism. Since M is a semi Hopfian R-module, f splits, and hence there exists an endomorphism $g : M \to M$, such that $fg = 1$. This implies that g is an injective endomorphism. Now since M is co-Hopfian, g is an automorphism. Therefore f is an automorphism and M becomes a μ-weakly Hopfian R-module. □

Corollary 2.10. *(i) Let M be an R-module with D2. If M is co-Hopfian, then it is μ-weakly Hopfian.*

(ii) Every semisimple co-Hopfian R-module is μ-weakly Hopfian.

(iii) Every quasi-projective co-Hopfian R-module is μ-weakly Hopfian.

Proposition 2.11. *Any direct summand of a μ-weakly Hopfian module M is μ-weakly Hopfian.*

Proof. Let $K \leq^{\oplus} M$, $\exists N \leq M$ such that $M = K \oplus N$. Let $f : K \to K$ be a μ-small surjective endomorphism, then f induces a surjective endomorphism of M, $f \oplus 1_N : M \to M$ with $(f \oplus 1_N)(k+n) = f(k) + n$, where $k \in K$ and $n \in N$. Thus by lemma 1.3, $Ker(f \oplus 1_N) = Ker(f) \oplus 0 \ll_\mu K \oplus N$. Since M is μ-weakly Hopfian, $f \oplus 1_N$ is an automorphism of M. Hence f is an automorphism of K. Therefore K becomes μ-weakly Hopfian. □

The next result gives a condition that a direct sum of two μ-weakly Hopfian modules is μ-weakly Hopfian.

Proposition 2.12. *Let $M = M_1 \oplus M_2$ and let M_1, M_2 be invariant submodules under any surjection of M. Then M is μ-weakly Hopfian if and only if M_1, M_2 are μ-weakly Hopfian.*

Proof. $\Rightarrow$) Clear by Proposition 2.11.

$\Leftarrow$) Let $f : M \to M$ be a μ-small epimorphism, then $f|_{M_i} : M_i \to M_i$ is a μ-small surjection where $i \in \{1; 2\}$. By assumption, $f|_{M_i}$ is an automorphism. Let $f(m_1 + m_2) = 0$, then $f(m_1) + f(m_2) = 0$ and so $m_1 = m_2 = 0$. Thus f is an injective endomorphism and M is μ-weakly Hopfian. □

Corollary 2.13. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is μ-weakly Hopfian if and only if M_1 and M_2 are μ-weakly Hopfian.*

It is clear that every μ-weakly Hopfian module is weakly Hopfian. The following examples shows that the converse is not true, in general.

Example 2.14. If M is a noncosingular semisimple R-module, since every nonozero homomorphic image of M is noncosingular, then every submodule of M is μ-small. Hence M is not μ-weakly Hopfian. But the only small submodule of M is the zero submodule. Thus M is weakly Hopfian.

Lemma 2.15. *Let M, N and L be modules. If $f : M \to N$ and $g : N \to L$ are two μ-small epimorphisms. Then gf is a μ-small epimorphism.*

Proof. Suppose that $Kergf + K = M$ with M/K is cosingular, then $Kerg + f(K) = f(M)$. Since M/K is cosingular, $f(M)/f(K)$ is cosingular. Now since $Kerg \ll_\mu f(M) = N$, $f(M) = f(K)$ and $M = Kerf + K$. As $Kerf \ll_\mu M$ and M/K is cosingular, $M = K$. Thus gf is a μ-small epimorphism.

□

Theorem 2.16. *Let M be an R-module with ACC on μ-small submodules. Then M is μ-weakly Hopfian.*

Proof. Let M be an R-module and $f : M \to M$ be a μ-small epimorphism of M. Then $Kerf \subseteq Kerf^2 \subseteq ... \subseteq Kerf^n \subseteq ...$ is an ascending chain of μ-small submodules of M by Lemma 2.15. Since M satisfies the ACC on μ-small submodules, there exists a positive number n such that $Kerf^n = Kerf^{n+1}$. Let $x \in Kerf$, then $f(x) = 0$. Since f is an epimorphism, there exists $x_1 \in M$ such that $f(x_1) = x$. Since f is an epimorphism, there exists $x_2 \in M$ such that $f(x_2) = x_1$. Repeating the process, we obtain that $x_{n-1} \in M$ with $f(x_n) = x_{n-1}$. Thus

$$x = f(x_1) = f^2(x_2) = ... = f^n(x_n).$$

Since $x \in Kerf$, $0 = f(x) = f(f^n(x_n))$, that is, $f^{n+1}(x_n) = 0$. So $x_n \in Kerf^{n+1} = Kerf^n$. Consequently, $f^n(x_n) = 0$, hence $x = 0$, thus $Kerf = 0$ and f is an isomorphism. Then M is μ-weakly Hopfian. □

3 Characterizations the class of rings R for which every R-module is μ-weakly Hopfian

Lemma 3.1. *Let M be a nonzero R-module. Then the following statements are equivalent.*
(i) M is μ-weakly Hopfian.
(ii) $M/K \cong M$ for every $K \ll_\mu M$ if and only if $K = 0$.

Proof. (i) $\Rightarrow$ (ii) Assume that $M \cong M/K$ for any $K \ll_\mu M$. Let $\varphi : M/K \to M$ be an isomorphism and $\pi : M \to M/K$ the canonical epimorphism. Then $\varphi\pi$ is an epimorphism with $Ker(\varphi\pi) = K$. Hence $\varphi\pi$ is a μ-small epimorphism. Then $\varphi\pi$ is an isomorphism by (i). Therefore $K = 0$.

(ii) $\Rightarrow$ (i) Let $f : M \to M$ be a μ-small epimorphism. Then $M \cong M/Ker(f)$ by the first isomorphism theorem. From (ii), we find $Ker(f) = 0$. This prove that f is an isomorphism. Therefore M is μ-weakly Hopfian. □

Proposition 3.2. *Let M be a μ-weakly Hopfian module. If $M \cong M \oplus N$ for some injective semisimple module N, then $N = 0$. More if M is projective, then we have the converse.*

Proof. Let M be a μ-weakly Hopfian module and $M \cong M \oplus N$ for some injective semisimple module N. It is easy to see that $M \cong K \oplus L$ where $K \cong N$ and $L \cong M$. Note that K is a μ-small submodule of M as N is semisimple injective by Lemma 1.2. Since $M/K \cong L \cong M$, $K = 0$ by Lemma 3.1.

For the moreover statement, assume that M is projective and f is a surjective endomorphisme of M, where $Ker(f) \ll_\mu M$. Then $M = Kerf \oplus T$, where $T \leq M$ and $T \cong M$. Since $Ker(f) \ll_\mu M$, we have $M = H \oplus T$ where H is an injective semisimple submodule of $Ker(f)$, by Lemma 1.2. Now, modular law implies that $Ker(f) = H$. Therefore $M \cong Ker(f) \oplus M$ and $Kerf$ is semisimple injective. Hence $Ker(f) = 0$ and M becomes μ-weakly Hopfian. □

Proposition 3.3. *Let R be a semisimple artinian ring. Then a free R-module F is μ-weakly Hopfian if and only if it has finite rank.*

Proof. Let F be a free μ-weakly Hopfian R-module. If F has infinite rank, then $R^{\mathbb{N}}$ is μ-weakly Hopfian (because $R^{\mathbb{N}}$ is a direct summand of F). Since $R^{\mathbb{N}} \cong R^{\mathbb{N}} \oplus R^{\mathbb{N}}$ and $R^{\mathbb{N}} \neq 0$, it is impossible, by Proposition 3.2. Hence F has finite rank. Conversely, If F has finite rank, then it is Hopfian and so it is μ-weakly Hopfian. □

In the following, we present some basic characterizations of projective μ-weakly Hopfian modules.

Theorem 3.4. *Let M be a projective R-module and $f \in \mathrm{End}_R(M)$. Then the following statements are equivalent:*

(i) M is μ-weakly Hopfian.

(ii) If f has a right inverse and $Ker(f)$ is semisimple injective, then f has a left inverse in $\text{End}_R(M)$.

(iii) If f has a right inverse and $Ker(f) \ll_\mu M$, then f has a left inverse in $\text{End}_R(M)$.

(iv) If f has a right inverse g and $(1-gf)M \ll_\mu M$, then f has a left inverse in $\text{End}_R(M)$.

(v) If f is a surjective endomorphism and $Ker(f)$ is semisimple injective, then f has a left inverse in $\text{End}_R(M)$.

Proof. If M be a projective module and $f \in \text{End}_R(M)$, then f is a surjective endomorphism if and only if f has a right inverse g. Therefore $Ker(f) = (1-gf)M$ and $M = Ker(f) \oplus (gf)M$.

(1) $\Rightarrow$ (2) If f has a right inverse g, then $fg = 1$. Since $Ker(f) \leq^{\oplus} M$, it is projective. Then $M \cong M \oplus Kerf$ where $Kerf$ is semisimple injective. Now by Proposition 3.2, $Ker(f) = 0$.

(2) $\Rightarrow$ (3) If f has a right inverse and $Ker(f) \ll_\mu M$. Since $Ker(f) \leq^{\oplus} M$, $Ker(f)$ is semisimple injective. Therefore f has a left inverse in $\text{End}_R(M)$.

(3) $\Rightarrow$ (4) It is clear, because $Ker(f) = (1-gf)M \ll_\mu M$

(4) $\Rightarrow$ (5) It is clear, because $Ker(f) = (1-gf)M \ll_\mu M$ if and only if $Ker(f)$ is semisimple injective.

(5) $\Rightarrow$ (1) Let f be a surjective endomorphism of M and $Ker(f) \ll_\mu M$. Since M is projective, f has a right inverse g and $Ker(f) = (1-gf)M \leq^{\oplus} M$. Hence $Ker(f)$ is semisimple injective. Therefore by (5), f has a left inverse and it is an automorphism.

□

Theorem 3.5. *Let R be a ring. Then the following statements are equivalent:*

(i) Every quasi-projective R-module is μ-weakly Hopfian.

(ii) Every projective R-module is μ-weakly Hopfian.

(iii) Every free R-module is μ-weakly Hopfian.

(iv) Every minimal right ideal of R is small in R_R.

(v) R has no nonzero semisimple injective R-module.

Proof. (1) $\Rightarrow$ (2) Is clear.

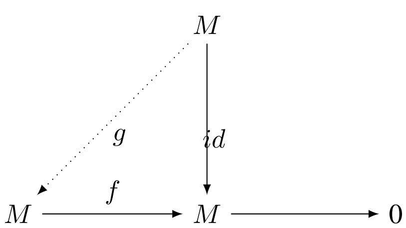
(2) $\Rightarrow$ (3) Is clear.

(3) $\Rightarrow$ (2) Is clear by Proposition 2.11

(2) $\Rightarrow$ (4) Assume that m is a minimal right ideal of R. Then either m is a direct summand of R_R or it is small in R_R. If m is a direct summand of R_R, then $M = (R/m)^{(\mathbb{N})}$ is semisimple. Hence R is semisimple. Then M is injective projective. Therefore M is μ-weakly Hopfian by (2). Since $M \cong M \oplus M$, $M = 0$, by Proposition 3.2, which is impossible, and so m is small in R_R.

(4) $\Rightarrow$ (5) Is clear.

(5) $\Rightarrow$ (1) Assume that M is a quasi-projective module and $f : M \to M$ is an epimorphism where $Ker(f) \ll_\mu M$. Since M is quasi-projective, there existe an endomorphism g of M which makes the following diagram commutative.



Therefore, $fg = id$ and $M = Kerf \oplus Img$. As $Ker(f) \ll_\mu M$, $M = N \oplus Img$, for some semisimple injective submodule N of $Ker(f)$, by Lemma 1.2. Then by modular law $Ker(f) = N \oplus (Im(g) \cap Ker(f)) = N$. Since R has no nonzero semisimple injective R-module, $N = 0$, hence $Kerf = 0$. Therefore f is an automorphism and M becomes μ-weakly Hopfian.

□

4 Conclusion

In this paper the notion of μ-weakly Hopfian modules are present. The relation between the class of μ-weakly Hopfian and other classes of Hopfian modules are given. Some basic characterizations of μ-weakly Hopfian modules are proved. And some other properties of μ-weakly Hopfian modules are also obtained with examples.

For further studies we shall be interested in the following problems:

- What is the structure of rings whose finitely generated right modules are μ-weakly Hopfian?
- Let R be a ring with identity, and M be a μ-weakly Hopfian module. Is $M[X, X^{-1}]$ μ-weakly Hopfian in $R[X, X^{-1}]$-module?
- Let R be a μ-weakly Hopfian ring and $n \geq 1$ an integer. Is the matrix ring $M_n(R)$ μ-weakly Hopfian?

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