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Rota-Baxter Operators on Complex Semi-simple Algebras

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Abstract This paper studies Rota-Baxter (RB) operators in complex semi-simple algebras A. They are certain $\mathbb{C}$-linear endomorphisms of A, when considered the latter as a $\mathbb{C}$-vector space. Properties as the nilpotency or the spectrum of such operator R are studied. Some examples are given when A is a Clifford algebra.

1 Introduction

Given an $F-$algebra A and a scalar $\lambda \in F$, where F is a field with characteristic different from two. A linear operator $R : A \longrightarrow A$ is called an **Rota-Baxter** operator (RB operator, shortly) on A of weight λ if the following relation

$$R(x)R(y) - \lambda xy = R(R(x)y + xR(y)), \tag{1.1}$$

holds for all $x, y \in A$.

Rota-Baxter operators or relations were introduced by G.Baxter [2]. It was introduced in order to solve certain analytic and combinatorial problems and then applied to many fields in mathematics and mathematical physics. In fact, the Rota-Baxter relation (1.1) generalizes the integration-by-parts formula.

The Lie algebraic version of equation (1.1) with $\lambda = 0$ is just the well-known classical Yang-Baxter equation which plays an important role in intergrable systems and when the weight $\lambda = 1$, it corresponds to the operator form of the modified classical Yang-Baxter equation.

There is a big difference between RB operators of nonzero weight and RB operators of weight zero. There are few known general constructions for the first ones such as splitting and triangular-splitting RB operators. In contrast, there are a lot of examples for the second ones, and it is not clear which of them are of most interest.

Also, RB operators have been studied by their own interest. In [4] it was proved that any RB operator of nonzero weight on an odd-dimensional simple Jordan algebra J of bilinear form is splitting. RB operators were classified on $sl_2(\mathbb{C})$ [10, 11], $M_2(\mathbb{C})$ [4], $sl_3(\mathbb{C})$ [11], the Grassmann algebra Gr2 [4, 8], the 3-dimensional simple Jordan algebra of bilinear form, the Kaplansky superalgebra K_3 [4], 3-dimensional solvable Lie algebras [11], low-dimensional Lie superalgebras [9], low-dimensional pre-Lie (super)algebras [1], low-dimensional semigroup algebras [7]. The classification of RB perators of special kind on polynomials, power series and Witt algebra was found [5, 6].

An application of Rota-Baxter (associative) algebras is to get some new algebraic structures. In fact, if the weight $\lambda = 0$, then from equations (1.1), it is obvious that the product

$$x * y := [R(x), y] = R(x)y - yR(x)$$

defines a pre-Lie algebra.

Throughout of the paper, without special saying, all algebras are considered associative with unit e.

By the trivial RB operators of weight λ, we mean the zero operator and -λid, where id denotes the identity map.

In this paper, we study Rota-Baxter operators on complex semi-simple algebras A: We prove that any RB operator of zero weight of A is nilpotent. Any eigenvector $x \in A$ of a RB operator

R of weight -1 with $R(x) = \lambda x$ where $\lambda \notin \{0, 1\}$ (if any) is nilpotent. When A is simple, then $\{0, 1\}$ is a subset of the spectrum of any non-trivial RB operator of weight -1. Some examples of RB operator of weight -1 on A are given, when A is a Clifford algebra.

The paper is organized as follows. After introduction, preliminary Section 2 consists to recall some basic results of RB operators. In section 3, we study Rota Baxter operators of zero weight on A. In section 4, we study eigenvalues and eigenvectors of RB operators of weight -1. In section 5, we make some examples of RB operators when A is a simple algebra, in particular, if A is a Clifford algebra.

2 PRELIMINARIES

We start by collecting some examples and some general properties of RB operators for later use.

We need recall that: An element t of A is said to be an idempotent element if $t^2 = t$. Two idempotent elements f and h such that $hf = fh = 0$ are called orthogonal. A non-zero idempotent element of A is said to be primitive if it is not a sum of two non-zero orthogonal idempotent elements.

Example 2.1.
1. Given an algebra A of continuous functions on $\mathbb{R}$, an integration operator $R(f)(x) := \int_0^x f$ is an RB operator on A of weight zero.
2. consider the algebra of sequences in a F-algebra, with componentwise addition and multiplication. Define an operator
$$R : (a_1, a_2, a_3, ..., a_n, ...) \longmapsto (0, a_1, a_1 + a_2, ..., \sum_{k<n} a_k, ...).$$
R is a Rota-Baxter operator of weight 1.
3. A linear map R_a on the polynomial algebra $F[x]$ defined as $R(x^n) = \frac{(x^{n+1} - a^{n+1})}{n+1}$ is an RB operator on $F[x]$ of weight zero, for any $a \in F$.

Proposition 2.2. *[4] Let A be an associative unital algebra.*

1. *Let R be an RB operator on A of weight λ. Then $-R - \lambda id$ is an RB operator of weight λ and the operator $\lambda^{-1}R$ is an RB operator of weight 1, provided that $\lambda \neq 0$.*
2. *Let R be an RB operator of weight λ on A, and let $\psi \in Aut(A)$. Then, $R_\psi = \psi^{-1}R\psi$ is an RB operator on A of weight λ. The same result is true when ψ is an antiautomorphism of A, i.e., a bijection from A to A satisfying $\psi(xy) = \psi(y)\psi(x)$ for all $x, y \in A$*

Proposition 2.3. *Let A be an algebra and $R : A \longrightarrow A$ be a linear isomorphism. Then R is a RB operator of weight zero on A if and only if R^{-1} is a derivation on A.*

Proof. For any $x, y \in A$, R is a RB operator on A if and only if $R(x)R(y) = R(R(x)y+xR(y))$, which is equivalent to $R^{-1}(uv) = uR^{-1}(v) + R^{-1}(u)v$, where $u = R(x), v = R(y)$. Therefore, the conclusion follows. □

Proposition 2.4. *Let $t \in A$ such that $t^2 = -\lambda t$ with $\lambda \in \mathbb{C}$. The linear map $R_t : x \longmapsto xt$ is an RB operator on A of weight λ.*

Proof. It follows, easily, from relation (1.1). □

Remark 2.5. A particular case of the previous proposition is when $t^2 = t$, (t is an idempotent element of A, the RB operator R_t is a projector of A; that is R_t satisfying the relation $R_t^2 = R_t$. In the same way, the map $L_t : x \longmapsto tx$ is an RB operator. It is the same for the composition map $L_t o R_t : x \longmapsto txt$.
If t and t' are two orthogonal idempotent elements of A then, the sum $R_t + R_{t'} = R_{t+t'}$ is an RB operator of A.

Proposition 2.6. *Let R be an RB operator on A of weight zero. Then*

1. *$e \notin Im(R)$.*
2. *If $R(e) \in \mathbb{C}e$, then $R(e) = 0$, and $R^2 = 0$.*
3. *$Ker(R^k)$ is an $Im(R^k)$-module for all $k \geq 1$.*

Proof.

1. Assume there exists $x \in A$ such that $R(x) = e$. We have $e = ee = R(x)R(x) = R(R(x)x + xR(x)) = 2R(x) = 2e$, a contradiction.
2. From (1), $R(e) = 0$. The other statement follows from $0 = R(e)R(x) = R(R(e)x + eR(x)) = R(R(x))$.
3. Let $x \in Ker(R)$ and $y = R(z) \in Im(R)$. We have $0 = R(x)R(z) = R(R(x)z + xR(z)) = R(xy) = R(yx)$. Then, $yx \in Ker(R)$ and so, $Ker(R)$ is an $Im(R)$-module. Suppose we have proved the result for all numbers less or equal to k. Let $x \in Ker(R^{k+1})$ and $y = R(z) \in Im(R)$, we get $R^{k+1}(xR^{k+1}(z)) = R^k(R(xR^{k+1}(z))) = R^k(R(x)R^k(y) - R(R(x)R^k(z)))$. By the induction hypothesis, $R(x)R^k(y), R(x)R^k(z) \in Ker(R^k)$. So, $R^{k+1}(xR^{k+1}(z)) = 0$ and so, the result is holds for $k + 1$. □

Corollary 2.7. *(i) There is no bijective derivation on A.*

(ii) Let R an RB operator R on A of weight zero, and let t be a non zero idempotent element of A. Then, $t \notin Im(R^k) \cap Ker(R)$, for $k \geq 2$. Moreover, if t is primitive and if $t = R(x) \in Im(R)$ then, either $txt = 0$ or $t \in Im(R) \cap Ker(R)$.

Proof.

(i) According to Proposition 2.3 and the first point of the previous proposition.

(ii) Suppose that $t \in Im(R^2) \cap Ker(R)$, i.e., $t = R(x)$ for some $x \in Im(R)$. Then $t = R(x) = R(x)R(x) = R(R(x)x + xR(x)) = R(tx + xt) = 0$. It is a contradiction. The same proof for $k \geq 2$. The second point follows from the previous proposition (4). □

Remark 2.8. If $y = R(x) \in Im(R)$ is an invertible element of A, where R is an RB operator of zero weight on A and $x \in A$, then $R(x)^{-1} \notin Im(R)$. In particular, $R(x)^{-1} \neq R(x^{-1})$, when x is an invertible element of A.

From now on, A denotes a complex semi-simple algebra with dimension $N \geq 1$.

3 Rota-Baxter operators of weight zero

In order to study the nilpoency of RB operators of zero weight of A, we use its semi-simplicity properties and the algebraic closure of $\mathbb{C}$. Let us start by giving some elementary properties that we can use thereafter.

Proposition 3.1. *Let R be an RB operator on A of weight zero. Then $Im(R)$ is a subalgebra of A. Moreover, if $Im(R)^2 = \{0\}$ then, R is nilpotent.*

Proof. Let $x = R(u), y = R(v) \in Im(R)$. Then $xy = R(u)R(v) = R(R(u)v + uR(v)) \in Im(R)$. So, $Im(R)$ is a subalgebra of A. On the other hand, it is easy to see that 0 is the only eigenvalue of R. Indeed: Suppose that x is an eigenvector of R with nonzero eigenvalue λ. We have

$$0 = R(x)R(e) = R(R(x) + xR(e)) = R[\frac{1}{\lambda}R(e)R(x) + R(x)] = R^2(x) = \lambda^2 x,$$

a contradiction. Hence, R is nilpotent. □

Proposition 3.2. *Let R be an RB operator on A of weight zero. Then*

1. *$R^k(e) = \frac{1}{k!}(R(e))^k$, for all integer k.*
2. *If $R(e) \neq 0$ then, $R(e)$ is a divisor of zero in A.*

Proof.

1. By induction on $k \in \mathbb{N}$. For $k = 0, 1$, it is true. Suppose that the statement holds true for all numbers less or equal to k. Using (1.1), we have

$$\begin{aligned} R^{k+1}(e) &= R(R^k(e)) = R[eR^k(e) + R(e)R^{k-1}(e) - R(e)R^{k-1}(e)] \\ &= R^k(e)R(e) - R(R(e)R^{k-1}(e)) \\ &= \tfrac{1}{k!}R(e)^{k+1} - \tfrac{k!}{(k-1)!}R^{k+1}(e); \text{ (by induction hypothesis).} \end{aligned}$$

Finally, we have $R^{k+1}(e) = \frac{1}{(k+1)!}R(e)^{k+1}$.

2. For dimensional-reason, there exists a non zero polynomial $P = \sum_{k=0}^{r} a_k X^k \in \mathbb{C}[X]$, of minimal degree $r \geq 1$, such that $P(R(e)) = 0$. By Proposition 2.6 (1), $a_0 = 0$. Hence, $R(e)[Q(R(e))] = 0$, where $Q = \sum_{k=1}^{r} a_k X^{k-1}$. By minimality of r, we have $Q(R(e)) \neq 0$ and we are done. □

Corollary 3.3. *Let R be an RB operator on A of weight zero. Then, $R(e)$ is nilpotent.*

Proof. Assume that $R(e)$ is not nilpotent. Then, there exists a non zero polynomial $P = a_1X + a_2X^2 + ... + X^r \in \mathbb{C}[X]$ of minimal degree $r \geq 1$, such that $P(R(e)) = 0 : (*)$. Using the previous proposition (first point), we have

$$(r+1)R(P(R(e))) - R(e)P(R(e)) = \frac{r-1}{2}a_1R(e)^2 + \cdots + \frac{1}{r}a_{r-1}R(e)^r = 0 : (**).$$

By minimality of r, we have $a_{r-1} \neq 0$ and hence, $(*) - \frac{r}{a_{r-1}}(**) = 0$. Again, the minimality of r implies $a_1 = ... = a_{r-1} = 0$, a contradiction. □

Lemma 3.4. *Let R be an RB operator on A of weight zero. Then,*

(i) $R^l(e)R^k(e) = \frac{(l+k)!}{k!l!}R^{(l+k)}(e)$, *for all* $l, k \in \mathbb{N}$.

(ii)

$$R(x)R^m(e) = \sum_{k=0}^{m} R^{m+1-k}\left(xR^k(e)\right) \tag{3.1}$$

For all $x \in A$ and for all $m \in \mathbb{N}$.

Proof.

(i) It follows from Proposition 3.2.

(ii) By induction on $m \in \mathbb{N}$. For $m = 0$, it is true. Let $m \in \mathbb{N}$. Suppose that the statement holds for m. Using Relation (1.1), we have

$$\begin{aligned} R(x)R^{m+1}(e) &= R\left(R(x)R^m(e) + xR^{m+1}(e)\right) \\ &= R\left(R(x)R^m(e)\right) + R\left(xR^{m+1}(e)\right) \\ &\overset{(*)}{=} R\left(\textstyle\sum_{k=0}^{m} R^{m+1-k}\left(xR^k(e)\right)\right) + R\left(xR^{m+1}(e)\right) \\ &= \textstyle\sum_{k=0}^{m+1} R^{m+2-k}\left(xR^k(e)\right). \end{aligned}$$

Equality $(*)$ is justified by induction hypothesis. □

Corollary 3.5. *Given a RB operator R on A of weight zero, and l, m non negative integers. Then, for any $x \in A$, we have the following relations*

$$R^l(x)R^m(e) = \sum_{k=0}^{m} a_k R^{m+l-k}\left(xR^k(e)\right), \tag{3.2}$$

for some non negative integers a_k.
If $R(e)^m = 0$, for some $m \geq 1$, then

$$R^l(x) = -\sum_{k=1}^{m-1} R^{l-k}\left(xR^k(e)\right), \text{ for all } l \geq m+1. \tag{3.3}$$

Proof. From Lemma 3.4, it follows that:

$$R^l(x)R^m(e) = \sum_{k_1=0}^{m} \sum_{k_2=0}^{k_1} ... \sum_{k_l=0}^{k_{(l-1)}} R^{m+l-k_l}\left(xR^{k_l}(e)\right).$$

By inverting the sums we obtain the first equality. The second one follows from Relation (3.1). □

Remark 3.6. 1. Analogously to the proof of Lemma 3.4, we obtain

$$R^m(e)R(x) = \sum_{k=0}^{m} R^{m+1-k}(R^k(e)x), \text{ and}$$

$$R(x)R^m(y) = R^m(R(x)y) + \sum_{k=1}^{m} R^{m+1-k}(xR^k(y)),$$

for all $x, y \in A$ and for all integer $m \geq 1$.

2. By Corollary 3.5, we give the following equality

$$R^l(x)R(e) = lR^{l+1}(x) + R^l(xR(e)).$$

To summaries, we get the following theorem which characterize RB operators of weight zero of A.

Theorem 3.7. *Any RB operator R on A of weight zero is nilpotent.*

Proof. It follows from Lemma 3.4 (i) and Formula (3.3). □

4 Rota-Baxter operators of weight -1

Let us compare the obtained results in the case of zero weight and nonzero case. By Proposition 2.2, any RB operator of nonzero weight on A can be assumed of weight -1. We are intersted in this section on RB operators of weight -1 on A.

Lemma 4.1. *Let R be an RB operator of weight -1 on A. Then*

1. *If $R(e) \in \mathbb{C}e$, then $R(e) \in \{0, e\}$. In this case, we have*

$$A = Ker(R) \oplus Im(R).$$

2. *$Ker(R^k)$ is a $Im(R - id)^k$-bi-module for all integer $k \geq 1$.*

Proof.

1. It follows, easily, from Relation (1.1).
2. Let $x \in Ker(R)$ and $y \in A$. By (1.1) we have
$R(x(R(y) - y)) = R(x)R(y) - R(R(x)y) = 0$.
Then, $xIm(R - id) \subset Ker(R)$. Analogously, we obtain $Im(R - id)x \subset Ker(R)$.
Let k be a non negative integer. Assume that, the result is provided for all numbers less or equal to k. Let $x \in Ker(R^{k+1})$ and $y \in A$. We have

$$\begin{aligned}
&R^{k+1}(x[(R - id)^{k+1}(y)]) \\
&= R^k\left(R(x[R - id](R - id)^k(y)\right) \\
&= R^k(R(x)R(R - id)^k(y)) - R^k(R(x)(R - id)^k(y)) \\
&= R^k(R(x)(R - id)^k(R(y))) - R^k(R(x)(R - id)^k(y)).
\end{aligned}$$

Since $R(x) \in Ker(R^k)$ then, Induction hypothesis gives

$$R^{k+1}(x[(R - id)^{k+1}(y)]) = 0.$$

So, $xIm(R-id)^{k+1} \subset Ker(R^{k+1})$. Analogously, we have $Im(R-id)^{k+1}x \subset Ker(R^{k+1})$. □

Corollary 4.2. *Let R be a nonzero RB operator of weight -1 on A. Then*

1. *If $R(e) \in \mathbb{C}e$ then, R is a projector operator. (That is $R^2 = R$).*
2. *If A is simple. Then id is the only bijective RB operator of weight -1 on A.*

Proof.

1. From Lemma 4.1, if $R(e) \in \mathbb{C}e$ then, $R(e) \in \{0, e\}$. Using Relation (1.1), we obtain $R(x)R(e) = R(R(x) + xR(e) - x)$. According to the value of $R(e)$, we obtain $R(R(x) - x) = 0$, for all $x \in A$. So, R is a projector operator on A.
2. Let R be a bijective RB operator of weight -1 on A. By Proposition 2.2 $(id - R)$ is an RB operator of weight -1 on A. By Lemma 4.1 $Ker(id - R)$ is a bi-ideal of $A = Im(R)$. Since A is assumed simple, then $Ker(id - R) = \{0\}$ or $Ker(id - R) = A$. That is $R - id$ is bijective or $R = id$. If $(R - id)$ is bijective, then $Im(R - id) = A = Im(R)$.
Thus, there are $x, y \in A$ such that $R(y) = e = R(x) - x$.
By Relation (1.1) we have
$$R(x) = R(x)R(y) = R[R(x)y + xR(y) - xy] = R[R(x)y + x - xy].$$
Since R is bijective then $R(x)y+x-xy = x$ and hence, $R(x)y = xy$. Since $R(x) = (x+e)$, then $(x + e)y = xy$ and so, $y = 0$, as a result, we obtain $e = R(y) = 0$, a contradiction. Consequently, $R = id$. □

Lemma 4.3. *Let R be an RB operator of weight -1 on A. Then*

1. *$R^m(e)R(e) = (m + 1)R^{m+1}(e) - mR^m(e)$, for all $m \geq 1$.*
2. *$R^m(e) = \sum_{k=1}^{m} \alpha_k^m R(e)^k$, for some non negative rational numbers α_k^m such that $\alpha_m^m = \alpha_1^m = \frac{1}{m!}$ and $\alpha_k^{m+1} = \frac{\alpha_{k-1}^m + m\alpha_k^m}{m+1}$, for all $m \geq 2$ and for all $k = 2, ..., m$.*
3. *$R(e)^m = \sum_{k=1}^{m} \beta_k^m R^k(e)$, for some non zero integers β_k^m such that $\beta_m^m = m!$, $\beta_1^m = (-1)^{m-1}$ and $\beta_k^{m+1} = (k + 1)\beta_{k-1}^m - k\beta_k^m$, for all $m \geq 2$ and for all $k = 2, ..., m$.*

Proof. By a simple calculation, using Identity (1.1) we obtain (1). The other statements follow by induction on $m \geq 1$, using (1). □

Corollary 4.4. *Assume that $R(e) \notin \mathbb{C}e$. We have*

1. *$span\{R^k(e), 0 \leq k \leq m\} = span\{R(e)^k, 0 \leq k \leq m\}$, for all integer m.*
2. *Let $m \geq 2$. $R(e), R^2(e), ..., R^m(e)$ are linearly independent if and only if the $R(e)$, $R^2(e)$, ..., $R^m(e)$ are.*
3. *If $R(e)$ is nilpotent then, R is not.*
4. *If $R^m(e) = 0$ for some $m \geq 2$ then, $R(e)$ is not nilpoent.*

Proof. The first two points follow immediately from Lemma 4.3. The proof of the third statement and that of the fourth are analogous. Let us show, therefore (3). Assume that $R(e)$ is nilpotent with nilpotnce index denoted $m \geq 2$. By Lemma 4.3, we have, for all $l \geq m$,

$$\begin{aligned} R^l(e) &= \sum_{k=1}^{l} \alpha_k^l R(e)^k \\ &= \sum_{k=1}^{m-1} \alpha_k^l R(e)^k \end{aligned}$$

Multiplying this equality by $R(e)^{m-2}$ gives

$$R^l(e)R(e)^{m-2} = \alpha_1^l R(e)^{m-1} \neq 0.$$

So, we get the result. □

Proposition 4.5. *Let R be an RB operator of weight -1 on A such that $R(e) \notin \mathbb{C}e$. Then $R(e)$ is not nilpotent.*

Proof. It follows from Corollary 4.4. □

Remark 4.6. 1. The result of the previous proposition is true for all RB operator of nonzero weight on A.

2. An RB operator R, of nonzero weight on A, can be nilpotent without $R(e)$ being.

3. An RB operator of nonzero weight on A is not necessarily nilpotent, (see Corollary 4.2). However that any RB operator of weight zero on A is strongly nilpotent, (by Theorem 3.7).

Our objective in the following is to describe the spectrum of an RB operator R on A, which we denote $Sp(R)$. For any eigenvalue λ of R with multiplicity-order which we denote m_λ, we set $E_\lambda = Ker(R - \lambda id)$ and $N_\lambda = Ker((R - \lambda id)^{m_\lambda})$. More hover, for any number $\gamma \notin Sp(R)$ we denote $E_\gamma = \{0\}$. It is well-know that, $E_\lambda \subset N_\lambda, \forall \lambda \in Sp(R)$ and $A = \oplus N_\lambda$.
In all the following, R denoted an RB operator of weight -1 on A.

Lemma 4.7. *Let $\lambda, \alpha \in Sp(R)$ such that $\{\alpha, \lambda\} \neq \{0, 1\}$, and let $x \in E_\lambda$, $y \in E_\alpha$. So, either $xy = 0$ or $xy \in E_\gamma$, where $\gamma = \frac{\lambda\alpha}{\lambda+\alpha-1}$. In particular, $E_\lambda E_\alpha \subset E_\gamma$. Moreover, if λ and α are different from 0 and 1 then, $\gamma \notin \{0, 1, \lambda, \alpha\}$.*

Proof. By Relation 1.1, we have $\lambda\alpha xy = (\lambda + \alpha - 1)R(xy)$. So, we get the result. □

Corollary 4.8. *Let $x \in Ker(R-\lambda_1 id)\backslash\{0\}$ for some number $\lambda_1 \notin \{0, 1\}$ (provided his existence) then, for all integer $k \geq 2$, either $x^k = 0$ or $x, x^2, ..., x^k$ are linearly independent eigenvectors of R. In particular, x is necessarily nilpotent.*

Proof. Let $k \geq 2$. Assume that $x^k \neq 0$. Then, $x^l \neq 0$ and $R(x^l) = \lambda_l x^l$, for all $2 \leq l \leq k$, with $\lambda_l = \frac{\lambda_1\lambda_{l-1}}{\lambda_{l-1}+\lambda_1-1}$. From the previous lemma, it follows that the $\lambda_1, \lambda_2, ..., \lambda_k$ are pairwise distinct eigenvalues of R. So, we get the result. □

Remark 4.9. For all numbers $\lambda \in \{0, 1\}$ and for all $\alpha \notin \{0, 1\}$. We have, $E_\lambda E_\alpha \subset E_\lambda$.

Proposition 4.10. *R is diagonalizable if, and only if, $R^2 = R$.*

Proof. If $R^2 = R$ then, R is easily, diagonalizable. Conversely, suppose that R is diagonalizable with eventual pairwise distinct eigenvalues $\lambda_0 = 0, \lambda_1 = 1, ..., \lambda_r$ for some $r \geq 1$. Let us write

$$e = x_0 + x_1 + \cdots + x_r, \; x_i \in E_{\lambda_i} \text{ for all } i = 0, ..., r.$$

For any $j \geq 2$ we have, $x_j = ex_j = \underbrace{x_0x_j}_{\in E_{\lambda_0}} + \underbrace{x_1x_j}_{\in E_{\lambda_1}} + \underbrace{... + x_rx_j}_{\in \oplus_{i\neq j} E_{\lambda_i}} \in E_{\lambda_j}$. So, $x_j = 0$ and so, $e = x_0 + x_1$. In the same way, we show that $x = \underbrace{x_0x + x_1x}_{\in E_{\lambda_0}\oplus E_{\lambda_1}} = 0$, for any $x \in E_{\lambda_j}$. Thus, $E_{\lambda_j} = \{0\}$, for all $j \geq 2$. Consequentially, $A = E_{\lambda_0} \oplus E_{\lambda_1}$, and hence, $R^2 = R$. □

Remark 4.11. If $R^2 = R$ then, the only elements $t \in Im(R)$ and $t' \in Ker(R)$ such that $e = t+t'$ are orthogonal idempotent elements of A.

Lemma 4.12. *For any eigenvalue $\lambda \neq 1$ of R we have $N_\lambda E_0 \subset E_0$ and $E_0 N_\lambda \subset E_0$.*

Proof. Let λ an eigenvalue of R different from 1 with multiplicity-order m. Given $x \in N_{\lambda_i}$ and $x_0 \in E_0$. If $x \in E_\lambda$ then, $xx_0 \in E_\lambda E_0 \subset E_0$, (by Lemma 4.7). If $x \notin E_\lambda$ then, $m \geq 2$ in which case, we have

$$(R - \lambda id)^m(x) = 0 = \sum_{k=0}^{m} C_m^k(-\lambda)^{m-k} R^k(x).$$

Thus,

$$-\sum_{k=0}^{m} C_m^k(-\lambda)^{m-k} x = \sum_{k=0}^{m} C_m^k(-\lambda)^{m-k}[R^k(x) - x].$$

That is,

$$-(1-\lambda)^m x = \sum_{k=1}^{m} C_m^k (-\lambda)^{m-k}[R^k(x) - x].$$

Hence,

$$-(1-\lambda)^m x x_0 = \sum_{k=1}^{m} C_m^k (-\lambda)^{m-k}[R^k(x)x_0 - x x_0].$$

and hence,

$$-(1-\lambda)^m R(x x_0) = \sum_{k=1}^{m} C_m^k (-\lambda)^{m-k} R[R^k(x)x_0 - x x_0].$$

Using Identity 1.1, we have

$$\begin{aligned} R[R^k(x)x_0 - x x_0] &= R\left(\textstyle\sum_{i=1}^{k}[R^i(x)x_0 - R^{i-1}(x)x_0]\right) \\ &= \textstyle\sum_{i=1}^{k} R[R^i(x)x_0 - R^{i-1}(x)x_0] \\ &= \textstyle\sum_{i=1}^{k} R^i(x)R(x_0) = 0. \end{aligned}$$

It follows that, $R(x x_0) = 0$, (since $(1-\lambda)^m \neq 0$) and hence, $x x_0 \in E_0$.
Consequentially, $N_\lambda E_0 \subset E_0$. In the same way, we show that, $E_0 N_\lambda \subset E_0$.
Let us now, show that, $N_0 E_0 \subset E_0$. Given the elements $x \in N_0$ and $x_0 \in E_0$. If $x \in E_0$ then, $x x_0 \in E_0$. (By Lemma 4.7). If $R(x) \neq 0$ then $m_0 \geq 2$, in this case, we have

$$-x x_0 = [R^{m_0}(x)x_0 - x x_0] = \sum_{k=1}^{m_0} \underbrace{[R^k(x) - R^{k-1}(x)]x_0}_{\in Ker(R)}.$$

So, $x x_0 \in E_0$ and so, $N_0 E_0 \subset E_0$. In the same way, one shows that, $E_0 N_0 \subset E_0$. □

Theorem 4.13. *If A is simple then, we have: $1 \notin Sp(R)$ if, and only if, $R = 0$.*

Proof It is easy, to see that, if $R = 0$ then, $1 \notin Sp(R)$.
Conversely, Assume that $1 \notin Sp(R)$. By the previous lemma, we have $Ker(R)$ is a nonzero bi-ideal of A. So, $Ker(R) = A$, since A is simple. We are done. □

Corollary 4.14. *When A is simple we have*

1. *R is nilpotent if, and only if, $R = 0$.*
2. *If R is non trivial (that is $R \neq 0$ and $R \neq id$) then, $\{0, 1\} \subset Sp(R)$.*

Proof It follows from Theorem 4.13 □

Proposition 4.15. *(i) $N_\lambda E_1 \subset E_1$ and $E_1 N_\lambda \subset E_1$, for all $\lambda \neq 0$.*

(ii) $E_\lambda N_0 \subset N_0$ and $N_0 E_\lambda \subset N_0$, for all $\lambda \neq 1$.

(iii) $E_\lambda N_1 \subset N_1$ and $N_1 E_\lambda \subset N_1$, for all $\lambda \neq 0$.

Proof

(i) Given a nonzero eigenvalue λ of R. Set $x \in N_\lambda$ and $x_1 \in E_1$. We have

$$-(-\lambda)^{m_\lambda} x x_1 = -\sum_{k=1}^{m_\lambda} (-\lambda)^{m_\lambda - k} C_{m_\lambda}^k \underbrace{R^k(x)x_1}_{\in E_1}.$$

Hence, $x x_1 \in E_1$ and hence, $N_\lambda E_1 \subset E_1$. In the same way, we obtain $E_1 N_\lambda \subset E_1$.

(ii) Let $\lambda \in Sp(R)$ with $\lambda \neq 0$ and $\lambda \neq 1$. If $m_0 = 1$ then, $E_\lambda N_0 \subset N_0$ by Remarque 4.9. Suppose that $m_0 \geq 2$. Let $v \in E_\lambda$ and $x \in N_0$. By induction on $k = 1, ..., m_0$, we prove that $vR^{m_0-k}(x) \in Ker(R^k)$. The result is true for $k = 1$, since $R^{m_0-1}(x) \in Ker(R)$ (see Remarque 4.9). Assume that the result is proved for a fixed $k \in \{1, m_0 - 1\}$. Since

$$\begin{aligned}\lambda vR^{m_0-k}(x) &= R(v)R^{m_0-k}(x)\\ &= R[R(v)R^{m_0-k-1}(x) + vR^{m_0-k}(x) - vR^{m_0-k-1}(x)]\\ &= R[\lambda vR^{m_0-k-1}(x) + vR^{m_0-k}(x) - vR^{m_0-k-1}(x)]\\ &= (1-\lambda)R[vR^{m_0-k-1}(x)] + R(vR^{m_0-k}(x))\end{aligned}$$

hence, by induction hypothesis, we have

$$0 = R^k(\lambda vR^{m_0-k}(x))) = (1-\lambda)R^{k+1}[vR^{m_0-k-1}(x)] + \underbrace{R^{k+1}(vR^{m_0-k}(x))}_{=0},$$

and hence, $vR^{m_0-k-1}(x) \in Ker(R^{k+1})$. In particular, for $k = m_0$ we obtain $vx \in Ker(R^{m_0})$. So, $E_\lambda N_0 \subset N_0$. In the same way, one can obtain $N_0 E_\lambda \subset N_0$.

(iii) It follows from (ii) and Proposition 2.2, (replacing R by $id - R$ and λ by $(1-\lambda)$). □

Corollary 4.16. *If A is simple of dimension 4 then, $Sp(R) \subset \{0, 1\}$. With equality, if and only if, R is non trivial.*

Proof Assume that $R \neq 0$ and $R \neq id$. By Corollary 4.14, we have $\{0, 1\} \subset Sp(R)$. By Proposition 4.10, $Sp(R)$ contains at most one eigenvalue other that 0 and 1 and. Suppose that R has an eigenvalue λ different from 0 and 1 with multiplicity-order m. If $m = 1$ then, $A = N_0 \oplus N_1 \oplus E_\lambda$. Set $e = \underbrace{x_0}_{\in N_0} + \underbrace{x_1}_{\in N_1} + \underbrace{x_\lambda}_{\in E_\lambda}$. Thus, $x_\lambda = \underbrace{x_0 x_\lambda + x_1 x_\lambda}_{\in N_0 \oplus N_1} + x_\lambda^2$. By Corollary 4.8, we have $x_\lambda^2 = 0$ hence, $x_\lambda = 0$ and hence $e = x_0 + x_1$. Consequently, $x = \underbrace{x_0 x + x_1 x}_{\in N_0 \oplus N_1} = 0$ for all $x \in E_\lambda$, a contradiction. So, $m = 2$. In this case, $E_\lambda \neq N_\lambda$ (see Proposition 4.10). Moreover, there exists $v_1, v_2 \in N_\lambda$ such that $R(v_1) = \lambda v_1$ and $R(v_2) = v_1 + v_2$. In one hand, we have $v_1^2 = 0$ (see Corollary 4.8). On the other hand, we have

$$R(v_1)R(v_2) = R(R(v_1)v_2 + v_1 R(v_2) - v_1 v_2).$$

That is,

$$\begin{aligned}\lambda v_1 v_2 &= R(\lambda v_1 v_2 + v_1 v_2 - v_1 v_2)\\ &= \lambda R(v_1 v_2).\end{aligned}$$

Thus, $v_1 v_2 \in E_1$.
Set $e = x_0 + x_1 + \alpha_1 v_1 + \alpha_2 v_2$, where $x_0 \in E_0$, $x_1 \in E_1$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. So,

$$v_1 = \underbrace{v_1 x_0 + v_1 x_1}_{\in E_0 \oplus E_1} + \alpha_1 \underbrace{v_1^2}_{=0} + \underbrace{\alpha_2 v_1 v_2}_{\in E_1} \in (E_0 \oplus E_1) \cap E_\lambda,$$

and so, $v_1 = 0$, a contradiction. □

Corollary 4.17. *If A is simple of dimension 4 then, for all non trivial RB operator R of weight -1 on A we have, $m_0 + m_1 \leq 3$.*

Proof At once, we have $0 \leq m_1, m_0 \leq 3$. By replacing R by $id - R$, it suffices to eliminate the cases where $m_0 = 3$ or $m_1 = m_0 = 2$.

(i) If $m_0 = 3$ then, $\dim(E_0) = \dim(E_1) = 1$ and $\dim(N_0) = 3$. Moreover, there exist nonzero vectors $u_0, u_1, u_2 \in N_0$ such that $R(u_0) = 0$, $R(u_1) = u_0$ and $R(u_2) = u_1 + u_0$.
Let v be a nonzero vector of E_1. By Formula (1.1), we have

$$u_0 v = R(u_1)R(v) = R[u_0 v + u_1 v - u_1 v] = R(u_0 v).$$

Hence, $u_0 v \in E_1$. In the same way we have $u_1 v \in E_1$.
On the other hand, we have

$$u_0^2 = R(u_1)R(u_1) = R[u_0 u_1 + u_1 u_0 - u_1^2] \in E_0,$$

then, $u_1^2 \in span(u_0, u_1)$. In the same way, we have

$$u_0 u_1 = R(u_1)R(u_2) = R[u_0 u_2 + u_1^2 - u_1 u_2] \in E_0,$$

then, $u_1^2 - u_1 u_2 \in span(u_0, u_1)$ from which we get $u_1 u_2 \in span(u_0, u_1)$.
Let us write $e = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha v$ for some numbers $\alpha_0, \alpha_1, \alpha_2$ and α.
• If $\alpha = \alpha_2 = 0$ then, $e = \alpha_0 u_0 + \alpha_1 u_1$ and so, $u_2 = u_2 e = \alpha_0 u_0 u_2 + \alpha_1 u_1 u_2 \in span(u_0, u_1)$, a contradiction.
• If $\alpha_2 = 0$ and $\alpha \neq 0$ then, $\underbrace{u_0 = \alpha_0 u_0^2 + \alpha_1 u_0 u_1}_{\in E_0} + \underbrace{\alpha u_0 v}_{\in E_1}$ and hence, $u_0 v = 0 \in E_0$ which gives $AE_0 \subset E_0$. On the other hand, we have $N_0 E_0 \subset E_0$, we deduce that E_0 is a nonzero bi-ideal of A and so, $E_0 = A$ a contradiction.
• If $\alpha = 0$ and $\alpha_2 \neq 0$ then,

$$\underbrace{v = \alpha_0 u_0 v + \alpha_1 u_1 v}_{\in E_1} + \alpha_2 u_2 v + \underbrace{\alpha v^2}_{\in E_1}$$

and hence, $u_2 v = 0 \in E_1$. According to (i) and (ii), we obtain $AE_1 \subset E_1$, we deduce that E_1 is a nonzero bi-ideal of A and so $E_1 = A$ a contradiction.
• If $\alpha\alpha_2 \neq 0$ then, as above, we have $u_2 v, v u_2 \in E_1$ and so, E_1 is a nonzero bi-ideal of A and so $E_1 = A$ a contradiction.

(ii) If $m_0 = m_1 = 2$. Then, $\dim(E_0) = \dim(E_1) = 1$ and there exist nonzero vectors $u_1, u_2 \in N_0$ such that $R(u_1) = 0$ and $R(u_2) = u_1$. Similarly, there are nonzero vectors $v_1, v_2 \in N_1$ such $R(v_1) = v_1$ and $R(v_2) = v_1 + v_2$. Formula (1.1) gives $v_1 u_1 = u_1 v_1 = 0$.
Let us write $e = \alpha_1 u_1 + \alpha_2 u_2 + \lambda_1 v_1 + \lambda_2 v_2$ for some numbers $\alpha_1, \alpha_2, \lambda_1$ and λ_2.
• If $\alpha_2 \neq 0$ then, by multiplying e by v_1 we get $u_2 v_1, v_1 u_2 \in E_1$ which makes E_1 a nonzero bi-ideal of A, a contradiction. Thus, $\alpha_2 = 0$.
• If $\lambda_2 = 0$ then, $e = \alpha_1 u_1 + \lambda_1 v_1$. In this case, α_1 and λ_1 are necessarily non-zero. (multiply e by v_1 and by u_1). By multiplying e by v_2 we get $v_2 = \alpha_1 u_1 v_2 + \lambda_1 v_1 v_2$. So, $u_1 v_2 = \frac{1}{\alpha_1}(v_2 - \lambda_1 v_1 v_2)$ and so, $R(u_1 v_2) = \frac{1}{\alpha_1}(v_1 + v_2 - \lambda_1 v_1 v_2)$. Calculating $R(u_1 v_2)$ we get $\lambda_1 v_1 = \alpha_1 u_1$ a contradiction. Thus, $\lambda_2 \neq 0$ which gives $u_1 v_2, v_2 u_1 \in E_0$. Consequently, E_0 is a nonzero bi-ideal of A, a contradiction.

Remark 4.18. $\dim(E_0) + \dim(E_1) \geq 3$.

5 Examples

Let A be a complex finite dimensional semi-simple algebra. Then, there exist $\tau_1, ..., \tau_r \in A$ such that

$$e = \sum_{i=1}^{r} \tau_i, \text{ with } \tau_i \tau_j = \delta_{ij} \tau_i \text{ for all } 1 \leq i, j \leq r. \text{ (Pierce decomposition of } e \text{).} \quad (5.1)$$

Here, δ_{ij} is the Kronecker symbol: ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$). For more details, see for example, [13].
We denote, For all $1 \leq i, j \leq r$, $A_i = A\tau_i := \{a\tau_i / a \in A\}$ and $A_{ij} = \tau_i A \tau_j$.
We will also need the following notations:

$$A_0 = \oplus_i A_{ii},\ A_- = \oplus_{i<j} A_{ij} \text{ and } A_+ = \oplus_{i>j} A_{ij}.$$

It is easy to see that:
• $A = \oplus_i A_i = \oplus_{ij} A_{ij}$.
• $A = A_- \oplus A_0 \oplus A_+$.

- $A_- A_0 = A_- = A_0 A_-$.
- $A_+ A_0 = A_+ = A_0 A_+$.
- $A_0 A_0 = A_0$ and $A_- A_+ = \{0\} = A_+ A_-$.
- $A_+ A_+ \subset A_+$ and $A_- A_- \subset A_-$.

In particular, A_0 is an unital subalgebra of A of which $(\tau_1, ..., \tau_r)$ is a basis, (since each A_i is minimal left ideal of A, see [3]).

Proposition 5.1. *If R_0 is an RB operator of weight λ on A_0, then an operator R defined as*

$$R(a_- + a_0 + a_+) = R_0(a_0) - \lambda a_\pm, \; a_\pm \in A_\pm, a_0 \in A_0,$$

is an RB operator on A of weight λ.

Proof. It follows from Formula (1.1), using the fact that $A_0 A_\pm \subset A_\pm$ and $A_+ A_- = \{0\}$. □

Theorem 5.2. *A linear operator $R(\tau_i) = \sum_{k=1}^{r} a_{ik}\tau_k$, $a_{ik} \in \mathbb{C}$, is an RB operator of weight 1 on A_0 if and only if the following conditions are satisfied:*

$$a_{ik}a_{jk} = a_{ji}a_{ik} + a_{ij}a_{jk} \text{ for } i \neq j, \; a_{ik}(a_{ik} - 2a_{ii} - 1) = 0 \text{ for } i = j. \tag{5.2}$$

Proof. For any $1 \leq i, j \leq r$, we have

$$R(\tau_i)R(\tau_j) = R(\tau_i R(\tau_j) + R(\tau_i)\tau_j + \tau_i\tau_j)$$

$$\text{if and only if, } \sum_{k=1}^{r} a_{ik}\tau_k \sum_{l=1}^{r} a_{il}\tau_l = R\left(\tau_i \sum_{k=1}^{r} a_{jk}\tau_k + \sum_{k=1}^{r} a_{ik}\tau_k\tau_j + \delta_{ij}\tau_i\right)$$

$$\text{if and only if, } \sum_{k=1}^{r}\left(\sum_{l=1}^{r} a_{ik}a_{jl}\tau_k\tau_l\right) = R(a_{ji}\tau_i + a_{ij}\tau_j + \delta_{ij}\tau_i)$$

$$\text{if and only if, } \sum_{k=1}^{r} a_{ik}a_{jk}\tau_k = \sum_{k=1}^{r} (a_{ji}a_{ik} + a_{ij}a_{jk} + \delta_{ij}a_{ik})\,\tau_k.$$

From which (5.2) follows. □

Using Theorem 5.2, we make some examples of RB operators on A, when A denotes Complex Clifford algebras $\mathbb{C}l(p, q)$. We use definitions and notations of complex Clifford algebras using in [12].

Example 5.3. 1. Consider $A = \mathbb{C}l(1,1)$ equipped with generators e_1, e_2: They satisfy $e_1^2 = e = -e_2^2$ and $e_1e_2 = 0$. Set $t_1 = \frac{1}{2}(e - e_1)$ and $t_2 = \frac{1}{2}(e + e_1)$. We have $e = t_1 + t_2$; a Pierce decomposition of identity.

An operator R_0 defined as $R_0(t_i) = \sum a_{ik}t_k$ is a RB operator on A_0 of weight 1 if ,and only if, one of the following cases is true:

a. $a_{11} = a_{22} = -1$ and ($a_{12} = -1, a_{21} = 0$ or $a_{21} = -1, a_{12} = 0$). That is

$$\begin{cases} R_0(t_1) = -t_1 - t_2 \quad \text{and} \quad R_0(t_2) = -t_2, \\ \text{or} \\ R_0(t_1) = -t_1 - t_2 \quad \text{and} \quad R_0(t_2) = -t_1 \end{cases}$$

b. $a_{11} = a_{22} = 0$ and ($a_{12} = 1, a_{21} = 0$, or $a_{21} = 1, a_{12} = 0$). That is

$$\begin{cases} R_0(t_1) = t_2 \quad \text{and} \quad R_0(t_2) = 0, \\ \text{or} \\ R_0(t_1) = 0 \quad \text{and} \quad R_0(t_2) = t_1 \end{cases}$$

On the other hand, we have $A_- = \mathbb{C}t_3$ and $A_+ = \mathbb{C}t_4$ where $t_3 = e_1e_2$ and $t_4 = (e_1e_2 - e_2)$. Proposition 5.1 gives RB operators on $\mathbb{C}l(1,1)$ of weight 1.

2. Let us consider $A = \mathbb{C}l(1,3)$ with generators e_1, e_2, e_3, e_4. They satisfy the following relation ($e_1^2 = e = -e_2^2 = -e_3^2 = -e_4^2$). The following operator is an RB operator of weight 1 on $A_0 = \oplus_{i=1}^{4}\mathbb{C}t_i$.

$$R_0(t_1) = 0,\ R_0(t_2) = -t_2,\ R_0(t_3) = -t_2 - t_3,\ R_0(t_4) = -t_2 - t_3 - t_4,$$

where $t_1 = \frac{1}{4}(e - e_1)(e - ie_2e_3)$, $t_2 = \frac{1}{4}(e - e_1)(e + ie_2e_3)$, $t_3 = \frac{1}{4}(e + e_1)(e - ie_2e_3)$ and $t_4 = \frac{1}{4}(e + e_1)(e + ie_2e_3)$ form a Pierce decomposition of identity: $e = t_1 + t_2 + t_3 + t_4$. Proposition 5.1 makes RB operators on A.

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