Palestine Journal of Mathematics

Vol. 12(3)(2023) , 332–340

FINSLERIAN HYPERSURFACES OF A FINSLER SPACE WITH SPECIAL GENERALIZED (α, β)-METRIC

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Communicated by Nouressadat Touafek

MSC 2010 Classifications: Primary 53B40 ; Secondary 53C60.

Keywords and phrases: Finsler space, Finslerian hypersurface, Special generalized metric, Riemannian metric and one form metric.

Abstract In the present paper, we study the Finslerian hypersurfaces of a special generalized (α, β)-metric in the form $L(\alpha, \beta) = \alpha\phi(\frac{\beta}{\alpha})$, with condition $\phi(0) = \phi^{'}(0) = \phi^{''}(0) = \phi^{'''}(0) = 1$. Further, we examined the hypersurfaces of this special metric as a hyperplane of first, second and third kinds with certain condition on the above Finsler metric.

1 Introduction

The concept of Finslerian hypersurface is first introduced by Matsumoto in the year 1985 and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further many authors studied these hyperplanes in different changes of the Finsler metric [2, 3, 5, 6, 10, 11, 13, 14] and obtained very interesting geometric results for the stand point of Finsler space.

A Finsler metric $L(\alpha, \beta)$ in an n-dimensional differentiable manifold M^n is known as an (α, β)-metric, if L is a positively homogeneous function of degree one in two variables Riemannian meric $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and a one-form metric $\beta = b_i(x)y^i$ on M^n. The Finsler space $F^n = \{M^n, L\}$ equiped with this metric is known as Finsler space with (α, β)-metric. The interesting and important examples of this (α, β)-metric are Randers metric $(\alpha + \beta)$, Kropina metric $\frac{\alpha^2}{\beta}$, Matsumoto metric $\frac{\alpha^2}{(\alpha-\beta)}$ and Z. Shen square metric $\frac{(\alpha+\beta)^2}{\beta}$. The notion of an (α, β)-metric was introduced by M. Matsumoto [8] and further it has been studied by many authors of the world. The (α, β)-metrics are very important class to study the property of Finsler space with this metric.

If an (α, β)-metric is expressed by the following form,

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$, is a Riemannian metric and $\beta = b_i(x)y^i$, is a 1-form with $||\beta_x||_\alpha < b_0$, for all $x \in M$.

It is well known that $L = \alpha\phi(s)$ is a regular (α, β)-metric if the function $\phi(s)$ is a function with $|s| < b_0$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0,$$

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$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0,$$

In the present paper, we consider a special generalized (α, β)-metric in the form

$$L = \alpha\phi(\frac{\beta}{\alpha}) = \alpha\phi(s), \tag{1.1}$$

where $\phi(s)$ satisfies the conditions,

$$\phi(0) = \phi^{'}(0) = \phi^{''}(0) = \phi^{'''}(0) = 1,$$

i.e. constant and examined the hypersurfaces of this metric as a hyperplane of first, second and third kinds.

2 Preliminaries

In the present paper, we have consider an n-dimensional Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$, equipped with special generalized (α,β)-metric given by equation (1.1).

Differentiating equation (1.1) partially with respect to α and β are given by

$$L_\alpha = \phi(\tfrac{\beta}{\alpha}) - \tfrac{\beta}{\alpha}\phi^{'}(\tfrac{\beta}{\alpha}), \qquad L_\beta = \phi^{'}(\tfrac{\beta}{\alpha})$$

$$L_{\alpha\alpha} = \tfrac{\beta^2}{\alpha^3}\phi^{''}(\tfrac{\beta}{\alpha}), \quad L_{\beta\beta} = \tfrac{1}{\alpha}\phi^{''}(\tfrac{\beta}{\alpha})$$

$$L_{\alpha\beta} = -\tfrac{\beta}{\alpha^2}\phi^{''}(\tfrac{\beta}{\alpha})$$

where $L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_\beta = \frac{\partial L}{\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial\alpha}, \quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial\beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial\beta}.$

In Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$ the normalized element of support $l_i = \dot{\partial_i}L$ and angular metric tensor h_{ij} are given by [9]:

$$l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2}Y_i Y_j$$

where $Y_i = a_{ij}y^j$. For the fundamental metric function (1.1) above constants are

$$\begin{cases} p = \phi(\frac{\beta}{\alpha})\{\phi(\frac{\beta}{\alpha}) - \frac{\beta}{\alpha}\phi^{'}(\frac{\beta}{\alpha})\}, \\ q_0 = \phi(\frac{\beta}{\alpha})\phi^{''}(\frac{\beta}{\alpha}), \\ q_{-1} = -\frac{\beta}{\alpha^2}\phi(\frac{\beta}{\alpha})\phi^{''}(\frac{\beta}{\alpha}), \\ q_{-2} = \frac{1}{\alpha^4}\phi(\frac{\beta}{\alpha})\{\beta^2\phi^{''}(\frac{\beta}{\alpha}) + \alpha\beta\phi^{'}(\frac{\beta}{\alpha}) - \alpha^2\phi(\frac{\beta}{\alpha})\}. \end{cases} \tag{2.1}$$

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha,\beta)$ are given by [9]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2}Y_i Y_j \tag{2.2}$$

where

$$\begin{cases} p_0 = q_0 + L_\beta^2 = \phi(\frac{\beta}{\alpha})\phi^{''}(\frac{\beta}{\alpha}) + \{\phi^{'}(\frac{\beta}{\alpha})\}^2, \\ p_{-1} = q_{-1} + L^{-1}pL_\beta \\ \quad = \frac{1}{\alpha^2}[\alpha\phi(\frac{\beta}{\alpha})\phi^{'}(\frac{\beta}{\alpha}) - \beta\{\phi^{'}(\frac{\beta}{\alpha})\}^2 - \beta\phi(\frac{\beta}{\alpha})\phi^{''}(\frac{\beta}{\alpha})], \\ p_{-2} = q_{-2} + p^2 L^{-2} \\ \quad = \frac{\beta}{\alpha^4}[\beta\phi(\frac{\beta}{\alpha})\phi^{''}(\frac{\beta}{\alpha}) - \alpha\beta\phi(\frac{\beta}{\alpha})\phi^{'}(\frac{\beta}{\alpha}) + \beta\{\phi^{'}(\frac{\beta}{\alpha})\}^2]. \end{cases} \tag{2.3}$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2}y^i y^j \tag{2.4}$$

where $b^i = a^{ij}b_j \quad and \quad b^2 = a_{ij}b^i b^j$

$$\begin{cases} s_0 = \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} = \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} = \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2)b^2\}, \\ \tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2). \end{cases} \tag{2.5}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [11]

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k \tag{2.6}$$

where,

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i \tag{2.7}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space R^n and $\bigtriangledown_k$ be the covariant derivative with respect to x^k relative to this christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \tag{2.8}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ -C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + \\ C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji} \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B^k_i = g^{kj} B_{ji} \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k \\ \lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i \end{aligned} \tag{2.10}$$

where $'0'$ denote contraction with y^i except for the quantities p_0, q_0 and s_o.

3 Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ where $\{\alpha = 1, 2, 3...(n-1)\}$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is [9],

$$y^i = B^i_\alpha(u) v^\alpha. \tag{3.1}$$

The metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma,$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v), y(u,v)\} B^i_\alpha N^j = 0, \qquad g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1.$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1. \tag{3.2}$$

(B^α_i, N_i) inverse of (B^i_α, N^i) is given by

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^\alpha_i N^i = 0, \qquad B^i_\alpha N_i = 0,$$

$$N_i = g_{ij} N^j, \quad B^k_i = g^{kj} B_{ji}, \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by [9]

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i (B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,$$

$$G^{\alpha}_{\beta}=B^{\alpha}_{i}(B^{i}_{0\beta}+\Gamma^{*i}_{0j}B^{j}_{\beta}),\qquad C^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma},$$

where

$$M_{\beta\gamma}=N_iC^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma},\quad M^{\alpha}_{\beta}=g^{\alpha\gamma}M_{\beta\gamma},\quad H_{\beta}=N_i(B^{i}_{0\beta}+\Gamma^{*i}_{oj}B^{j}_{\beta}),$$

and

$$B^{i}_{\beta\gamma}=\frac{\partial B^{i}_{\beta}}{\partial u^{\gamma}},\qquad B^{i}_{0\beta}=B^{i}_{\alpha\beta}v^{\alpha}.$$

The quantities $M_{\beta\gamma}$ and H_{β} are called the second fundamental v-tensor and normal curvature vector respectively [9]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [9]

$$H_{\beta\gamma}=N_i(B^{i}_{\beta\gamma}+\Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma})+M_{\beta}H_{\gamma},\tag{3.3}$$

where

$$M_{\beta}=N_iC^{i}_{jk}B^{j}_{\beta}N^{k}.\tag{3.4}$$

The relative h and v-covariant derivatives of projection factor B^{i}_{α} with respect to $IC\Gamma$ are given by

$$B^{i}_{\alpha|\beta}=H_{\alpha\beta}N^{i},\qquad B^{i}_{\alpha}|_{\beta}=M_{\alpha\beta}N^{i}.$$

It is obvious from the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma}-H_{\gamma\beta}=M_{\beta}H_{\gamma}-M_{\gamma}H_{\beta}.\tag{3.5}$$

The above equation yield

$$H_{0\gamma}=H_{\gamma},\quad H_{\gamma0}=H_{\gamma}+M_{\gamma}H_{0}.\tag{3.6}$$

We shall use following lemmas which are due to Matsumoto [9] in the coming section

Lemma 3.1. *The normal curvature $H_0=H_{\beta}v^{\beta}$ vanishes if and only if the normal curvature vector H_{β} vanishes.*

Lemma 3.2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_{\alpha}=0$.*

Lemma 3.3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_{\alpha}=0$ and $H_{\alpha\beta}=0$.*

Lemma 3.4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_{\alpha}=0$ and $H_{\alpha\beta}=M_{\alpha\beta}=0$.*

4 Hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α,β)-metric

In this section, we considered the vector field $b_i(x)=\frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$ in a special generalized metric $L(\alpha,\beta)=\alpha\phi(\frac{\beta}{\alpha})$. Now we consider a Finslerian hypersurface $F^{(n-1)}(c)$ given by a equation $b(x)=c$, a constant [11].

From the parametric equation $x^i=x^i(u^{\alpha})$ of a Finslerian hypersurface $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^{\alpha}}=0$$

$$\frac{\partial b(x)}{\partial x^{i}}\frac{\partial x^{i}}{\partial u^{\alpha}}=0$$

$$b_iB^{i}_{\alpha}=0$$

Above shows that $b_i(x)$ are covarient component of a normal vector field of a Finslerian hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad and \quad b_i y^i = 0 \quad i.e \quad \beta = 0 \tag{4.1}$$

and induced metric $L(u, v)$ of a Finslerian hypersurface $F^{n-1}(c)$ is given by

$$L(u, v) = \sqrt{a_{\alpha\beta} v^\alpha v^\beta}, \qquad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta, \tag{4.2}$$

which is a Riemannian metric.

Writing $\beta = 0$ and taking $\phi(0) = \phi^{'}(0) = \phi^{''}(0) = \phi^{'''}(0) = 1$ in the equations (2.1), (2.3) and (2.5) we get

$$\begin{aligned} p = 1, \quad q_0 = 1, \quad q_{-1} = 0, \quad q_{-2} = -\alpha^{-2}, \\ p_0 = 2, \quad p_{-1} = \alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 1 + b^2, \\ s_0 = \frac{1}{1+b^2}, \quad s_{-1} = \frac{1}{\alpha(1+b^2)}, \quad s_{-2} = \frac{-b^2}{\alpha^2(1+b^2)}. \end{aligned} \tag{4.3}$$

From equation (2.4) we get,

$$g^{ij} = a^{ij} - \frac{1}{1+b^2} b^i b^j - \frac{1}{\alpha(1+b^2)}(b^i y^j + b^j y^i) + \frac{b^2}{\alpha^2(1+b^2)} y^i y^j. \tag{4.4}$$

Thus along Finslerian hypersurface $F^{n-1}(c)$, equations (4.4) and (4.1) leads to

$$g^{ij} b_i b_j = \tfrac{b^2}{1+b^2}.$$

So we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+b^2}} N_i, \qquad b^2 = a^{ij} b_i b_j, \tag{4.5}$$

where b is the length of the vector b^i.

Again from equations (4.4) and (4.5), we get

$$b^i = a^{ij} b_j = (1+b^2) N^i + \frac{b^2}{\alpha} y^i. \tag{4.6}$$

Thus we have,

Theorem 4.1. *The Induced Riemannian metric in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric is given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).*

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of a Finsler space F^n are given by

$$h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \quad and \quad g_{ij} = a_{ij} + 2 b_i b_j + \frac{1}{\alpha}(b_i Y_j + b_j Y_i). \tag{4.7}$$

From equations (4.1), (4.7) and (3.2) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along a Finslerian hypersurface $F^{n-1}_{(c)}, \quad h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Thus along $F^{n-1}_{(c)}, \quad \frac{\partial p_0}{\partial \beta} = \frac{4}{\alpha}$,

from equation (2.7), we get

$$\gamma_1 = \tfrac{1}{\alpha}, \qquad m_i = b_i,$$

then hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{1}{2\alpha}b_ib_jb_k, \tag{4.8}$$

in special generalized metric of Finsler hypersurface $F_{(c)}^{(n-1)}$. Due to fact from equations (3.2), (3.3), (3.5), (4.1) and (4.8), we have

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{1+b^2}}h_{\alpha\beta} \quad and \quad M_\alpha = 0. \tag{4.9}$$

Therefore, from equation (3.6), it follows that $H_{\alpha\beta}$ is symmetric. Thus, we have

Theorem 4.2. *The second fundamental v-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from equation (4.1), we have $b_iB^i_\alpha = 0$. Then, we have

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0.$$

Therefore, from equation (3.5) and using $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_jN^jH_\beta$, we have

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0, \tag{4.10}$$

since $b_i|_j = -b_hC^h_{ij}$, we get

$$b_{i|j}B^i_\alpha N^j = 0.$$

Therefore, from equation (4.10) we have,

$$\sqrt{\frac{b^2}{1+b^2}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0, \tag{4.11}$$

because $b_{i|j}$ is symmetric. Now contracting equation (4.11) with v^β and using equation (3.1), we get

$$\sqrt{\frac{b^2}{1+b^2}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0. \tag{4.12}$$

Again contracting by v^α equation (4.12) and using equation (3.1), we have

$$\sqrt{\frac{b^2}{1+b^2}}H_0 + b_{i|j}y^iy^j = 0. \tag{4.13}$$

From lemma (3.1) and (3.2), it is clear that the special generalized metric of Finsler hypersuface $F_{(c)}^{(n-1)}$ is a hyperplane of first kind, if and only if $H_0 = 0$. Thus from equation (4.13), it is obvious that Finslerian hypersurface $F_{(c)}^{n-1}$ is a hyperplane of first kind, if and only if $b_{i|j}y^iy^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n defined on y^i, but $b_{ij} = \bigtriangledown_jb_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ where $b_{ij} = \bigtriangledown_jb_i$ in the following. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by equation (2.9). Since b_i is a gradient vector, then from equation (2.8), we have

$$E_{ij} = b_{ij} \qquad F_{ij} = 0 \qquad and \qquad F^i_j = 0.$$

Thus equation (2.9) reduces to

$$D^i_{jk} = B^ib_{jk} + B^i_jb_{0k} + B^i_kb_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + \tag{4.14}$$
$$C_{jkm}A^m_sg^{is} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}),$$

where

$$B_i = 2b_i + \frac{1}{\alpha}Y_i, \quad B^i = \frac{1}{1+b^2}b^i + \frac{1}{\alpha(1+b^2)}y^i, \tag{4.15}$$

$$B_iB^i = \frac{1+2b^2}{1+b^2}, \quad \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{1}{2\alpha}(a_{ij} - \alpha^{-2}Y_iY_j) + \frac{2}{\alpha}b_ib_j,$$

$$B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-2}y^iY_j) + \frac{3}{2\alpha(1+b^2)}b^ib_j - \frac{1}{\alpha^2(1+b^2)}b^iY_j - \frac{(1+4b^2)}{2\alpha^2(1+b^2)}b_jy^i,$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0}.$$

In view of equations (4.3) and (4.4), the relation in equation (2.10) becomes by virtue of equation (4.15), we have $B^i_0 = 0,\ B_{i0} = 0$, which leads $A^m_0 = B^m b_{00}$.

Now contracting equation (4.14) by y^k we get

$$D^i_{j0} = B^ib_{j0} + B^i_jb_{00} - B^mC^i_{jm}b_{00}.$$

Again contracting the above equation with respect to y^j, we have

$$D^i_{00} = B^ib_{00} = \{\tfrac{1}{1+b^2}b^i + \tfrac{1}{\alpha(1+b^2)}y^i\}b_{00}.$$

Paying attention to equation (4.1), along a Finslerian hypersurface $F^{(n-1)}_{(c)}$, we get

$$b_iD^i_{j0} = \frac{b^2}{(1+b^2)}b_{j0} + \frac{(1+4b^2)}{2\alpha(1+b^2)}b_jb_{00} + \frac{1}{(1+b^2)}b_ib^mC^i_{jm}b_{00} - \frac{b^2}{\alpha^2(1+b^2)}Y_jb_{00}. \tag{4.16}$$

Now we contracting equation (4.16) by y^j, we have

$$b_iD^i_{00} = 0. \tag{4.17}$$

From equations (3.3), (4.5), (4.6), (4.9) and $M_\alpha = 0$, we have

$$b_ib^mC^i_{jm}B^j_\alpha = b^2M_\alpha = 0.$$

Thus the relation $b_{i|j} = b_{ij} - b_rD^r_{ij}$ the equations (4.16) and (4.17) gives

$$b_{i|j}y^iy^j = b_{00} - b_rD^r_{00} = b_{00}.$$

Consequently equations (4.12) and (4.13) can be written as

$$\sqrt{\frac{b^2}{1+b^2}}H_\alpha + b_{i0}B^i_\alpha = 0, \tag{4.18}$$

$$\sqrt{\frac{b^2}{1+b^2}}H_0 + b_{00} = 0.$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_iy^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij}y^iy^j = b_iy^ib_jy^j$, for some $c_j(x)$. Thus, we can expressed

$$2b_{ij} = b_ic_j + b_jc_i. \tag{4.19}$$

Now from equations (4.1) and (4.19), we get

$$b_{00} = 0, \quad b_{ij}B^i_\alpha B^j_\beta = 0, \quad b_{ij}B^i_\alpha y^j = 0.$$

Hence from equation (4.18), we get $H_\alpha = 0$, again from equations (4.19) and (4.15), we get $b_{i0}b^i = \frac{c_0b^2}{2},\ \lambda^m = 0,\ A^i_jB^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = \frac{1}{2\alpha}h_{\alpha\beta}$.

Now we use equations (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14) then we have

$$b_rD^r_{ij}B^i_\alpha B^j_\beta = -\frac{c_0b^2}{4\alpha(1+b^2)^{\frac{3}{2}}}h_{\alpha\beta}. \tag{4.20}$$

Thus the equation (4.11) will become to

$$\sqrt{\frac{b^2}{1+b^2}}H_{\alpha\beta} + \frac{c_0 b^2}{4\alpha(1+b^2)^{\frac{3}{2}}}h_{\alpha\beta} = 0. \tag{4.21}$$

Hence the Finslerian hypersurface $F^{n-1}_{(c)}$ is an umbilic.

Theorem 4.3. *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric to be a hyperplane of first kind is (4.19).*

Corollary 4.4. *The second fundamental h-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric is directly proportional to its angular metric tensor.*

Now from lemma (3.3), Finslerian hypersurface $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from equation (4.20), we get

$$c_0 = c_i(x)y^i = 0.$$

Therefore, there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x)b_i(x).$$

Therefore, from equation (4.19) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x).$$

This can also be written as

$$b_{ij} = \psi(x)b_ib_j.$$

Using lemma (3.3) and equation (4.21), we have

Theorem 4.5. *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric to be a hyperplane of second kind is (4.21).*

Again from lemma (3.4), together with equation (4.9) and $M_\alpha = 0$ shows that Finslerian hypersurface $F^{n-1}_{(c)}$ is not a hyperplane of third kind.

Theorem 4.6. *The Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with special generalized (α, β)-metric is not a hyperplane of the third kind.*

5 Conclusions

The Lorentz force law are the case of inhomogeneous and isotropic medium which can be written in terms of geodesic equation of Finsler space with Rander's metric while the measurement of slope of Mountain surface with respect to time are considered by Finsler space with Matsumoto metric. In the present paper, the class of special generalized (α, β)-metric is a significant non-Riemannian Finsler metric and generalization of Rander's metric, Kropina metric, Matsumoto metric, Z. Shen square metric and some others special metric.

The special generalized (α, β)-metric $L = \alpha\phi(\frac{\beta}{\alpha}) = \alpha\phi(s)$ [4] is also written in the form of Exponential (α, β)-metric as $L = \alpha e^{(\frac{\beta}{\alpha})} = \alpha e^s$.If we apply the certain condition in exponential form metric, we have a special (α, β)-metric where, $\phi(s) = \phi(0)+s\phi'(0)+\frac{s^2}{2}\phi''(0)+\frac{s^3}{6}\phi'''(0) = 1 + s + \frac{s^2}{2} + \frac{s^3}{6}$.Therefore, the condition of special generalized (α, β)-metric is very important and applicable in future work of Finsler Geometry.

In this paper, we obtained Finslerian hypersurface of a special generalized (α, β)-metric with certain conditions in the form $L(\alpha, \beta) = \alpha\phi(\frac{\beta}{\alpha})$. Further, we obtained the necessary and sufficient condition for a Finslerian Hypersurface $F^{(n-1)}(c)$ of a Finsler space $F^n(c)$ equipped with special generalized (α, β) metric will be hyperplane of first, second and third kind in the theorem (4.3), (4.5) and (4.6) respectively.

Acknowledgement : The author is very thankful to the Referee's for their valuable comments and suggestions.

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Received: 2022-05-19
Accepted: 2022-12-09