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A NEW SYMMETRIC APPROACH TO FIBONACCI NUMBERS AND THEIR PROPERTIES

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Abstract In this paper, we exploit the concept of a particular $n \times n$ symmetric matrix of the form $F = [F_k]_{n\times n}$, where $k = \max(i, j)+1$ and F_K is the kth Fibonacci number. We investigate some special properties of this new matrix. In addition, we construct the Hadamard exponential form of this new matrix. We compute the spectral norm, determinant, principal minors, some upper and lower bounds of this matrix and its Hadamard exponential. Finally, we prove that its Hadamard inverse is positive definite, and investigate some properties of its Hadamard inverse.

1 Introduction

Recently, numerous researchers [8, 9, 10, 11, 12, 13, 14, 15] studied many properties of particular matrices involving special numbers. Specifically, Akbulak and Bozkort [1] investigated the properties of Toeplitz matrices involving Fibonacci and Lucas numbers. Akbulak [2] studied Hadamard exponential matrix of the form $e^{\circ H_n} = [e^{i+j}]_{i,j}$. Bozkurt [3] found l_p norms of almost Cauchy-Toeplitz matrices. Solak and Bozkort [4] determined bounds for the spectral and l_p norms of Cauchy-Hankel matrices of the form $H_n = [\frac{1}{g+kh}]_{i,j=0}^n$, where k is defined by $i + j = k$ and g, h are any positive numbers. Civciv and Turkmen [6] established a lower and upper bound for the l_p norms of the Khatri-Rao product of Cauchy-Hankel matrix of the form $H_n = [\frac{1}{0/5+i+j}]_{i,j=0}^n$.

Koken and Bozkurt [7] defined the Lucas QL-matrix similar to the Fibonacci Q-matrix and found some well-known equalities and a Binet-like formula for the Lucas numbers. In [11], Petroudi and Pirouz investigated some properties of particular circulant matrix involving Van Der Laan hybrid sequence. Solak and Bahsi [14] studied the matrix of the form $B = [b_{ij}]$ where$b_{ij} = a+\min(i, j)-1$. They studied some properties of this matrix. For more information of the concerned literature, one can go through these references [1, 2, 3, 4, 5]

2 Preliminiaries and Definitions

If we start from $n = 0$, then the Fibonacci and Lucas numbers F_n and L_n are given respectively by

$$F_0 = 0, F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2;$$
$$L_0 = 2, L_1 = 1,\ L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

From [1, 7, 8], we have the following relations for Fibonacci and Lucas numbers

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1; \quad F_{m+n} = F_{m+1}F_n + F_mF_{n-1}, \tag{2.1}$$

$$\sum_{i=1}^{n-1} F_i^2 = F_nF_{n-1}; \quad L_n = 2\alpha^n - \sqrt{5}F_n. \tag{2.2}$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \alpha\beta = -1,$$
$$L_n = \alpha^n + \beta^n, \alpha^2 = \frac{3+\sqrt{5}}{2}, \beta = \frac{3-\sqrt{5}}{2}, \tag{2.3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

Let $A = [a_{ij}]$ is an $n \times n$ matrix then the maximum column length norm $C_1(.)$ and maximum row length norm $r_1(.)$ of A [1] is defined by

$$C_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2} \quad , r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}. \tag{2.4}$$

Hadamard exponential and Hadamard inverse of this matrix are defined respectively by $e^{\circ A} = e^{a_{ij}}$ and $A^{\circ(-1)} = (\frac{1}{a_{ij}})$. Accoeding to [3], the l_p norm of A can be defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}. \tag{2.5}$$

For $p = 2$, this norm is called Frobenius or Euclidean norm and denoted by $\|A\|_F$or$\|A\|_E$. By [3], the spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} |\lambda_i|}, \tag{2.6}$$

where λ_i are the eigenvalues of matrix AA^H. Also, A^H is conjugate transpose of A. The inequality between the Frobenius and spectral norm [6] can be given as follows

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F. \tag{2.7}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices. Hadamard product of A and B is defined by $A \circ B = [a_{ij}b_{ij}]$. Let A, B and C are $m \times n$ matrices and $A = B \circ C$. Then we have [7]

$$\|A\|_2 \le r_1(B).c_1(C). \tag{2.8}$$

It is known that

$$\begin{aligned} &\textstyle\sum_{k=1}^{n-1} x^k = x + x^2 + \cdots + x^{n-1} = \frac{x^n - x}{x-1}, \\ &\textstyle\sum_{k=1}^{n-1} kx^k = \frac{(n-1)x^{n+1} - nx^{n-1} + x}{(x-1)^2}. \end{aligned} \tag{2.9}$$

In 2012, Abulak and Ipek [2] defined Hadamard inverse $F^{\circ(-1)}$ and Hadamard exponential $H_n = e^{\circ[F]_{n\times n}}$ of F as follows

$$F^{\circ(-1)} = \begin{bmatrix} \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \frac{1}{F_3} & \frac{1}{F_3} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \frac{1}{F_4} & \frac{1}{F_4} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{F_{n+1}} & \frac{1}{F_{n+1}} & \frac{1}{F_{n+1}} \cdots & \frac{1}{F_{n+1}} \end{bmatrix}, \tag{2.10}$$

and

$$H_n = e^{\circ[F]_{n\times n}} = \begin{bmatrix} e^{F_2} & e^{F_3} & e^{F_4} \cdots & e^{F_{n+1}} \\ e^{F_3} & e^{F_3} & e^{F_4} \cdots & e^{F_{n+1}} \\ e^{F_4} & e^{F_4} & e^{F_4} \cdots & e^{F_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ e^{F_{n+1}} & e^{F_{n+1}} & e^{F_{n+1}} \cdots & e^{F_{n+1}} \end{bmatrix}. \tag{2.11}$$

3 Main Results

In this section, we introduce a particular $n \times n$ matrix $F = [F_k]_{n\times n} = \left[F_{\max(i,j)+1}\right]_{i,j=1}^{n}$, where F_k (for $k = 0, 1, 2, \ldots, n$), is the kth Fibonacci number, in the following form

$$F = \begin{bmatrix} 1 & 2 & 35\cdots & F_{n+1} \\ 2 & 2 & 35\cdots & F_{n+1} \\ 3 & 3 & 35\ldots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n+1} & F_{n+1} & F_{n+1}\,F_{n+1}\cdots & F_{n+1} \end{bmatrix} = \begin{bmatrix} F_2 & F_3 & F_4\cdots & F_{n+1} \\ F_3 & F_3 & F_4\cdots & F_{n+1} \\ F_4 & F_4 & F_4\ldots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n+1} & F_{n+1} & F_{n+1}\cdots & F_{n+1} \end{bmatrix}. \tag{3.1}$$

We observe that this matrix is symmetric. First, we find determinant of this matrix. Then, we show that this matrix is invertible and find its inversion.

Theorem 3.1. *Let F be a matrix as in (3.1). Then we have*

$$\det(F) = (-1)^{n+1}F_{n+1}\prod_{i=2}^{n}(F_{i+1}-F_i) = (-1)^{n+1}F_{n+1}\prod_{i=2}^{n}F_{i-1}. \tag{3.2}$$

Proof. By definition of F, we have

$$\det(F) = \det\begin{bmatrix} F_2 & F_3 & F_4\cdots & F_{n+1} \\ F_3 & F_3 & F_4\cdots & F_{n+1} \\ F_4 & F_4 & F_4\ldots & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n+1} & F_{n+1} & F_{n+1}\cdots & F_{n+1} \end{bmatrix} \tag{3.3}$$

If we use elementary row operations then we get

$$\det(F) = \det\begin{bmatrix} F_2 & F_3 & F_4 & \cdots & F_{n+1} \\ F_3-F_2 & 0 & 0 & \cdots & 0 \\ F_4-F_2 & F_4-F_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{n+1}-F_2 & F_{n+1}-F_3 & F_{n+1}-F_4 & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

By expanding this determinant, we obtain

$$\det(F) = (-1)^{n+1}F_{n+1}\prod_{i=2}^{n}(F_{i+1}-F_i) = (-1)^{n+1}F_{n+1}\prod_{i=2}^{n}F_{i-1}. \tag{3.5}$$

Theorem 3.2. *Let F be a Matrix as in (3.1). Then, F is invertible and the inversion of F is a tridiagonal matrix of the form*

$$G = [g_{ij}] = F^{-1} = \begin{cases} g_{11} = -F_2 \\ g_{ij} = 0 \quad for\,|i-j| > 1 \\ g_{ii} = -\frac{F_i}{F_{i-1}F_{i-2}} \quad for 1 < i = j < n \\ g_{ij} = \frac{1}{F_{i-1}} \quad for\,|i-j| = 1 \\ g_{nn} = \frac{-F_n}{F_{n-1}F_{n+1}} \end{cases}. \tag{3.6}$$

Proof. By theorem 3.1 it is clear that F is nonsingular. So, F is invertible. Now, to proving the theorem, we need a lemma from matrix algebra.

Lemma 3.3. *Let A is an $n \times n$ nonsingular matrix, b is an $n \times 1$ matrix and c is a real number. If we take $M = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$. Then we have*

$$M^{-1} = \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}bb^TA^{-1} & -\frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix}, \tag{3.7}$$

where $k = c - b^TA^{-1}b$.

Proof. By multiplying of two matrices M, M^{-1}, we have

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} . \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}bb^TA^{-1} & -\frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix} = I_{n+1}. \tag{3.8}$$

Thus the proof is completed.

Now, by principal of mathematical induction on n, we prove the theorem 3.2.
Proof. The result is true for $n = 2$ that is

$$F = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} F_2 & F_3 \\ F_3 & F_3 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -F_2 & 1 \\ 1 & \frac{F_2}{F_3F_1} \end{bmatrix}.$$

Now, assume that the result is true for n, that is

$$F = [F_{\max(i,j)+1}]_{i,j=1}^n, F^{-1} = ([F_{\max(i,j)+1}]_{i,j=1}^n)^{-1}.$$

We prove that the result is true for $n+1$.

By taking $M = [F_{\max(i,j)+1}]_{i,j=1}^{n+1}$ and $A = [F_{\max(i,j)+1}]_{i,j=1}^n$, we have

$$b^T = \begin{bmatrix} F_{n+2} & F_{n+2} & \cdots & F_{n+2} \end{bmatrix}, c = F_{n+2}, \frac{1}{k} = -\frac{F_{n+1}}{F_nF_{n+2}}, \frac{1}{k}A^{-1}b = \begin{bmatrix} 0 & 0 \cdots \frac{1}{F_n} \end{bmatrix}^T,$$

$$-\frac{1}{k}b^TA^{-1} = \begin{bmatrix} 0 & 0 \cdots \frac{1}{F_n} \end{bmatrix}.$$

By substituting these values along with lemma 3.3, we get the result.

Theorem 3.4. *Let F be as in (3.1), then the Euclidean norm of F is*

$$\|F\|_E = \sqrt{\frac{2}{5}(nL_{2n+4} - (n+1)L_{2n+2} + L_2) + \frac{2n(-1)^n}{5} - \frac{1}{5}(L_{2n+3} - L_3)}. \tag{3.9}$$

Proof. By definition of F and Euclidean norm, we have

$$\|F\|_E^2 = (2n-1)F_{n+1}^2 + (2n-3)F_n^2 + (2n-5)F_{n-1}^2 + \ldots + 3F_3^2 + F_2^2 = \sum_{k=1}^n (2k-1)F_{k+1}^2.$$

By using (2.11), we have

$$\begin{aligned} \|F\|_E^2 &= \sum_{k=1}^n (2k-1)\left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right)^2 = 2\sum_{k=1}^n k\left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right)^2 - \sum_{k=1}^n \left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right)^2 \\ &= \frac{2}{5}\sum_{k=1}^n k\left(\alpha^{2k+2} + \beta^{2k+2} - 2(\alpha\beta)^{k+1}\right) - \frac{1}{5}\sum_{k=1}^n \left(\alpha^{2k+2} + \beta^{2k+2} - 2(\alpha\beta)^{k+1}\right). \end{aligned}$$

Now, according to (2.9), we get

$$\begin{aligned}&\|F\|_E^2 = \tfrac{2}{5}\left[\alpha^2\left(\tfrac{n(\alpha^2)^{n+2}-(n+1)(\alpha^2)^{n+1}+\alpha^2}{(\alpha^2-1)^2}\right)+\beta^2\left(\tfrac{n(\beta^2)^{n+2}-(n+1)(\beta^2)^{n+1}+\beta^2}{(\beta^2-1)^2}\right)\right]\\ &+\tfrac{4}{5}\left(\tfrac{n(\alpha\beta)^{n+2}-(n+1)(\alpha\beta)^{n+1}+\alpha\beta}{(\alpha\beta-1)^2}\right)-\tfrac{1}{5}\left[\alpha^2\left(\tfrac{(\alpha^2)^{n+1}-\alpha^2}{\alpha^2-1}\right)+\beta^2\left(\tfrac{(\beta^2)^{n+1}-\beta^2}{\beta^2-1}\right)-2\alpha\beta\left(\tfrac{(\alpha\beta)^{n+1}-\alpha\beta}{\alpha\beta-1}\right)\right].\end{aligned}$$

If we set $\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}$, then, we have

$$\alpha\beta=1, \alpha^2-1=\left(\frac{1+\sqrt{5}}{2}\right)^2-1=\frac{1+\sqrt{5}}{2}=\alpha, \beta^2-1=\left(\frac{1-\sqrt{5}}{2}\right)^2-1=\frac{1-\sqrt{5}}{2}=\beta.$$

So, we get

$$\begin{aligned}\|F\|_E^2 &= \tfrac{2}{5}\left(n(\alpha^{2n+4}+\beta^{2n+4})-(n+1)(\alpha^{2n+2}+\beta^{2n+2})+(\alpha^2+\beta^2)\right)+\tfrac{2n(-1)^n}{5}\\ &\quad-\tfrac{1}{5}\left((\alpha^{2n+3}+\beta^{2n+3})-(\alpha^3+\beta^3)\right)\\ &= \tfrac{2}{5}\left(nL_{2n+4}-(n+1)L_{2n+2}+L_2\right)+\tfrac{2n(-1)^n}{5}-\tfrac{1}{5}\left(L_{2n+3}-L_3\right).\end{aligned}$$

By taking $\frac{1}{2}$th power from the both sides of above equalities, we have

$$\|F\|_E=\sqrt{\frac{2}{5}\left(nL_{2n+4}-(n+1)L_{2n+2}+L_2\right)+\frac{2n(-1)^n}{5}-\frac{1}{5}\left(L_{2n+3}-L_3\right)}. \tag{3.10}$$

Theorem 3.5. *Let F be as in (3.1), then we have the following upper and lower bounds for the spectral norm of F.*

$$\begin{aligned}&\tfrac{1}{\sqrt{n}}\sqrt{\tfrac{2}{5}\left(nL_{2n+4}-(n+1)L_{2n+2}+L_2\right)+\tfrac{2n(-1)^n}{5}-\tfrac{1}{5}\left(L_{2n+3}-L_3\right)}\le\|F\|_2\\ &\le\sqrt{\tfrac{2}{5}\left(nL_{2n+4}-(n+1)L_{2n+2}+L_2\right)+\tfrac{2n(-1)^n}{5}-\tfrac{1}{5}\left(L_{2n+3}-L_3\right)}\end{aligned}$$

Proof. It follows from theorem 3.4 and statement (2.7).

Theorem 3.6. *Let F be as in (3.1), then we have the following upper bound for spectral norm of F.*

$$\|F\|_2\le F_{n+1}\sqrt{n(F_nF_{n+2}-1)}. \tag{3.11}$$

Proof. By definition of A, we have

$$\begin{aligned}&\begin{bmatrix}F_2 & F_3 & F_4\cdots & F_{n+1}\\ F_3 & F_3 & F_4\cdots & F_{n+1}\\ F_4 & F_4 & F_4\ldots & F_{n+1}\\ \vdots & \vdots & \vdots & \vdots\\ F_{n+1} & F_{n+1} & F_{n+1}\cdots & F_{n+1}\end{bmatrix}\\ &=\begin{bmatrix}F_2 & 1 & 1\cdots & 1\\ F_3 & F_3 & 1\cdots & 1\\ F_4 & F_4 & F_4\ldots & 1\\ \vdots & \vdots & \vdots & \vdots\\ F_{n+1} & F_{n+1} & F_{n+1}\cdots & 1\end{bmatrix}\circ\begin{bmatrix}1 & F_3 & F_4\cdots & F_{n+1}\\ 1 & 1 & F_4\cdots & F_{n+1}\\ 1 & 1 & 1\ldots & F_{n+1}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1\cdots & F_{n+1}\end{bmatrix}\\ &=A\circ B.\end{aligned}$$

By definition of row maximum length norm and column maximum length norm, we have

$$r_1(A)=\max_i\sqrt{\sum_j|a_{ij}|^2}=\sqrt{\sum_{i=2}^{n+1}F_i^2}=\sqrt{\sum_{i=2}^{n}F_i^2+F_{n+1}^2}. \tag{3.12}$$

By applying (2.1), we get

$$r_1(A) = \sqrt{F_nF_{n+1} - F_1 + F_{n+1}^2} = \sqrt{F_{n+1}(F_n + F_{n+1}) - 1} = \sqrt{F_nF_{n+2} - 1}. \tag{3.13}$$

$$c_1(B) = \max_j \sqrt{\sum_i |b_{ij}|^2} = \sqrt{nF_{n+1}^2} = \sqrt{n}F_{n+1}. \tag{3.14}$$

From (2.8) we know $\|F\|_2 \leq r_1(A)c_1(B)$. Thus, we have

$$\|F\|_2 \leq \sqrt{F_nF_{n+2} - 1}\sqrt{n}F_{n+1} = F_{n+1}\sqrt{n(F_nF_{n+2} - 1)}. \tag{3.15}$$

Theorem 3.7. *Let F be as in (3.1), then determinant of Hadamard inverse of F is*

$$\det(F^{\circ(-1)}) = \frac{1}{F_{n+1}} \prod_{2}^{n} \frac{1}{F_{i+1}}. \tag{3.16}$$

Proof. By definition of Hadamard inverse, we have

$$\det\left(F^{\circ(-1)}\right)$$

$$= \det \begin{bmatrix} \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \frac{1}{F_3} & \frac{1}{F_3} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \frac{1}{F_4} & \frac{1}{F_4} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{F_{n+1}} & \frac{1}{F_{n+1}} & \frac{1}{F_{n+1}} \cdots & \frac{1}{F_{n+1}} \end{bmatrix} = \begin{bmatrix} \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} \cdots & \frac{1}{F_{n+1}} \\ \frac{1}{F_3} - \frac{1}{F_2} & 0 & 0 \cdots & 0 \\ \frac{1}{F_4} - \frac{1}{F_2} & \frac{1}{F_4} - \frac{1}{F_3} & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{F_{n+1}} - \frac{1}{F_2} & \frac{1}{F_{n+1}} - \frac{1}{F_3} & \frac{1}{F_{n+1}} - \frac{1}{F_4} \cdots & 0 \end{bmatrix}$$

$$= (-1)^{n+1}\frac{1}{F_{n+1}}\left(\frac{1}{F_3} - \frac{1}{F_2}\right)\left(\frac{1}{F_4} - \frac{1}{F_3}\right)\cdots\left(\frac{1}{F_{n+1}} - \frac{1}{F_n}\right)$$

$$= \frac{(-1)^{n+1}}{F_{n+1}}\left(\frac{F_2 - F_3}{F_2F_3}\right)\left(\frac{F_3 - F_4}{F_3F_4}\right)\cdots\left(\frac{F_n - F_{n+1}}{F_nF_{n+1}}\right)$$

$$= \frac{(-1)^{n+1}}{F_{n+1}}\left(\frac{-F_1}{F_2F_3}\right)\left(\frac{-F_2}{F_3F_4}\right)\cdots\left(\frac{-F_{n-1}}{F_nF_{n+1}}\right)$$

$$= \frac{(-1)^{n+1}}{F_{n+1}}(-1)^{n-1}\frac{1}{F_3F_4F_5\cdots F_nF_{n+1}}$$

$$= \frac{1}{F_{n+1}}\prod_{i=2}^{n}\frac{1}{F_{i+1}}.$$

Thus, the proof is completed.

Corollary 3.8. *Let $F^{\circ(-1)}$ be as in (2.10), then $F^{\circ(-1)}$is a positive definite matrix.*

Proof. According to theorem 3.7, all leading principal minors of $F^{\circ(-1)}$ are positive, thus the result follows from [16]. □

Corollary 3.9. *Let $F^{\circ(-1)}$ be as in (2.10), then all eigenvalues of $F^{\circ(-1)}$ are positive.*

Proof. Since $F^{\circ(-1)}$ is positive definite, so all eigenvalues of $F^{\circ(-1)}$ are positive [16].

Example 3.10. Let $F^{\circ(-1)}$ be as in (2.10). We represent in Table 1, determinants and eigenvalues of $F^{\circ(-1)}$ for some values of n.

n	$\det(F^{\circ(-1)})$	Eigen values of each A (is rounded off to four decimal places)
2	0.2500	1.3090, 0.1910
3	0.0275	0.0682, 0.2731, 1.4921
4	0.0022	0.0454, 0.0981, 0.3172, 1.5727
5	1.0417×10^{-4}	0.0279, 0.0550, 0.1225, 0.3437, 1.6091
6	3.0819×10^{-6}	0.0174, 0.0358, 0.0616, 0.1379, 0.3581, 1.6244

Table1. Determinants and eigenvalues of $F^{\circ(-1)}$

Theorem 3.11. *Let $F^{\circ(-1)}$ be as in (2), then $F^{\circ(-1)}$ is invertible and the inversion of $F^{\circ(-1)}$ is a tridiagonal matrix of the form*

$$G = [g_{ij}] = \left(F^{\circ(-1)}\right)^{-1} = \begin{cases} g_{11} = F_2 \\ g_{ij} = 0 \;\; for\, |i-j| > 1 \\ g_{ii} = \frac{F_{i-1}^3}{F_{i-3}F_{i-2}} for 1 < i = j < n \\ g_{ij} = -\frac{F_{i-1}F_i}{F_{i-2}} for\, |i-j| = 1 \\ g_{nn} = \frac{F_n^2}{F_{n-2}} \end{cases} . \tag{3.17}$$

Proof. The proof is similar to theorem 3.2.

Theorem 3.12. *Let $H_n = e^{\circ[F]_{n\times n}}$be a matrix as in (2.11), then*

$$\begin{aligned} &\det\left(e^{\circ[F]_{2\times2}}\right) = e^3(1-e), \\ &\det\left(e^{\circ[F]_{3\times3}}\right) = e^6(1-e)^2, \\ &\det\left(e^{\circ[F]_{n\times n}}\right) = -e^{F_{n+1}}\left(e^{F_{n-1}}-1\right)\det\left(e^{\circ[F]_{(n-1)\times(n-1)}}\right) ; for n \geq 4. \end{aligned} \tag{3.18}$$

Proof. For $n = 2, 3$, determinant of F can be easily calculated by definition of determinant. For $n \geq 4$, we have

$$\det(H_n) = \det(e^{\circ[F]_{n\times n}}) = \det \begin{bmatrix} e^{F_2} & e^{F_3} & e^{F_4}\dots & e^{F_{n+1}} \\ e^{F_3}-e^{F_2} & 0 & 0\cdots & 0 \\ e^{F_4}-e^{F_2} & e^{F_4}-e^{F_3} & 0\dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ e^{F_{n+1}}-e^{F_2} & e^{F_{n+1}}-e^{F_3} & e^{F_{n+1}}-e^{F_4}\dots & 0 \end{bmatrix}$$

$$\begin{aligned} &= (-1)^{n+1}e^{F_{n+1}}\left(e^{F_3}-e^{F_2}\right)\left(e^{F_4}-e^{F_3}\right)\left(e^{F_5}-e^{F_4}\right)\cdots\left(e^{F_{n+1}}-e^{F_n}\right) \\ &= (-1)^{n+1}e^{F_{n+1}}\prod_{i=2}^{n}\left(e^{F_{i+1}}-e^{F_i}\right) \\ &= \left[(-1)^n\left(e^{F_3}-e^{F_2}\right)\left(e^{F_4}-e^{F_3}\right)\left(e^{F_5}-e^{F_4}\right)\cdots\left(e^{F_n}-e^{F_{n-1}}\right)\right]\left(-e^{F_n}\left[e^{F_{n+1}}(e^{F_{n-1}}-1)\right]\right) \\ &= -e^{F_{n+1}}(e^{F_{n-1}}-1)\det\left(e^{\circ[F]_{(n-1)\times(n-1)}}\right). \end{aligned}$$

Theorem 3.13. *Let $H_n = e^{\circ[F]_{n\times n}}$ be as in (2.11), then H_n is invertible and we have*

$$H_2^{-1} = \left(e^{\circ[F]_{2\times2}}\right)^{-1} = \begin{bmatrix} -\frac{1}{e^2-e} & \frac{1}{e^2-e} \\ \frac{1}{e^2-e} & \frac{1}{e^3-e^2} \end{bmatrix}, \tag{3.19}$$

$$H_3^{-1} = \left(e^{\circ[F]_{3\times3}}\right)^{-1} = \begin{bmatrix} -\frac{1}{e^2-e} & \frac{1}{e^2-e} & 0 \\ \frac{1}{e^2-e} & -\frac{(e+1)}{e^2(e-1)} & \frac{1}{e^3-e^2} \\ 0 & \frac{1}{e^3-e^2} & -\frac{1}{e^3(e-1)} \end{bmatrix}. \tag{3.20}$$

And for $n \geq 4$ we have

$$H_n^{-1} = \left(e^{\circ[F]_{n\times n}}\right)^{-1} = \begin{bmatrix} \frac{-1}{e^2-e} & \frac{1}{e^2-e} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{e^2-e} & \frac{-(e+1)}{e^2-e} & \frac{1}{e^3-e^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{e^3-e^2} & \frac{e^2+e+1}{e^3-e^5} & \frac{-1}{e^3-e^5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{e^3-e^5} & D & \frac{-1}{e^5-e^8} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{e^{F_{n-1}}-e^{F_n}} & E & \frac{-1}{e^{F_n}-e^{F_{n+1}}} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{e^{F_n}-e^{F_{n+1}}} & \frac{-1}{e^2-e} \end{bmatrix},$$

(3

where

$$D = \frac{e^4+e^3+e^2+e+1}{e^5(-e^4-e^3+e+1)}, E = \frac{\sum_{k=0}^{F_n-1} e^k}{e^{F_n}\left(-\sum_{k=F_{n-1}}^{F_n-1} e^k + \sum_{k=0}^{F_{n-2}-1} e^k\right)}.$$

Proof. By theorem 3.12, $H_n = e^{\circ[F]_{n\times n}}$ is nonsingular. So, it is invertible. For $n = 2, 3$ the inversion of $H_n = e^{\circ[F]_{n\times n}}$ can be easily computed by definition. We can prove this theorem for $n \geq 4$, by applying similar method which is used in theorem 3.2.

Corollary 3.14. *If we set $D_n = |\det(F)|$, in particular $D_1 = 1, D_2 = |\det([F]_{2\times 2})|, \ldots, D_n = |\det([F]_{n\times n})|$, then we have*

$$D_n D_{n-2} = \frac{F_{n+1}F_{n-1}^2}{F_n^2 F_{n-2}} D_{n-1}^2.$$

Proof. By definition of D_n, we have

$$\begin{aligned} D_n D_{n-2} &= \left(F_{n+1} \textstyle\prod_{i=2}^{n} (F_{i+1}-F_i)\right)\left(F_{n-1} \textstyle\prod_{i=2}^{n-2} (F_{i+1}-F_i)\right) \\ &= F_{n+1}F_{n-1}(F_{n+1}-F_n)(F_n-F_{n-1})\left(\textstyle\prod_{i=2}^{n-2}(F_{i+1}-F_i)\right)^2 \\ &= \tfrac{F_{n+1}F_{n-1}^2}{F_n^2F_{n-2}}\left(F_n^2F_{n-2}^2\left(\textstyle\prod_{i=2}^{n-2}(F_{i+1}-F_i)\right)^2\right) = \tfrac{F_{n+1}F_{n-1}^2}{F_n^2F_{n-2}} D_{n-1}^2. \end{aligned}$$

Thus, the proof is completed.

Corollary 3.15. *Let Δ_i denotes the leading principal minors of $F^{\circ(-1)}$. In particular we take $\Delta_1 = 1, \ldots, \Delta_n = \det(F^{\circ(-1)})$. Then we have*

$$1)\Delta_n\Delta_{n-2} \leq \Delta_{n-1}^2,$$

$$2)\Delta_n\Delta_{n-2} = \frac{F_n^3}{F_{n+1}^2F_{n-1}}\Delta_{n-1}^2,$$

$$3)\Delta_1\Delta_2\Delta_3\cdots\Delta_n = \prod_{k=3}^{n+1}\frac{1}{F_k^{n-k+3}} = \prod_{k=3}^{n+1}\frac{F_k^{k-3}}{F_k^n}.$$

Proof. By definition of principal minors and theorem 3.7, we have

$$\begin{aligned} (1)\Delta_n\Delta_{n-2} &= \left(\tfrac{1}{F_{n+1}}\textstyle\prod_{i=3}^{n+1}\tfrac{1}{F_i}\right)\left(\tfrac{1}{F_{n-1}}\textstyle\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right) = \tfrac{1}{F_{n-1}}\tfrac{1}{F_{n+1}}\tfrac{1}{F_{n+1}}\tfrac{1}{F_n}\left(\textstyle\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right)^2 \\ &\leq \left(\tfrac{1}{F_n}\tfrac{1}{F_n}\textstyle\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right)^2 = \Delta_{n-1}^2. \end{aligned}$$

Thus, the proof is completed.

$$
\begin{aligned}
(2)\Delta_n\Delta_{n-2} &= \left(\tfrac{1}{F_{n+1}}\prod_{i=3}^{n+1}\tfrac{1}{F_i}\right)\left(\tfrac{1}{F_{n-1}}\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right) = \tfrac{1}{F_{n-1}}\tfrac{1}{F_{n+1}^2}\tfrac{1}{F_n}\left(\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right)^2 \\
&= \tfrac{F_n^3}{F_{n+1}^2F_{n-1}}\left(\tfrac{1}{F_n}\tfrac{1}{F_n}\prod_{i=3}^{n-1}\tfrac{1}{F_i}\right)^2 = \tfrac{F_n^3}{F_{n+1}^2F_{n-1}}\left(\tfrac{1}{F_n}\prod_{i=3}^{n}\tfrac{1}{F_i}\right)^2 = \tfrac{F_n^3}{F_{n+1}^2F_{n-1}}\Delta_{n-1}^2.
\end{aligned}
$$

Thus, the proof is completed.
Proof of (3) is straightforward by multiplying all Δ_k for $k = 1, \cdots, n$.

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