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Generating dual Rickart (Baer) modules via the cosingular submodule

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Abstract We introduces the concept of dual Rickart (Baer) modules in relation to the cosingular submodule. The paper demonstrates that a module is considered to be $\overline{Z}$-dual Rickart only if its submodule, $\overline{Z}(M)$, is a dual Rickart direct summand of the module M. Additionally, it is proven that a module is considered dual Baer with respect to $\overline{Z}(M)$ only when it is dual Rickart with respect to $\overline{Z}(M)$, and the module has the strong summand sum property for direct summands included in $\overline{Z}(M)$. Lastly, we present a characterization of right $\overline{Z}$-dual Baer rings.

1 Introduction

All rings considered in this paper will be associative with an identity element and all modules will be unitary right modules unless otherwise stated. Let R be a ring and M an R-module. $S = End_R(M)$ will denote the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M)$, $\phi(N) \subseteq N$). $Rad(M)$ and $Soc(M)$ denote the radical and the socle of a module M, respectively.

Let $L \subseteq K \leq M$. We say that K *lies above* L in M if $K/L \ll M/L$. A module M is called *lifting* if every submodule A of M lies above a direct summand D of M ([2]).

Let M be a module. Following [5], M is called *(dual) Rickart* in case for every endomorphism φ of M, ($Im\varphi$) $Ker\varphi$ is a direct summand of M. Researchers in module theory discovered the significance of idempotents in the ring of all endomorphisms of a module through the study of (dual) Rickart modules. A well-known outcome of this research is that a module M is considered Rickart and dual Rickart only when $End_R(M)$ is a von Neumann regular ring

Several studies have been conducted on dual Rickart modules and their extensions. However, this particular research delves into the overall characteristics of $\overline{Z}$-dual Rickart (Baer) modules. The paper presents various conditions that can be used to determine whether a module is $\overline{Z}$-dual Rickart (Baer).

The singular submodule of a module M consists of elements $m \in M$ such that, for some essential right ideal I of R, $mI = 0$. Talebi and Vanaja have introduced the concept of the dual of the singular submodule, denoted as $\overline{Z}(M)$, as the intersection of the kernels of all module homomorphisms $f : M \to U$ such that U is a small right R-module. A module M is referred to as a cosingular module if $\overline{Z}(M) = 0$ and noncosingular if $\overline{Z}(M) = M$. A ring R is called a right V-ring if every simple right R-module is injective, which is equivalent to $Rad(M) = 0$ for all right R-modules M. Any unfamiliar terminology can be found in [6] and [11].

A new approach to generalizing lifting modules has been introduced in [1], which involves utilizing a fixed fully invariant submodule of a given module. In this approach, a module M is considered $\mathcal{I}_F$-lifting (where F is a fully invariant submodule of M) if for every endomorphism g of M, the submodule $g(F)$ is above a direct summand of M. The authors of [1] also investigate various properties of such modules. Building upon this work, Moniri and Amouzegar study H-supplemented modules using the same approach in [7]. A module M is $\mathcal{I}_F$-H-supplemented if for every $g \in End_R(M)$, there exists a direct summand D of M such that $g(F) + X = M$ if and only if $D + X = M$, for all submodules X of M. Additionally, the authors provide some conditions for a $\mathcal{I}_F$-H-supplemented module to be $\mathcal{I}_F$-lifting and investigate the relationship between these and other similar classes of modules. They also study direct sums of $\mathcal{I}_F$-H-supplemented modules. Another related approach is studied in [10], where the authors investigate a version of $\oplus$-supplemented modules using a two-sided ideal of the related ring, namely, I-$\oplus$-supplemented modules, and utilize fully invariant submodules such as IK (where I is an ideal of R and K is a direct summand of M).

2 $\overline{Z}$-dual Rickart modules and $\overline{Z}$-dual Baer modules

One way to begin the section is by providing the main explanation.

Definition 2.1. Let M be a module. We say M is $\overline{Z}$-dual Rickart if for every $\varphi \in End_R(M)$, the submodule $\varphi(\overline{Z}(M))$ is a direct summand of M.

It is important to note that a dual Rickart module M might not be dual Rickart with respect to $\overline{Z}$.

Example 2.2. If a module M has a non-zero small submodule $\overline{Z}(M)$, then for any endomorphism φ of M, the submodule $\varphi(\overline{Z}(M))$ is also a small submodule of M. However, it is possible that $\varphi(\overline{Z}(M))$ is not a direct summand of M for some φ, which means that M is not a $\overline{Z}$-dual Rickart module. An example of such a module is $M = \mathbb{Z}_4$, where $\overline{Z}(M) = J(\mathbb{Z}_4) = \{0, 2\}$.

The following provides an important characterization of $\overline{Z}$-dual Rickart modules which will be used freely throughout the paper.

Theorem 2.3. *Let M be a module. Then the following conditions are equivalent:*
(1) *M is $\overline{Z}$-dual Rickart;*
(2) *$M = \overline{Z}(M) \oplus L$ where $\overline{Z}(M)$ is a dual Rickart module.*

Proof. (1) $\Rightarrow$ (2) Let M be $\overline{Z}$-dual Rickart. Then it is clear that $\overline{Z}(M)$ is a direct summand of M. Set $M = \overline{Z}(M) \oplus L$ for a submodule L of M. Suppose that g is an endomorphism of $\overline{Z}(M)$. Then $h = j \circ g \circ \pi$ is an endomorphism of M such that j is the inclusion from $\overline{Z}(M)$ to M and π is the projection of M on $\overline{Z}(M)$. Being M a $\overline{Z}$-dual Rickart module implies $h(\overline{Z}(M)) = Img$ is a direct summand of M and hence a direct summand of $\overline{Z}(M)$ as $h(\overline{Z}(M))$ is contained in $\overline{Z}(M)$.

(2) $\Rightarrow$ (1) Let $M = \overline{Z}(M) \oplus L$ such that $\overline{Z}(M)$ is dual Rickart. Suppose that φ is an endomorphism of M. Then $\lambda = \pi \circ \varphi \circ j$ will be an endomorphism of $\overline{Z}(M)$ where $j : \overline{Z}(M) \to M$ is the inclusion and $\pi : M \to \overline{Z}(M)$ is the projection on $\overline{Z}(M)$. As $\lambda(\overline{Z}(M)) = \varphi(\overline{Z}(M))$ and $\overline{Z}(M)$ is a dual Rickart module, then $\varphi(\overline{Z}(M))$ is a direct summand of $\overline{Z}(M)$ and consequently of M, as required.

Remark 2.4. If we have an indecomposable module M and its submodule $\overline{Z}(M)$ is not zero, then M is considered $\overline{Z}$-dual Rickart if and only if its submodule $\overline{Z}(M)$ is the same as the whole module M and M is dual Rickart. This means that if the submodule $\overline{Z}(M)$ is nontrivial, then M cannot be $\overline{Z}$-dual Rickart. For example, a module M that is local and has a non-zero submodule $\overline{Z}(M)$ that is not equal to M is not $\overline{Z}$-dual Rickart. An example of such a module is $\mathbb{Z}_{p^k}$ where p is prime and $k \geq 2$.

We will now attempt to examine a direct summand of a $\overline{Z}$-dual Rickart module inherits the property.

Proposition 2.5. *Let M be a module and N a direct summand of M. If M is $\overline{Z}$-dual Rickart, then N is $\overline{Z}$-dual Rickart.*

Proof. Set $M = N \oplus K$. Consider an arbitrary endomorphism λ of N. Then $f = j \circ \lambda \circ \pi$ will be an endomorphism of M, so that $f(\overline{Z}(M)) = \lambda(\overline{Z}(N))$ is a direct summand of M as M is a dual $\overline{Z}$-Rickart module. Note that $j : N \to M$ is the inclusion and $\pi : M \to N$ is the projection of M on N. It follows that $\lambda(\overline{Z}(N))$ is a direct summand of N, which completes the proof.

Following [4], we present an analogue for dual Baer modules in $\overline{Z}$-case.

Definition 2.6. Let M be a module. We say that M is $\overline{Z}$-dual Baer provided for every right ideal I of $End_R(M)$ the submodule $I\overline{Z}(M) = \sum_{\varphi \in I} \varphi(\overline{Z}(M))$ is a direct summand of M.

The following Theorem introduces some equivalent conditions for a module to be $\overline{Z}$-dual Baer.

Theorem 2.7. *Let M be a module. Then the following are equivalent:*
(1) *M is $\overline{Z}$-dual Baer;*
(2) *$\overline{Z}(M)$ is a dual Baer direct summand of M;*
(3) *M is $\overline{Z}$-dual Rickart and M has $SSSP$ for direct summands of M contained in $\overline{Z}(M)$;*
(4) *For every subset B of $End_R(M)$, the submodule $\sum_{\varphi \in B} \varphi(\overline{Z}(M))$ is a direct summand of M.*

Proof. (1) $\Rightarrow$ (2) Consider $S = End_R(M)$ as an ideal of itself. Then by (1), $S\overline{Z}(M) = \sum_{\varphi \in S} \varphi(\overline{Z}(M)) = \overline{Z}(M)$ is a direct summand of M. Now, let I be a right ideal of $End_R(\overline{Z}(M))$ and consider the inclusion $j : \overline{Z}(M) \to M$ and the projection $\pi_{\overline{Z}(M)} : M \to \overline{Z}(M)$. Consider the subset $I_0 = \{j \circ \lambda \circ \pi_{\overline{Z}(M)} \mid \lambda \in I\}$ of S. Then $J = I_0 S$ is a right ideal of S. As $I\overline{Z}(M) = \sum_{\varphi \in I} \varphi(\overline{Z}(M)) = \sum_{\varphi \in J} \varphi(\overline{Z}(M)) = J\overline{Z}(M)$ and M is a $\overline{Z}$-dual Baer module, we conclude that $I\overline{Z}(M) = J\overline{Z}(M)$ is a direct summand of M and consequently is a direct summand of $\overline{Z}(M)$, as well. It follows from [4, Theorem 2.1], $\overline{Z}(M)$ is a dual Baer module.

(2) $\Rightarrow$ (1) Let I be a right ideal of S and $B = \{\pi_{\overline{Z}(M)} \circ \varphi |_{\overline{Z}(M)} | \varphi \in I\}$. Note that $J = BEnd_R(\overline{Z}(M))$ is a right ideal of $End_R(\overline{Z}(M))$. Since $J\overline{Z}(M) = I\overline{Z}(M)$ and $\overline{Z}(M)$ is a dual Baer module, we conclude that $J\overline{Z}(M)$ is a direct summand of $\overline{Z}(M)$ and hence a direct summand of M.

(1) $\Rightarrow$ (3) Let $\varphi \in S$. As M is $\overline{Z}$-dual Baer and $< \varphi > \overline{Z}(M) = \varphi(\overline{Z}(M))$, then $\varphi(\overline{Z}(M))$ is a direct summand of M. Let $\{e_\gamma \mid \gamma \in \Gamma\}$ be a set of idempotents of S such that $Im e_\gamma \subseteq \overline{Z}(M)$ for each $\gamma \in \Gamma$. Suppose $I =< \sum_{\gamma \in \Gamma} e_\gamma >$ that is an ideal of S. Now, $I\overline{Z}(M) = \sum_{\varphi \in I} \varphi(\overline{Z}(M)) \subseteq \sum_{\gamma \in \Gamma} e_\gamma(M)$. As $e_\gamma(M)$ is contained in $\sum_{\varphi \in I} \varphi(\overline{Z}(M))$, it follows that $\sum_{\gamma \in \Gamma} e_\gamma(M) = \sum_{\varphi \in I} \varphi(\overline{Z}(M)) = I\overline{Z}(M)$ is a direct summand of M (note that M is $\overline{Z}$-dual Baer).

(3) $\Rightarrow$ (4) It follows from the fact that $\overline{Z}(M)$ is fully invariant in M.

(4) $\Rightarrow$ (1) It is obvious.

By Theorem 2.7, every $\overline{Z}$-dual Baer module is $\overline{Z}$-dual Rickart.

Proposition 2.8. *Let M be a regular module. If M satisfies $SSSP$ on direct summands of M contained in $\overline{Z}(M)$, then M is $\overline{Z}$-dual Baer.*

Proof. Let φ be an arbitrary endomorphism of M. As $\varphi(\overline{Z}(M)) = \sum_{x \in \varphi(\overline{Z}(M))} xR$, and M is regular, it follows that $\varphi(\overline{Z}(M))$ is a direct summand of M.

As a consequence of Theorem 2.7 and Proposition 2.8, if M is a regular $\overline{Z}$-dual Baer module then $\overline{Z}(M)$ is a semisimple module.

In the light of Theorem 2.7, we have the following remark.

Remark 2.9. Let M be an indecomposable module such that $\overline{Z}(M) \neq 0$. Then M is $\overline{Z}$-dual Baer if and only if $\overline{Z}(M) = M$ is dual Baer.

We next present an equivalent condition for a module to be $\overline{Z}$-dual Baer.

Theorem 2.10. *Let M be a module. Then M is $\overline{Z}$-dual Baer if and only if for every direct summand N of M is $\overline{Z}$-dual Baer.*

Proof. Let M be $\overline{Z}$-dual Baer and $M = N \oplus N'$ for a submodule N' of M. Then $\overline{Z}(M) = \overline{Z}(N) \oplus \overline{Z}(N')$. Suppose that A is a subset of $End_R(N)$. Then $B = \{j \circ \varphi \circ \pi_N \mid \varphi \in A\}$ in which $\pi_N : M \to N$ is the projection of M on N and j is the inclusion from N to M, is a subset of $End_R(M)$. It is straightforward to check that $A\overline{Z}(N) = \sum_{\varphi \in A} \varphi(\overline{Z}(N)) = \sum_{g \in B} g(\overline{Z}(M))$. Being M, a $\overline{Z}$-dual Baer module implies that $A\overline{Z}(N)$ is a direct summand of M and hence a direct summand of N. The result follows from Theorem 2.7. The converse is straightforward.

Corollary 2.11. *Let M be a module, P a projective module and $f : M \to P$ be an epimorphism such that $Kerf$ is contained in $\overline{Z}(M)$. Then, if M is $\overline{Z}$-dual Baer, then P is $\overline{Z}$-dual Baer.*

3 Relatively $\overline{Z}$-dual Rickart modules

In this section we shall define relative $\overline{Z}$-dual Rickart modules and we will apply this concept to study finite direct sums of $\overline{Z}$-dual Rickart modules.

Definition 3.1. Let M and N be R-modules. We say M is N-$\overline{Z}$*-dual Rickart* if for every homomorphism $\phi : M \to N$, the submodule $\phi(\overline{Z}(M))$ is a direct summand of N.

We provide an equivalent condition for relatively $\overline{Z}$-dual Rickart modules.

Theorem 3.2. *Let M and N be right R-modules. Then M is N-$\overline{Z}$-dual Rickart if and only if for every direct summand L of M and every submodule K of N, L is K-$\overline{Z}$-dual Rickart.*

Proof. Let M be N-$\overline{Z}$-dual Rickart. Suppose that $L = eM$ for some $e^2 = e \in End_R(M)$ and let K be a submodule of N. Assume that $\psi \in Hom(L, K)$. Since $\psi \circ e(M) = \psi(L) \subseteq K \subseteq N$ and M is N-$\overline{Z}$-dual Rickart, $\psi \circ e(\overline{Z}(M))$ is a direct summand of N. As $\psi \circ e(\overline{Z}(M))$ is contained in K, we conclude that $\psi \circ e(\overline{Z}(M))$ is a direct summand of K. We shall prove that $\psi(\overline{Z}(L))$ is a direct summand of K. Suppose that $M = L \oplus L'$. Next, we have $\overline{Z}(M) = \overline{Z}(L) \oplus \overline{Z}(L')$. Then $e(\overline{Z}(M)) = e(\overline{Z}(L)) = \overline{Z}(L)$. Now $\psi \circ e(\overline{Z}(M)) = \psi(\overline{Z}(L))$ combining with M is $\overline{Z}$-dual Rickart relative to N, we come to a conclusion that $\psi(\overline{Z}(L))$ is a direct summand of K.

Proposition 3.3. *Let M be a $\overline{Z}$-dual Rickart module. Then*

(1) *If L and K are direct summands of M with $L \subseteq \overline{Z}(M)$, then $L + K$ is a direct summand of M.*

(2) *M has SSP for direct summands of M that are contained in $\overline{Z}(M)$.*

Proof. (1) Let $K = eM$ and $L = fM$ for some $e^2 = e \in End_R(M)$ and $f^2 = f \in End_R(M)$. Since $M = fM \oplus (1-f)M$, $L = fM \subseteq \overline{Z}(M)$, we have $\overline{Z}(M) = fM \oplus \overline{Z}((1-f)M)$. Then $((1-e)f)(\overline{Z}(M)) = (1-e)fM$. As M is a $\overline{Z}$-dual Rickart module, $((1-e)f)(\overline{Z}(M)) = (1-e)fM$ is a direct summand of M. Since $(1-e)fM = (fM + eM) \cap (1-e)M$, $M = ((fM + eM) \cap (1-e)M) \oplus T$ for some $T \leq M$. Hence $(1-e)M = ((fM + eM) \cap (1-e)M) \oplus (T \cap (1-e)M)$. So $M = eM \oplus (1-e)M = eM + ((fM + eM) \cap (1-e)M) \oplus (T \cap (1-e)M) = (fM + eM) + (T \cap (1-e)M)$. Since $(fM + eM) \cap (T \cap (1-e)M) = 0$, $M = (eM + fM) \oplus (T \cap (1-e)M)$. Hence $K + L$ is a direct summand of M.

(2) It is clear by (1).

Theorem 3.4. *Let M be a module. Then M is $\overline{Z}$-dual Rickart if and only if $\sum_{\phi \in I} \phi(\overline{Z}(M))$ is a direct summand of M for every finitely generated right ideal I of $End_R(M)$.*

Proof. Assume that I is a finitely generated right ideal of $End_R(M)$ generated by $\phi_1, \ldots, \phi_n$. As M is $\overline{Z}$-dual Rickart, $\phi_i(\overline{Z}(M))$ is a direct summand of M for each $1 \leq i \leq n$. By Proposition 3.3, M has SSP for direct summands which are contained in $\overline{Z}(M)$. Since $\phi_i(\overline{Z}(M)) \subseteq \overline{Z}(M)$, $\sum_{\phi \in I} \phi(\overline{Z}(M)) = \phi_1(\overline{Z}(M)) + \cdots + \phi_n(\overline{Z}(M))$ is a direct summand of M. The converse is obvious.

4 Applications of $\overline{Z}$-dual Baer modules to rings

We will now apply the concept of $\overline{Z}$-dual Baer, which was initially introduced for modules, to rings.

Definition 4.1. Let R be a ring. Then R is called a right $\overline{Z}$-dual Baer ring if it is $\overline{Z}$-dual Baer as a right R-module.

A left $\overline{Z}$-dual Baer ring R is defined similarly. The property of being a $\overline{Z}$-dual Baer ring is not left-right symmetric as the following example shows.

Example 4.2. ([8, Example 3.3]) Let D be a commutative local integral domain with field of fractions Q (for example, we might take D the localization of the integers $\mathbb{Z}$ by a prime number p, i.e., D is the subring of the field of rational numbers consisting of fractions a/b such that b

is not divisible by p). Let $R = \begin{pmatrix} D & Q \\ 0 & Q \end{pmatrix}$. The operations are given by the ordinary matrix operations. Since D is local it has a unique maximal ideal, say m and the Jacobson radical of R is $J(R) = \begin{pmatrix} m & Q \\ 0 & 0 \end{pmatrix}$. Then $R/J(R) \cong (D/m) \times Q$. Thus R is semilocal. On the other hand, if we suppose that D has zero socle, then R has zero left socle and so $\overline{Z}(R_R) = Soc({}_RR) = 0$. Hence R_R is $\overline{Z}$-dual Baer. But R has non-zero right socle, namely, $\overline{Z}({}_RR) = Soc(R_R) = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$. It is known that, $\overline{Z}({}_RR) = Soc(R_R)$ is essential in ${}_RR$ (see [3]). It follows that ${}_RR$ can not be $\overline{Z}$-dual Baer.

It is easy to show that all semisimple rings are right $\overline{Z}$-dual Baer. The following provides a way to describe right $\overline{Z}$-dual Baer rings using semisimple direct summands.

Theorem 4.3. *Let R be a ring. Then the following are equivalent.*
(1) *R is right $\overline{Z}$-dual Baer.*
(2) *$R = \overline{Z}(R_R) \oplus K$ for some right ideal K of R and $\overline{Z}(R_R)$ is dual Baer as an R-module.*
(3) *$R = \overline{Z}(R_R) \oplus K$ for some right ideal K of R and $\overline{Z}(R_R)$ is semisimple as an R-module.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.7.
(1) $\Rightarrow$ (3) The ring R has a decomposition $R = \overline{Z}(R_R) \oplus K$ where K is a right ideal of R. Assume that B is a submodule of $\overline{Z}(R_R)$. We claim that B is a direct summand of $\overline{Z}(R_R)$. Since B has the form $\sum_{b\in B} bR$ and R is $\overline{Z}$-dual Baer, $\sum_{b\in B} bI$ is a direct summand of R. Therefore, $B\overline{Z}(R_R)$ is a direct summand of R. As B is contained in $\overline{Z}(R_R)$, we conclude that $B = BI$ is a direct summand of $\overline{Z}(R_R)$. It follows that $\overline{Z}(R_R)$ is semisimple.
(3) $\Rightarrow$ (1) Suppose that $R = \overline{Z}(R_R) \oplus K$ with a right ideal K of R and $\overline{Z}(R_R)$ is semisimple. Since $\overline{Z}(R_R)$ is semisimple, we conclude that $\overline{Z}(R_R)$ is dual Baer. Therefore, R is $\overline{Z}$-dual Baer by Theorem 2.7.

Theorem 4.4. *The following are equivalent for a ring R.*
(1) *R is right $\overline{Z}$-dual Baer.*
(2) *Every cyclic projective right R-module M is $\overline{Z}$-dual Baer.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is a cyclic projective right R-module. Then, $M = mR \cong R/r_R(m)$ for some $m \in M$. Therefore, $r_R(m)$ is a direct summand of R. Hence, $R = r_R(m) \oplus J$ where J is a right ideal of R. As R is right $\overline{Z}$-dual Baer, by Theorem 2.10 J is $\overline{Z}$-dual Baer. Hence M is $\overline{Z}$-dual Baer.
(2) $\Rightarrow$ (1) It is obvious.

5 Direct sum of $\overline{Z}$-dual Rickart modules and direct sum of $\overline{Z}$-dual Baer modules

This section focuses on exploring the properties of direct sums of $\overline{Z}$-dual Rickart modules and direct sums of $\overline{Z}$-dual Baer modules.

We will demonstrate that when the direct sum of $\overline{Z}$-Rickart modules that are $\overline{Z}$-dual Rickart.

Proposition 5.1. *Let $M = \oplus_{i=1}^n M_i$ and N be modules. If N has SSP for direct summands which are contained in $\overline{Z}(N)$, then M is N-$\overline{Z}$-dual Rickart if and only if M_i is N-$\overline{Z}$-dual Rickart for all $1 \le i \le n$.*

Proof. The sufficiency is obvious from Theorem 3.2. For the necessity, let ϕ be a homomorphism from M to N. Then $\phi = (\phi_i)_{i=1}^n$ where ϕ_i is a homomorphism from M_i to N for each $1 \le i \le n$. By hypothesis, $\phi_i(\overline{Z}(M_i))$ is a direct summand of N for each $1 \le i \le n$. Since N has SSP for direct summands which are contained in $\overline{Z}(N)$, we have
$\phi(\overline{Z}(M)) = \phi(\oplus_{i=1}^n \overline{Z}(M_i)) = \phi_1(\overline{Z}(M_1)) + \phi_2(\overline{Z}(M_2)) + \cdots + \phi_n(\overline{Z}(M_n)) \le^{\oplus} N$. Therefore M is N-$\overline{Z}$-dual Rickart.

Corollary 5.2. *Let $M = \oplus_{i=1}^{n} M_i$. Then M is $\overline{Z}$-dual Rickart relative to M_j $(1 \leq j \leq n)$ if and only if M_i is $\overline{Z}$-dual Rickart relative to M_j for each $1 \leq i \leq n$.*

Theorem 5.3. *Let $\{M_i\}_{i=1}^{n}$ and N be modules. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then N is $\oplus_{i=1}^{n} M_i$-$\overline{Z}$-dual Rickart if and only if N is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$.*

Proof. The sufficiency is obvious from Theorem 3.2. For the necessity, suppose that N is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$. We prove by induction on n. Assume that $n = 2$ and N is $\overline{Z}$-dual Rickart relative to M_1 and M_2. Let ϕ be a homomorphism from N to $M_1 \oplus M_2$. Then $\phi = \pi_1\phi + \pi_2\phi$, where π_i is the natural projection from $M_1 \oplus M_2$ to M_i $(i = 1, 2)$. As N is M_2-$\overline{Z}$-dual Rickart, $\pi_2\phi(\overline{Z}(N))$ is a direct summand of M_2. Let $M_2 = \pi_2\phi(\overline{Z}(N)) \oplus M_2'$ for some $M_2' \leq M_2$. Hence $M_1 \oplus M_2 = M_1 \oplus \pi_2\phi(\overline{Z}(N)) \oplus M_2'$. As M_2 is M_1-projective, $\pi_2\phi(\overline{Z}(N))$ is M_1-projective. Since $M_1 + \phi(\overline{Z}(N)) = M_1 \oplus \pi_2\phi(\overline{Z}(N))$ is a direct summand of $M_1 \oplus M_2$, there exists $T \subseteq \phi(\overline{Z}(N))$ such that $M_1 + \phi(\overline{Z}(N)) = M_1 \oplus T$, by [**?**, Lemma 4.47]. Thus $\phi(\overline{Z}(N)) = (\phi(\overline{Z}(N)) \cap M_1) \oplus T$. Since N is M_1-$\overline{Z}$-dual Rickart, $\pi_1\phi(\overline{Z}(N)) = M_1 \cap (M_2 + \phi(\overline{Z}(N))) = M_1 \cap \phi(\overline{Z}(N))$ is a direct summand of M_1. Therefore $\phi(\overline{Z}(N))$ is a direct summand of $M_1 \oplus T$. Since $M_1 \oplus T = M_1 \oplus \phi(\overline{Z}(N)) \leq^{\oplus} M_1 \oplus M_2$, $\phi(\overline{Z}(N))$ is a direct summand of $M_1 \oplus M_2$. Thus N is $\overline{Z}$-dual Rickart relative to $M_1 \oplus M_2$. Now, assume that N is $\overline{Z}$-dual Rickart relative to $\oplus_{i=1}^{n} M_i$. We show that N is $\overline{Z}$-dual Rickart relative to $M_{n+1} \oplus (\oplus_{i=1}^{n} M_i)$. Since M_{n+1} is M_j-projective for each $1 \leq j \leq n$, M_{n+1} is $\oplus_{i=1}^{n} M_i$-projective. As N is M_{n+1}-$\overline{Z}$-dual Rickart, N is $\oplus_{i=1}^{n+1} M_i$-$\overline{Z}$-dual Rickart by a similar argument for the case n = 2.

The above theorem incorporates concepts from the proof of Theorem 5.5 in [5].

Corollary 5.4. *Let $\{M_i\}_{i=1}^{n}$ be modules. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then $\oplus_{i=1}^{n} M_i$ is $\overline{Z}$-dual Rickart if and only if M_i is M_j-$\overline{Z}$-dual Rickart for all $1 \leq i, j \leq n$.*

Proof. The sufficiency is obvious from Theorem 3.2. For the necessity, assume that M_i is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$. Now $\oplus_{i=1}^{n} M_i$ is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$ by Corollary 5.2. Therefore, by Theorem 5.3, $\oplus_{i=1}^{n} M_i$ is $\overline{Z}$-dual Rickart.

Theorem 5.5. *Let $M = \oplus_{i=1}^{n} M_i$ be a module and $M_i \trianglelefteq M$ for all $i \in \{1, \dots, n\}$. Then M is a F-dual Rickart module if and only if M_i is $F \cap M_i$-dual Rickart for all $i \in \{1, \dots, n\}$.*

Proof. The necessity follows from Proposition 2.5. Conversely, let M_i be a $\overline{Z}$-dual Rickart module for all $i \in \{1, \dots, n\}$. Then $\overline{Z}(M) = \oplus_{i=1}^{n} \overline{Z}(M_i)$. Let $\phi = (\phi_{ij})_{i,j \in \{1,\dots,n\}} \in End_R(M)$ be arbitrary, where $\phi_{ij} \in Hom(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \{1, \dots, n\}$ and $\overline{Z}(M) = \oplus_{i=1}^{n} \overline{Z}(M_i)$, $\phi(\overline{Z}(M)) = \oplus_{i=1}^{n} \phi_{ii}(\overline{Z}(M_i))$. As M_i is $\overline{Z}$-dual Rickart, $\phi_{ii}(\overline{Z}(M_i))$ is a direct summand of M_i and so $\phi(\overline{Z}(M))$ is a direct summand of M. Therefore M is a $\overline{Z}$-dual Rickart module.

In the following we study some conditions that ensure us direct sums of $\overline{Z}$-dual Baer modules inherit the property.

Theorem 5.6. *Let $M = \oplus_{i=1}^{n} M_i$ be a module and $M_i \trianglelefteq M$ for all $i \in \{1, \dots, n\}$. Then M is a $\overline{Z}$-dual Baer module if and only if M_i is $\overline{Z}$-dual Baer for all $i \in \{1, \dots, n\}$.*

Proof. The necessity follows from Theorem 2.10. Conversely, let M_i be a $\overline{Z}$-dual Baer module for all $i \in \{1, \dots, n\}$ and I be a subset of $End_R(M)$. Then $\overline{Z}(M) = \oplus_{i=1}^{n}(\overline{Z}(M_i))$. Let $\phi = (\phi_{ij})_{i,j \in \{1,\dots,n\}} \in End_R(M)$ be arbitrary, where $\phi_{ij} \in Hom(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \{1, \dots, n\}$ and $\overline{Z}(M) = \oplus_{i=1}^{n}(\overline{Z}(M_i))$, we have $\phi(\overline{Z}(M)) = \oplus_{i=1}^{n} \phi_{ii}(\overline{Z}(M_i))$. Hence $\sum_{\phi \in I} \phi(\overline{Z}(M)) = \sum_{\phi \in I_i} \oplus_{i=1}^{n} \phi_{ii}(\overline{Z}(M_i)) = \oplus_{i=1}^{n} \sum_{\phi \in I_i} \phi_{ii}(\overline{Z}(M_i))$ where $I_i = \{\phi|_{M_i} : \phi \in I\} \subseteq End_R(M_i)$. As M_i is $\overline{Z}$-dual Baer for all $i \in \{1, \dots, n\}$, $\sum_{\phi \in I_i} \phi_{ii}(\overline{Z}(M_i))$ is a direct summand of M_i and so $\sum_{\phi \in I} \phi(\overline{Z}(M))$ is a direct summand of M. Therefore M is a $\overline{Z}$-dual Baer module.

We can prove the following proposition similar to the proof of Theorem 5.6.

Proposition 5.7. *Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of R-modules for an index set $\mathcal{I}$. If for every $i \in \mathcal{I}$, M_i is a fully invariant submodule of $\bigoplus_{i\in\mathcal{I}} M_i$, then $\bigoplus_{i\in\mathcal{I}} M_i$ is $\overline{Z}$-dual Baer if and only if M_i is $\overline{Z}$-dual Baer for every $i \in \mathcal{I}$.*

We now define relatively $\overline{Z}$-dual Baer modules and then we study direct sums of $\overline{Z}$-dual Baer modules applying this definition.

Definition 5.8. Let M and N be R-modules. Then, M is called *N-$\overline{Z}$-dual Baer* if for every subset I of $Hom_R(M,N)$, $\sum_{\phi\in I}\phi(\overline{Z}(M))$ is a direct summand of N.

Theorem 5.9. *Let $M = M_1 \oplus M_2$ and N be R-modules. If M is N-$\overline{Z}$-dual Baer, then for any direct summand K of N, M_i is K-$\overline{Z}$-dual Baer for $i = 1, 2$.*

Proof. As $\overline{Z}(M)$ is a fully invariant submodule of M, we have $\overline{Z}(M) = \overline{Z}(M_1) \oplus \overline{Z}(M_2)$. Suppose that A is a subset of $Hom_R(M_1,K)$. Then $B = \{j \circ \varphi \circ \pi_{M_1} \mid \varphi \in A\}$ in which $\pi_{M_1} : M \to M_1$ is the projection of M on M_1 and j is the inclusion from K to N, is a subset of $Hom_R(M,N)$. It is easy to check that $A\overline{Z}(M_1) = \sum_{\varphi\in A}\varphi(\overline{Z}(M_1)) = \sum_{g\in B} g(\overline{Z}(M))$. As M is a N-$\overline{Z}$-dual Baer module, $A\overline{Z}(M_1)$ is a direct summand of N and hence a direct summand of K.

Proposition 5.10. *Let $\{M_i\}_{i\in\mathcal{J}}$ be a class of R-modules for an index set $\mathcal{J}$, N an R-module. Then, the following hold.*

(1) *Let N have the SSP for direct summands which are contained in $\overline{Z}(N)$ and $\mathcal{J}$ be finite. Then, $\bigoplus_{i\in\mathcal{J}} M_i$ is N-$\overline{Z}$-dual Baer if and only if M_i is N-$\overline{Z}$-dual Baer for all $i \in \mathcal{J}$.*

(2) *Let N have the $SSSP$ for direct summands which are contained in $\overline{Z}(N)$, and $\mathcal{J}$ be arbitrary. Then, $\bigoplus_{i\in\mathcal{J}} M_i$ is N-$\overline{Z}$-dual Baer if and only if M_i is N-$\overline{Z}$-dual Baer for all $i \in \mathcal{J}$.*

Proof. (1) The sufficiency is obvious from Theorem 5.9. For the necessity, suppose that A is a subset of $Hom_R(\bigoplus_{i\in\mathcal{J}} M_i, N)$. Then $B_i = \{\phi j_i \mid \phi \in A\}$ in which j_i is the inclusion from M_i to $\bigoplus_{i\in\mathcal{J}} M_i$, is a subset of $Hom_R(M_i,N)$.

Assume that ϕ is a homomorphism from $\bigoplus_{i\in\mathcal{J}} M_i$ to N. Then $\phi = (\phi_i)_{i\in\mathcal{J}}$ where $\phi_i = \phi j_i$ is a homomorphism from M_i to N for each $i \in \mathcal{J}$. By hypothesis, $\sum_{\phi_i\in B_i}\phi_i(\overline{Z}(M_i))$ is a direct summand of N for each $i \in \mathcal{J}$. Since N has SSP for direct summands which are contained in $\overline{Z}(N)$, we have

$$\textstyle\sum_{\phi\in A}\phi(\overline{Z}(M)) = \sum_{\phi\in A}\phi(\oplus_{i=1}^{n}(\overline{Z}(M_i))) = \sum_{i\in\mathcal{J}}\sum_{\phi_i\in B_i}\phi_i(\overline{Z}(M_i)) \leq^{\oplus} N.$$

Therefore $\bigoplus_{i\in\mathcal{J}} M_i$ is N-$\overline{Z}$-dual Baer.

(2) Similar to (1).

Corollary 5.11. *Let $\{M_i\}_{i\in\mathcal{J}}$ be a class of R-modules for an index set $\mathcal{J}$. Then, for each $j \in \mathcal{J}$, $\bigoplus_{i\in\mathcal{J}} M_i$ is M_j-$\overline{Z}$-dual Baer if and only if M_i is M_j-$\overline{Z}$-dual Baer for all $i \in \mathcal{J}$.*

Proof. It follows from Proposition 5.10 and Theorem 2.7.

Similar to the proof of Theorem 5.3, one can prove the following theorem.

Theorem 5.12. *Let $\{M_i\}_{i=1}^{n}$ and N be modules. Assume that for each $i \geq j$ with $1 \leq i,j \leq n$, M_i is M_j-projective. Then N is $\oplus_{i=1}^{n}M_i$-$\overline{Z}$-dual Baer if and only if N is M_j-$\overline{Z}$-dual Baer for all $1 \leq j \leq n$.*

Corollary 5.13. *Let $\{M_i\}_{i=1}^{n}$ be modules. Assume that for each $i \geq j$ with $1 \leq i,j \leq n$, M_i is M_j-projective. Then $\oplus_{i=1}^{n}M_i$ is $\overline{Z}$-dual Baer if and only if M_i is M_j-$\overline{Z}$-dual Baer for all $1 \leq i,j \leq n$.*

Proof. The sufficiency is obvious from Theorem 5.9. For the necessity, assume that M_i is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$. Now $\oplus_{i=1}^{n}M_i$ is M_j-$\overline{Z}$-dual Rickart for all $1 \leq j \leq n$ by Corollary 5.11. Therefore, by Theorem 5.12, $\oplus_{i=1}^{n}M_i$ is $\overline{Z}$-dual Rickart.

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