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Spectral Theory and K-frames in Hilbert C^*-modules

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Abstract Frame theory is recently an active research area in mathematics and engineering with many exciting applications in a variety of different fields. In the current paper, we devoted to study the invariance of K-frames under the class of semi-regular operators in Hilbert $\mathbb{C}^*$-module.

1 Introduction and preliminaries

Frames, redundant systems in separable Hilbert spaces, which provide non-unique representations of vectors, were first introduced by Duffin and Schaeffer[10] and used them as a tool in the study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer[9]. Nowadays, Frames has been a useful tool in many areas such signal processing [7], sampling theory [27] and so on.

In recent years, research on a special class of frames, named K-frames, were first introduced by L. Găvruta [12] as a generalization of discrete frames due to some potential applications in sampling theory. Indeed, K-frames reconstruct the elements from the range of a bounded linear operator K in a separable Hilbert spaces. Frank and Larson [11] introduced the notion of frames in Hilbert C^*-module as a generalization of frames in Hilbert spaces.

In this section, we first present a brief account of basic definitions and some properties of Hilbert C^*-modules and their frames. For background material on frame theory and related topics, we refer to [8, 6, 5, 20].

Definition 1.1. [17] A left Hilbert C^*-module over the unital C^*-Algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{H}$ equipped with an $\mathcal{A}$-valued inner product

$$\langle .,. \rangle \quad : \quad \mathcal{H}\times\mathcal{H} \quad \longrightarrow \quad \mathcal{A}$$

satisfying the following conditions:

1. $\langle x,x\rangle_{\mathcal{A}} \geq 0$, for all $x\in\mathcal{H}$ and $\langle x,x\rangle_{\mathcal{A}} = 0$ if and only if $x=0$.
2. $\langle ax+y,z\rangle_{\mathcal{A}} = a\langle x,z\rangle_{\mathcal{A}} + \langle y,z\rangle_{\mathcal{A}}$, for all $a\in\mathcal{A}$ and $x,y,z\in\mathcal{H}$.
3. $\langle x,y\rangle_{\mathcal{A}} = \langle y,x\rangle_{\mathcal{A}}^*$, for all $x,y\in\mathcal{H}$.
4. $\mathcal{H}$ is complete with respect to the norm $||x|| = ||\langle x,x\rangle_{\mathcal{A}}||^{\frac{1}{2}}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T:\mathcal{H}\to\mathcal{K}$ is said to be adjointable if there exists a map $T^*:\mathcal{K}\to\mathcal{H}$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x\in\mathcal{H}$ and $y\in\mathcal{K}$. We denote $End_{\mathcal{A}}^*(\mathcal{H},\mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$. Throughout this paper, we suppose that $\mathcal{H}$ is a Hilbert C^*-module and $\mathbb{J}$ a countable index set of $\mathbb{N}$.

Example 1.2. Let us consider the following set

$$l^2(\mathcal{A}) = \{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{J}} a_j a_j^* \ converge \ in \ ||\ .\ ||\}.$$

It is easy to see that $l^2(\mathcal{A})$ with pointwise operations and the inner product

$$\langle \{a_j\}, \{b_j\} \rangle = \sum_{j \in \mathbb{J}} a_j b_j^*,$$

is a Hilbert C^*-module which is called the standard Hilbert C^*-module over $\mathcal{A}$.

For $T \in End_{\mathcal{A}}^*(\mathcal{H})$, we denote by $R(T)$ and $N(T)$ the range and the kernel subspaces of T respectively and I is the identity operator. We will say that T is positive, if $\langle Tx, x\rangle \geq 0,\ for\, all\, x \in \mathcal{H}$ [31].

It is well-known that each adjointable operator is necessarily bounded $\mathcal{A}$-linear in the sense $T(ax) = aT(x)$, for $a \in \mathcal{A}$ and $x \in \mathcal{H}$, but it is important to realize that the converse is false [17, 19].

Definition 1.3. [11] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A sequence $\{x_j\}_{j\in\mathbb{J}}$ is said to be a frame for $\mathcal{H}$, if there exist constant $\alpha, \beta > 0$ such that

$$\alpha\langle x, x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq \beta\langle x, x\rangle,\ for\, all\ x \in \mathcal{H}.$$

The constants α, β are called frame bounds.
If just the right inequality in the above definition holds, we say that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. The operator

$$\Phi : l^2(\mathcal{A}) \to \mathcal{H}, \quad defined\ \ by, \ \ \Phi(a) = \sum_{i\in\mathbb{J}} a_j x_j, \quad a = (a_i)_{j\in\mathbb{J}} \in l^2(\mathcal{A})$$

is called synthesis operator. The adjoint operator is given by

$$\Phi^* : \mathcal{H} \to l^2(\mathcal{A}) \quad defined\ \ by, \ \ \Phi^*(x) = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$$

is called the analysis operator. By composing Φ with its adjoint Φ^* we obtain the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \ \ S(x) = \Phi(\Phi^*(x)) = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle x_j$$

For each x $\in \mathcal{H}$, we have

$$\langle Sx, x\rangle = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle.$$

Then, S is bounded, positive and self-adjoint. Moreover, S verify

$$\alpha I \leq S \leq \beta I.$$

Thus, S is invertible.

Definition 1.4. [25] Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$. We shall say that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$, if there exist constants $\alpha, \beta > 0$ such that

$$\alpha\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq \beta\langle x, x\rangle,\ for\, all\ x \in \mathcal{H}.$$

Example 1.5. Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis for $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$ be defined as follows

$$Ke_1 = 3e_1, \ \ Ke_2 = e_2, \ \ Ke_3 = e_3, \ \ Ke_i = 0, \ \ for\ i \geq 4.$$

And

$$\theta_i = ie_i,\ for\, i = 1, 2, 3, \ \ \theta_i = 0,\ for\ i \geq 4.$$

Obviously, we have

$$K^*e_1 = 3e_1, \ \ K^*e_i = e_i, i = 2, 3\ and\ K^*e_i = 0,\ for\, i \geq 4.$$

Hence

$$\| K^*x \|^2 = \| \sum_{i\geq 1} \langle x, \theta_i \rangle K^* e_i \|^2 = 9 \mid \langle x, \theta_1 \rangle \mid^2 + \mid \langle x, \theta_2 \rangle \mid^2 + \mid \langle x, \theta_3 \rangle \mid^2 .$$

Thus

$$\frac{1}{9} \| K^*x \|^2 \leq \sum_{i\geq 1} \mid \langle x, \theta_i \rangle \mid^2 \leq 9 \| x \|^2 .$$

Which implies that $\{\theta_i\}_{i\geq 1}$ is a K-frame for $\mathcal{H}$.

Lemma 1.6. *[25] Let $\{x_j\}_{j\in\mathbb{J}}$ be a Bessel sequence in $\mathcal{H}$ and $K \in End^*_{\mathcal{A}}(\mathcal{H})$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ if and only if there exists $\alpha > 0$ such that*

$$S \geq \alpha K K^*.$$

where S is the frame operator for $\{x_j\}_{j\in\mathbb{J}}$.

Lemma 1.7. *[26] Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$. Then*

$$\langle Tx, Tx \rangle \leq \| T \|^2 \langle x, x \rangle, \ for \ all \ x \in \mathcal{H}.$$

Lemma 1.8. *[32] Let $T, G \in End^*_{\mathcal{A}}(\mathcal{H})$. If $R(G)$ is closed, then the following statements are equivalent:*

1. $R(T) \subseteq R(G)$.
2. $\alpha\langle T^*x, T^*x \rangle \leq \langle G^*x, G^*x \rangle$, *for some* $\alpha > 0$.

It is interesting to note that the concept of regularity is at the heart of the Kordula-Müller axiomatic spectral theory, that is given as follows

Definition 1.9. ([15]) A non-empty subset $\mathcal{R}$ of $End^*_{\mathcal{A}}(\mathcal{H})$ is called a regularity if the following two conditions hold :
(i) if $T \in \mathcal{R}$ and $n \geq 1$, then $T^n \in \mathcal{R}$;
(ii) if T, G, C, D are mutually commuting operator of $End^*_{\mathcal{A}}(\mathcal{H})$ satisfying $TC + GD = I$. Then

$$TG \in \mathcal{R} \Leftrightarrow T, G \in \mathcal{R}.$$

Proposition 1.10. *[23] Let $\mathcal{R}$ be a non-empty set of $End^*_{\mathcal{A}}(\mathcal{H})$ satisfying*

$$TG \in \mathcal{R} \Leftrightarrow T \in \mathcal{R} \ and \ G \in \mathcal{R},$$

for all commuting elements $T, G \in End^*_{\mathcal{A}}(\mathcal{H})$. Then, $\mathcal{R}$ is a regularity.

Example 1.11. The set of invertible operators in $End^*_{\mathcal{A}}(\mathcal{H})$ is a regularity.

Following Saphar in [30] the algebraic core $C(T)$ of T, is the greatest subspace $\mathcal{M}$ of $\mathcal{H}$ for which $T(\mathcal{M}) = \mathcal{M}$. Obviously, if T is surjective, then $C(T) = \mathcal{H}$.

Proposition 1.12. *[23] Suppose that T, G, C, D are mutually commuting in $End^*_{\mathcal{A}}(\mathcal{H})$ such that $TC + GD = I$. Then*

$$C(TG) = C(T) \cap C(G).$$

In addition, we pay attention that the concept of the conorme $\gamma(T)$ plays a fundamental role in the perturbation theory of Fredholm operators.

Definition 1.13. [22] For an operator $T \in End^*_{\mathcal{A}}(\mathcal{H})$, the conorme of T is defined by

$$\gamma(T) := inf\{\| Tx \|, \ x \in \mathcal{H}, \ dist(x, N(T)) = 1\}.$$

Formally, we set $\gamma(0) := \infty$. Clearly $\gamma(T) > 0$ if and only if $R(T)$ is closed.

Example 1.14. Let $T \in \mathcal{B}\left(\mathbb{C}^2\right)$ be defined as follows

$$\begin{array}{rcccl} T & : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & & (x_1, x_2) & \longmapsto & (x_1, x_1)\,. \end{array}$$

We have $\| Tx \| = \sqrt{2} \mid x_1 \mid$ and $\mathrm{dist}(x, N(T)) = \mid x_1 \mid$, where $x = (x_1, x_2)$.
Then $\gamma(T) = \sqrt{2}$.

The concept of invertibility admits several generalizations, for instance an operator $K \in End^*_{\mathcal{A}}(\mathcal{H})$ admits a generalized inverse $L \in End^*_{\mathcal{A}}(\mathcal{H})$ if :

$$KLK = K \text{ and } LKL = L.$$

In general, the generalized inverse is not unique, [21]
The Moore-Penrose inverse plays an important role in theoretical study and numerical analysis in many areas, such as the optimization problems and also in statistical problems.

Let us consider the operator

$$K_0 = K/N(K)^{\perp} \quad : \quad N(K)^{\perp} \quad \longrightarrow \quad R(K)$$

that is clearly bijective. Define $K^{\dagger}$ by

$$\begin{cases} K^{\dagger}x &= K_0^{-1}x \;\; if \;\; x \in R(K), \\ K^{\dagger}x &= \quad 0 \;\; if \;\; x \in R(K)^{\perp}. \end{cases}$$

Then, $K^{\dagger} = K_0^{-1} P_{R(K)}$ is called Moore-Penrose inverse of K.
Recall from [31], that the Moore-Penrose inverse of an operator $K \in End^*_{\mathcal{A}}(\mathcal{H})$ with closed range is a unique operator K^{+} such that

$$KK^{+}(u) = u, \quad for\, all\, u \in R(K).$$

The reader is referred to [29, 13] for more details.

In addition, the notion of semi-regularity of operators in Banach spaces, was originated classical treatment of perturbation theory owed to Kato [14] and it has been benefited from the work of many authors in the last years, in particular from the work of M. Mbekhta and Ouahab [22], Müller [24], Rakocevič [28].

Definition 1.15. [22] An operator $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is said to be semi-regular if $R(T)$ is closed and $N(T) \subset R(T^n)$, for every $n \geq 1$.

Example 1.16. Clearly, all injective operators with closed range and all surjective operators are semi-regular. Some other examples may be found in [16].

Proposition 1.17. *[4] Assume that $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is semi-regular and $L \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $TLT = T$. Then*

$$T^n L^n T^n = T^n, \; for\, all\, n \geq 1.$$

Now, we collect some useful properties of semi-regular operators. We refer to [1], [22], for further information.

Proposition 1.18. *[1] Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular. Then we have*

1. *$C(T)$ is closed;*
2. *T^n is semi-regular, for all $n \in \mathbb{N}$;*
3. *$T - \lambda I$ is semi-regular and $C(T) \subset C(T - \lambda I)$, for all $|\lambda| < \gamma(T)$.*

Remark 1.19. [1] If T is semi-regular and $C(T) = \{0\}$, then T is bounded below.

Recall that the semi-regular resolvent of a bounded operator T is defined by

$$reg\left(T\right)=\{\lambda\in\mathbb{C}:T-\lambda I\ is\ semi\ -\ regular\}.$$

Notice that $reg\left(T\right)$ is an open subset of $\mathbb{C}$. $(see[1])$, for more information.
In the following result, we show that the subspaces $C\left(T-\lambda I\right)$ are constant as λ ranges through a component Ω of $reg\left(T\right)$.

Theorem 1.20. *[1] Let $T\in End^{*}_{\mathcal{A}}\left(\mathcal{H}\right)$ and Ω be a connected component Ω of $reg\left(T\right)$. If $\lambda_0\in\Omega$, then*

$$C\left(T-\lambda I\right)=C\left(T-\lambda_0 I\right)$$

for every $\lambda\in\Omega$.

Theorem 1.21. *[23] The set of all semi-regular operators is a regularity.*

The main purpose of the present paper is to study the invariance of K-frames in Hilbert $\mathbb{C}^*$-modules under the class of semi-regular operators introduced by M. Mbekhta [22].

2 Main Results

For given $T\in End^{*}_{\mathcal{A}}(\mathcal{H})$, We fix the next notations

$$D\left(T\right)=\{\lambda\in\mathbb{C}:\mid\lambda\mid<\gamma\left(T\right)\}.$$

$$P_\lambda\left(T\right)=T^n-\lambda T^{n-1},\ \ \lambda\in\mathbb{C}\ and\ n\geq 1.$$

and we assume that $C\left(T\right)\neq\{0\}$.

Theorem 2.1. *Let $T,K\in End^{*}_{\mathcal{A}}\left(\mathcal{H}\right)$ be two semi-regular operators such that $KT=TK$. Let $\{x_i\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$. Then $\{T\left(x_i\right)\}_{j\in\mathbb{J}}$ is a K-frame for $C\left(T\right)$ with frame operator defined by $S_T=TST^*$.*

Proof. Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$ with frame bounds α and β. Then, for each $x\in C\left(T\right)$

$$\alpha\langle K^*x,K^*x\rangle\leq\sum_{j\in\mathbb{J}}\langle x,x_j\rangle.\langle x_j,x\rangle\leq\beta\langle x,x\rangle.$$

Now, let $u\in C\left(T\right)$, then there exists $v\in C\left(T\right)$ such that $u=T\left(v\right)$.
This give

$$K\left(u\right)=K\left(Tv\right)=\left(KT\right)\left(v\right)=\left(TK\right)\left(v\right).$$

It follows from Theorem 1.21, that $R\left(KT\right)$ is closed.
By Lemma 1.8, there exists $\alpha'>0$ such that

$$\alpha'\langle K^*x,K^*x\rangle\leq\langle\left(TK\right)^*x,\left(TK\right)^*x\rangle.$$

This implies that

$$\alpha\alpha'\langle K^*x,K^*x\rangle\leq\sum_{j\in\mathbb{J}}\langle x,Tx_j\rangle\langle Tx_j,x\rangle\leq\beta\langle T^*x,T^*x\rangle,\ \ \left(x\in C\left(T\right)\right).$$

Using Lemma 1.7, we have

$$\langle T^*x,T^*x\rangle\leq\mid\mid T\mid\mid^2\langle x,x\rangle,$$

So, there exists $A=\alpha\alpha'>0$ and $B=\beta\mid\mid T\mid\mid^2>0$ such that

$$A\langle K^*x,K^*x\rangle\leq\sum_{j\in\mathbb{J}}\langle x,Tx_j\rangle.\langle Tx_j,x\rangle\leq B\langle x,x\rangle.$$

Then, $\{T\left(x_j\right)\}_{j\in\mathbb{J}}$ is a K-frame for $C\left(T\right)$.
On the other hand, we have for every $x\in C\left(T\right)$

$$S\left(x\right)=\sum_{j\in\mathbb{J}}\langle x,x_j\rangle.x_j$$

It follows that

$$TST^*(x) = \sum_{j\in\mathbb{J}} \langle T^*x, x_j\rangle .Tx_j = \sum_{j\in\mathbb{J}} \langle x, Tx_j\rangle .Tx_j = S_T(x).$$

Thus, the frame operator for $\{T(x_j)\}_{j\in\mathbb{J}}$ is $S_T = TST^*$.
This completes the proof. □

Remark 2.2. S_T is bounded, positive and self-adjoint.

Corollary 2.3. *Assume that $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is semi-regular. Then $\{P_\lambda(T)(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $C(T)$, for every $\lambda \in D(T)$.*

Proof. By Proposition 1.12, we have

$$\begin{aligned} C(P_\lambda(T)) &= C\left(T^{n-1}\right) \cap C(T-\lambda I) \\ &= C(T) \cap C(T-\lambda I) \\ &= C(T). \end{aligned}$$

It follows from Proposition 1.18 and Theorem 1.21, that $P_\lambda(T)$ is a semi-regular operator. Therefore, By Theorem 2.1, we deduce that $\{(P_\lambda(T))(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $C(T)$. □

Under assumptions of the Theorem 1.20, we put $C_0(T) = C(T-\lambda I)$.

Corollary 2.4. *Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular. Then, $\{(T-\lambda I)(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $C_0(T)$, for every $\lambda \in \Omega$.*

Proof. It follows immediately from Theorem 1.20 and Theorem 2.1 □

Motivated by the work of Mbekhta [21], we exhibit some examples for which there is exists a bounded operator L such that $KLK = K$.

Example 2.5. Let $K, L \in \mathcal{B}\left(\mathbb{C}^2\right)$ defined by:

$$K = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, it is easy to get $KLK = K$.

Recall that an operator V is said to be a partial isometry if $VV^*V = V$.

Example 2.6. Let K $\in End^*_{\mathcal{A}}(\mathcal{H})$ such that $K = UVU^{-1}$ with V is a partial isometry and U an invertible operator, then

$$U^{-1}KU = V = VV^*V = U^{-1}KUU^*K^*U^{*-1}U^{-1}KU.$$

Therefore

$$K = KUU^*K^*U^{*-1}U^{-1}K = KLK.$$

where $L = (UU^*)K^*(UU^*)^{-1}$.

Theorem 2.7. *Assume $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ and $K, L \in End^*_{\mathcal{A}}(\mathcal{H})$. If L is such that $KLK = K$, then $\{(KL)x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ with frame bounds α and β. Then, for all $x \in \mathcal{H}$

$$\alpha\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq \beta\langle x, x\rangle,$$

since

$$KLK = K,$$

hence

$$\alpha\langle K^*L^*K^*x, K^*L^*K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, (KL)\,x_j\rangle\langle (KL)\,x_j, x\rangle \leq \beta\langle L^*K^*x, L^*K^*x\rangle.$$

By taking $\beta' = \beta \mid\mid KL \mid\mid^2$, we get

$$\alpha\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, (KL)\,x_j\rangle\langle (KL)\,x_j, x\rangle \leq \beta'\langle x, x\rangle.$$

Then $\{(KL)\,x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. □

Corollary 2.8. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be with closed range and $L \in End^*_{\mathcal{A}}(\mathcal{H})$ be a generalized inverse of K. If $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$, then $\{(KL)\,x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.*

Proof. Results from Theorem 2.7. □

Proposition 2.9. *Assume that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. Then, $\{x_j\}_{j\in\mathbb{J}}$ is also a K^n-frame for $\mathcal{H}$, for each $n \geq 1$.*

Proof. The first, there exist $\alpha, \beta > 0$ such that

$$\alpha\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \beta\langle x, x\rangle, \forall x \in \mathcal{H},$$

By Lemma 1.7, we have

$$\langle K^n x, K^n x\rangle \leq \mid\mid K^{n-1} \mid\mid^2 \langle Kx, Kx\rangle, \forall x \in \mathcal{H}.$$

Thus

$$\alpha \mid\mid K^{n-1} \mid\mid^{-2} \langle K^{n*}x, K^{n*}x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \beta\langle x, x\rangle.$$

This complete the proof. □

Proposition 2.10. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular and $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$. If L is such that $KLK = K$, then $\{(K^nL^n)\,x_j\}_{j\in\mathbb{J}}$ is a K^n-frame for $\mathcal{H}$.*
Morover, $\{\left(K^nK^{\dagger n}\right)x_j\}_{j\in\mathbb{J}}$ is also a K^n-frame for $\mathcal{H}$, for every $n \geq 1$.

Proof. The result follows from Proposition 1.17, Proposition 2.9 and Theorem 2.7. □

Example 2.11. Let $\{e_j\}_{j\geq 1}$ be a orthonormal basis of $l^2(\mathbb{C})$ and let K $\in \mathcal{B}\left(l^2(\mathbb{C})\right)$ defined as follows

$$K(x_1, x_2, ...) = (x_2, x_3, ...).$$

For $(y_j)_{j\geq 1} \in l^2(\mathbb{C})$, we have

$$K^*(y_1, y_2, ..) = (0, y_1, y_2, ..).$$

Thus

$$KK^*K(x_1, x_2, ..) = K(0, x_2, x_3, ..) = (x_2, x_3, ..) = K(x_1, x_2, ..),$$

hence

$$KK^*K = K \text{ and } K^*KK^* = K^*.$$

Then $K^* = K^\dagger$.
On the other hand, we have

$$\langle K^*x, K^*x\rangle = \sum_{j\geq 1} \mid x_j \mid^2 = \sum_{j\geq 1}\langle x, e_j\rangle\langle e_j, x\rangle.$$

Thus

$$\langle K^*x, K^*x\rangle \leq \sum_{j\geq 1}\langle x, e_j\rangle\langle e_j, x\rangle \leq \langle x, x\rangle.$$

Then $\{e_j\}_{j\in\mathbb{N}}$ is a K-frame for $l^2(\mathbb{C})$.
By a simple calculation, we deduce that $\{\left(K^nK^{\dagger n}\right)e_j\}_{j\geq 1}$ is a K^n-frame for $l^2(\mathbb{C})$, $(\forall n \geq 1)$.

In what follows, we are concerning with the construction of New K-frames. To this interest, we recall the following definition

Definition 2.12. [3] A sequence of $\mathcal{A}$-modules and $\mathcal{A}$-homomorphisms

$$...\longrightarrow M_{i-1} \xrightarrow{T_i} M_i \xrightarrow{T_{i+1}} M_{i+1} \longrightarrow ...$$

is said to be exact at M_i if $R(T_i) = N(T_{i+1})$.The sequence is exact if it is exact at each M_i.

Let us consider the following set

$$\mathbb{E}_T = \{K \in End^*_{\mathcal{A}}(\mathcal{H}) : \mathcal{H} \xrightarrow{K} \mathcal{H} \xrightarrow{T} \mathcal{H} \text{ is a sequence exact at } \mathcal{H}\}$$

Proposition 2.13. *Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be a semi-regular operator and $K \in \mathbb{E}_T$. Then there exists a constant $\alpha_p > 0$ such that $T^p T^{*p} \geq \alpha_p KK^*$, for every $p \geq 1$.*

Proof. Let T be semi-regular and $K \in \mathbb{E}_T$, then $R(K) = N(T)$. Since $R(T^p)$ is closed, we have

$$R(K) \subset R(T^p), \ \text{for all } p \geq 1.$$

Using Lemma 1.8, there exists $\alpha_p > 0$ such that $T^p T^{*p} \geq \alpha_p KK^*$.
This complete the proof □

Theorem 2.14. *Let $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular and $K \in \mathbb{E}_T$. If $\{x_j\}_{j\in I}$ is a frame for $\mathcal{H}$. Then $\{T^p(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$, for all $p \geq 1$.*

Proof. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{H}$ with frame bounds α and β. Then

$$\alpha\langle T^{p*}x, T^{p*}x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, T^p x_j\rangle.\langle T^p x_j, x\rangle \leq \beta\langle T^{p*}x, T^{p*}x\rangle, \ \text{for all } x \in \mathcal{H}.$$

By Lemma 1.8, there exists $\alpha' > 0$ such that

$$\alpha'\langle K^*x, K^*x\rangle \leq \langle T^{*p}x, T^{*p}x\rangle,$$

and from Lemma 1.7, we have

$$\langle T^{*p}x, T^{*p}x\rangle \leq \| T \|^{2p} \langle x, x\rangle, \ \text{for all } x \in \mathcal{H}.$$

Therefore, there exist $A = \alpha\alpha' > 0$ and $B = \beta \| T \|^{2p} > 0$ such that

$$A\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, T^p x_j\rangle.\langle T^p x_j, x\rangle \leq B\langle x, x\rangle, \ \text{for all } x \in \mathcal{H}.$$

which implies that $\{T^p x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. □

Corollary 2.15. *Let T be semi-regular and positive on $\mathcal{H}$ such that $\| T \| < 1$ and $K \in \mathbb{E}_T$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame for $\mathcal{H}$ with S frame operator such that $ST = TS$. Then, $\{e^T(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$, where $e^T = \sum_{p\in\mathbb{N}} \frac{1}{p!}T^p$.*

Proof. Let $K \in \mathbb{E}_T$, we have

$$\begin{aligned} S_{e^T} &= e^T S e^T \\ &= \sum_{k,p\geq 0} \frac{1}{k!}\frac{1}{p!} T^k S T^p \\ &= \frac{1}{(k'!)^2} S_{T^{k'}} + \sum_{k\neq k', p\neq k'} \frac{1}{k!}\frac{1}{p!} S T^{k+p}. \end{aligned}$$

So, by Proposition 2.13, there exists $\lambda_{k'} > 0$ such that

$$S_{T^{k'}} \geq \lambda_{k'} KK^*.$$

Then

$$S_{e^T} \geq \frac{\lambda_{k'}}{(k'!)^2} KK^*.$$

Obviously, $\{e^T(x_j)\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}$.
It follows from Lemma 1.6 that $\{e^T(x_j)\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. □

Remark 2.16. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $l^2(\mathbb{C})$. Let T $\in \mathcal{B}\left(l^2(\mathbb{C})\right)$ be defined as follows

$$T(a_0, a_1, ..) = (a_1, a_2, ., ., .) .$$

Obviously, we have

$$N(T) = \{(b, 0, 0, ., .) : b \in \mathbb{C}\}.$$

Now, let K $\in \mathcal{B}\left(l^2(\mathbb{C})\right)$ defined by

$$K(a_0, a_1, a_2, ..) = (a_0, a_1, 0, 0, ...) ,$$

hence

$$R(K) \simeq \mathbb{C}^2 \;\; and \;\; N(T) \simeq \mathbb{C}.$$

Therefore

$$R(K) \neq N(T) .$$

Consequently, we obtain $K \notin \mathbb{E}_T$.
By setting $a = (a_0, a_1, a_2, ., .)$, we get

$$\langle K^*(a), K^*(a)\rangle = \mid a_0 \mid^2 + \mid a_1 \mid^2 .$$

By some straightforward computations, we obtain that

$$\begin{aligned}\sum_{j\in\mathbb{N}} \langle T^p(a), e_j\rangle.\langle e_j, T^p(a)\rangle &= \sum_{j\in\mathbb{N}} \langle (a_p, a_{p+1}, ..), e_j\rangle.\langle e_j, (a_p, a_{p+1}, ..)\rangle \\ &= \sum_{j\in\mathbb{N}} \mid a_{p+j} \mid^2 .\end{aligned}$$

We take $p \geq 3$ and $a = (a_0, 0, 0, ..)$ such that $a_0 \neq 0$, we get

$$\sum_{j\in\mathbb{N}} \langle T^p(a), e_j\rangle.\langle e_j, T^p(a)\rangle < \langle K^*(a), K^*(a)\rangle.$$

Therefore $\{T^p(a_j)\}_{j\in\mathbb{N}}$ is not a K-frame for $\mathcal{H}$.

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