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EXISTENCE OF SOLUTIONS FOR THE AGGREGATION EQUATIONS WITH INITIAL DATA IN MORREY SPACES

Marta L. Suleiman, Juliana C. Precioso and Andrea C. Prokopczyk

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Abstract In this work we consider a class of nonlinear viscous transport equations describing aggregation phenomena in biology. We demonstrate the existence of solutions with initial data in Morrey spaces. We analyze the asymptotic stability of solutions persistence at large times.

1 Introduction

In this work, we consider the Cauchy problem for the heat equation corrected by the nonlocal and nonlinear transport term

$$u_t = \Delta u - \nabla \cdot (u(\nabla K * u)),\ x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R}^n. \tag{1.2}$$

This model has been used to describe collective motion and aggregation phenomena in biology and mechanics of continuous media, where the unknown $u = u(x,t)$ represents either the population density of a species or the density of particles in a granular media. Here, $n \geq 2$ and the Kernel $\nabla K \in L^1(\mathbb{R}^n)$ is a given function (the symbol " $*$ " denotes the convolution with respect to the variable x).

In the literature, aggregation equations have been greatly studied. For the case of (1.1) without the diffusion term, Laurent [18] proved several local and global existence results for a class of Kernels K with different regularity. We refer the reader to [7, 8, 9, 10] and references therein for results on existence and blowup of solutions to the IVP for the inviscid aggregation equation.

Moreover, we refer the reader to the works [2, 3, 11, 12, 13, 14, 21, 23] for the case of (1.1) and its generalizations considered either in whole space or in a bounded domain. It is worth highlighting that (1.1) contains, as a particular case, the elliptical chemotaxis parabolic system, whose simplified formula is described by

$$u_t = \Delta u - \nabla \cdot (u \nabla v),\ x \in \mathbb{R}^n,\ t > 0, \tag{1.3}$$

$$0 = \Delta v - \alpha v + u, \tag{1.4}$$

where $\alpha > 0$ is a given constant. In fact, taking the kernel $K = K(x)$ as the fundamental solution of the operator $-\Delta + \alpha Id$, one can rewrite (1.4) as $v = K * u$. Then, substitute this formula into (1.3) we obtain (1.1). We refer the reader to works [5, 6, 17, 19, 22] and to the references therein for mathematical results on systems modeling chemotaxis. In particular, in [4], it was obtained a global existence of solutions to the parabolic-elliptic system of chemotaxis on Morrey spaces framework.

We can also cite [13], where Karch studies the local/global-in-time existence of solutions to (1.1) – (1.2), with initial data in Lebesgue space, for a class of kernels $K : \mathbb{R}^n \to \mathbb{R}$ strongly singular, that is, $\nabla K \in L^{q'}(\mathbb{R}^n)$ for a some $q' \in [1, n]$ and $\nabla K \notin L^p(\mathbb{R}^n)$ for all $p > n$. Notice that any functions ∇K satisfying $|\nabla K| \sim |x|^{1-n}$ as $|x| \to 0$ and rapidly decreasing if $|x| \to \infty$ is strongly singular when $n \geq 2$. Therefore, as we are assuming $\nabla K \in L^1(\mathbb{R}^n)$, this kind of functions are particular examples of admissible Kernels K for our model.

Furthermore, in this work we consider the Cauchy problem (1.1) – (1.2) and show global-in-time existence of solutions with initial data in Morrey spaces, which makes our problem more general than those existing in the literature up to the present moment, by our knowledge.

Still in connection with this type of equation, it is worth highlighting that in [15], Karch developed the general theory to study the Cauchy problem for the parabolic equation

$$u_t = \Delta u + B(u, u), \tag{1.5}$$

$$u(x, 0) = u_0(x), \tag{1.6}$$

what encompasses a lot of models. Karch showed existence of global in time solutions assuming some scaling property of the equation as well as of the norm of the Banach space in which the solutions are constructed. However the bilinear form $B(\cdot, \cdot)$ in (1.5) has the scaling order less than 2 (see [15], page 535 for more details). In our work, we got a result of existence where the bilinear form does not need to be homogeneous, since the only assumption made for the operator K is that its gradient lies in $L^1(\mathbb{R}^n)$. In addition, our initial condition belongs to a bigger space than the Karch's.

Moreover, we analyze the asymptotic stability of solutions persistence for large times. Qualitative aspects, like symmetry of solutions, also are demonstrated. For results of this type for semilinear fractional heat equations we refer [1].

This manuscript is organized as follows. In the next section we review basic properties about Morrey spaces and the notion of mild solution for the IVP (1.1) – (1.2). We state our results in Section 3 (see Theorem 3.1 and Theorem 3.3) and prove them in Section 4.

2 Preliminaries

2.1 Function spaces and definitions

In this section, we review some properties about Morrey spaces. The reader is referred to [16] for further details about them.

For $1 \leq p < \infty$ and $0 \leq \lambda < n$, the Morrey space $\mathcal{M}_{p,\lambda} = \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ is defined as

$$\mathcal{M}_{p,\lambda} = \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty\}, \tag{2.1}$$

where

$$\|f\|_{p,\lambda} = \sup_{x_0 \in \mathbb{R}^n,\ R>0} \{R^{-\frac{\lambda}{p}} \|f\|_{L^p(B_R(x_0))}\} \tag{2.2}$$

and $B_R(x_0) \subset \mathbb{R}^n$ is the closed ball with center x_0 and radius R. The space $\mathcal{M}_{p,\lambda}$ endowed with $\|\cdot\|_{p,\lambda}$ is a Banach space. In particular, $\mathcal{M}_{p,0} = L^p$ for $p > 1$, and also $\mathcal{M}_{1,0}$ is the Banach space of finite measures, which can also be denoted as $\mathcal{M}$. We include $L^\infty = L^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_\infty < \infty \ a.e. \text{ in } \mathbb{R}^n\}$ between Morrey spaces, taking $p = \infty$ or $\lambda = n$ in the notation $\mathcal{M}_{p,\lambda}$.

If $\frac{n}{q} = \frac{n-\lambda}{p}$, we obtain the continuous inclusion

$$L^q \subset \mathcal{M}_{p,\lambda}. \tag{2.3}$$

We have the following scaling for $\|\cdot\|_{p,\lambda}$

$$\|f(\alpha \cdot)\|_{p,\lambda} = \alpha^{-\frac{n-\lambda}{p}} \|f\|_{p,\lambda}, \text{ for all } \alpha > 0. \tag{2.4}$$

Hölder inequality holds true in the framework of Morrey spaces. Precisely, if $1 \leq p_i \leq \infty$ and $0 \leq \lambda_i < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_2}{p_2} + \frac{\lambda_1}{p_1}$, then

$$\|fg\|_{p_3,\lambda_3} \leq \|f\|_{p_1,\lambda_1} \|g\|_{p_2,\lambda_2}. \tag{2.5}$$

Young type inequalities hold true in Morrey spaces, that is, if $1 \leq p \leq \infty$ e $0 \leq \lambda < n$, then for $g \in L^1(\mathbb{R}^n)$ and $f \in \mathcal{M}_{p,\lambda}$

$$\|g * f\|_{p,\lambda} \leq \|g\|_1 \|f\|_{p,\lambda}. \tag{2.6}$$

The notation $\|\cdot\|_1$ denotes the norm in $L^1(\mathbb{R}^n)$.

2.2 Mild solutions

The linearization of (1.1) – (1.2) is the Cauchy problem for the linear heat equation

$$u_t - \Delta u = 0, \ x \in \mathbb{R}^n, \ t > 0, \tag{2.7}$$

$$u(x,0) = u_0(x), \ x \in \mathbb{R}^n, \tag{2.8}$$

which solution is given by

$$u(t) = G(t)u_0,$$

where $G(t)$ is the convolution operator with kernel $g(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.

Then, according Duhamel's principle, the problem (1.1) – (1.2) is formally equivalent to the integral system

$$u(t) = G(t)u_0 + B(u,u)(t), \ \ t > 0, \tag{2.9}$$

where

$$B(u,v)(t) = -\int_0^t \nabla_x G(t-s)(u(\nabla K * v))(s)ds. \tag{2.10}$$

Throughout this paper, solutions of (2.9) are called mild ones for (1.1) – (1.2).

3 Results

The aim of this section is to state our results for the Cauchy problem (1.1) – 1.2. We start by performing a scaling analysis in order to find the correct indexes for Kato-Fujita type norms based on Morrey spaces. If u is a classical solution for (1.1) – (1.2), then the rescaled function

$$u_\alpha(x,t) := \alpha u(\alpha x, \alpha^2 t), \text{ for all } \alpha > 0, \tag{3.1}$$

is also solution since the ∇K is a homogeneous function of degree $-n$. With this motivation, the scaling map of (1.1) – (1.2) can be defined as

$$u \longrightarrow u_\alpha. \tag{3.2}$$

We observe that a solution u is called self-similar when it is invariant by (3.2), that is, $u \equiv u_\alpha$ for all $\alpha > 0$.

Formally, making $t \to 0^+$ in (3.2), one obtains the scaling map associated to the initial condition

$$u_0(x) \longrightarrow \alpha u_0(\alpha x). \tag{3.3}$$

We want that initial data u_0 be in a Morrey spaces such that its norm is invariant for (3.2). Then, from the last observation, we choose the space $\mathcal{M}_{p,\lambda}$, with $p = n - \lambda$.

Let $n \in \mathbb{N}$, $n \geq 2$, $0 \leq \lambda < n-1$, $1 < p < q < \infty$. Consider the parameters $p = n - \lambda$ and $\eta = \frac{1}{2} - \frac{n-\lambda}{2q}$, and let $BC((0,\infty), X)$ stand for the class of continuous and bounded functions from $(0,\infty)$ to a Banach space X. Global-in-time solution $u = u(x,t)$ will be sought in the scaling-invariant Kato-Fujita class

$$E_q = \{u \in BC((0,\infty), \mathcal{M}_{p,\lambda}); \ t^\eta u \in BC((0,\infty), \mathcal{M}_{q,\lambda})\}, \tag{3.4}$$

which is a Banach space with the norm

$$\|u\|_{E_q} = \sup_{t>0} \|u(\cdot,t)\|_{p,\lambda} + \sup_{t>0} t^\eta \|u(\cdot,t)\|_{q,\lambda}. \tag{3.5}$$

Our well-posedness result reads as follows.

Theorem 3.1. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $0 \leq \lambda < n-1$, $\nabla K \in L^1$, $1 < p < q < \infty$, with $1 < \frac{q}{2}$, $p = n-\lambda$, $\frac{1}{p} + \frac{1}{q} < 1$ and $\eta = \frac{1}{2} - \frac{n-\lambda}{2q}$. Suppose that $u_0 \in \mathcal{M}_{p,\lambda}$. Let M_1, M_2 be as in Lemma 4.3, $M = max\{M_1, M_2\}$ and $0 < \varepsilon < \frac{1}{4M}$.*

(i) (Existence and uniqueness) There exists $\delta = \delta(\varepsilon) > 0$ such that (1.1)–(1.2) has a global solution $u \in E_q$ provided $\|u_0\|_{p,\lambda} \leq \delta$. This solution is the unique one satisfying $\|u\|_{E_q} \leq 2\varepsilon$.

(ii) (Continuous dependence) The solution u depends continuously on initial data u_0. Moreover, $u(\cdot,t) \rightharpoonup u_0(\cdot)$ in $D'(\mathbb{R}^n)$ when $t \to 0^+$.

(iii) (Symmetry) The solution u is odd (even) in $\mathbb{R}^n$, $t > 0$, whenever u_0 and K are odd (even).

It is observed that, in this case, the solution u is not self-similar ($u = u_\alpha$), otherwise, K would be homogeneous of degree $1-n$, which implies ∇K (belonging to L^1) as homogeneous. This would be incongruous, since L^p spaces does not contain homogeneous functions.

Remark 3.2. (i) (Local-in-time well-posedness) In order to obtain a local-in-time version of Theorem 3.1, the smallness hypothesis on the initial condition must be replaced by that on existence time $T > 0$.

Here, the space E_q previously defined must also be redefined by the following local spaces:

$$E_{q,T} = \{u \in BC((0,T), \mathcal{M}_{p,\lambda}); \limsup_{t\to 0^+} t^\eta \|u(\cdot,t)\|_{q,\lambda} = 0\}.$$

Moreover, the initial data $u_0 \in \mathcal{M}_{p,\lambda}$ is such that

$$\limsup_{t\to 0^+} t^\eta \|G(t)u_0\|_{q,\lambda} = 0. \tag{3.6}$$

However, in particular, u_0 satisfies (3.6) when it belongs to subspace $\ddot{\mathcal{M}}_{p,\lambda} \subsetneq \mathcal{M}_{p,\lambda}$, which is defined by the following condition

$$f \in \ddot{\mathcal{M}}_{p,\lambda} \Leftrightarrow \lim_{y\to 0} \|\tau_y f - f\|_{p,\lambda} = 0,$$

where τ_y indicates the translation $\tau_y(x) = f(x-y)$ for $y \in \mathbb{R}^n$.

Since that the semigroup $\{G(t)\}_{t\geq 0}$ is not strongly continuous at $t = 0^+$ on $\mathcal{M}_{p,\lambda}$ we need to restrict ourselves to $\ddot{\mathcal{M}}_{p,\lambda}$, which is the maximal closed subspace of $\mathcal{M}_{p,\lambda}$, where $\{\tau_y\}_{y\in\mathbb{R}^n}$ is strongly continuous (see [16], Lemma 3.1).

(ii) (Alternative blow up) As discuss in (i), the local-in-time version of Theorem 3.1 is obtained considering the initial data $u_0 \in \ddot{\mathcal{M}}_{p,\lambda}$, which implies the existence of the solution u in the space $C([0,T_{max}), \mathcal{M}_{p,\lambda})$, where $T_{max} > 0$ stands for the maximal existence time.

Moreover, if $T_{max} < \infty$ then we have the solution blows up at finite time, i.e., $\|u(\cdot,t)\|_{p,\lambda} \to \infty$ when $t \to T_{max}^-$; otherwise we have $T_{max} = \infty$, and the solution u is global-in-time. We can cite [1, 24, 25, 26] for more results about blow up for nonlinear diffusion equations.

Theorem 3.3. *(Asymptotic stability) Under the hypotheses of Theorem 3.1, let u and v be two solutions as in the Theorem 3.1 with initial data u_0 and v_0, respectively. We have that*

$$\lim_{t\to+\infty} \|G(t)(u_0 - v_0)\|_{p,\lambda} = \lim_{t\to+\infty} t^\eta \|G(t)(u_0 - v_0)\|_{q,\lambda} = 0 \tag{3.7}$$

if and only if

$$\lim_{t\to+\infty} \|u(\cdot,t) - v(\cdot,t)\|_{p,\lambda} = \lim_{t\to+\infty} t^\eta \|u(\cdot,t) - v(\cdot,t)\|_{q,\lambda} = 0. \tag{3.8}$$

4 Proofs of the results

In this part, we prove the results that were stated in the previous section. Initially, we will start by recalling an abstract fixed point lemma which will be useful for our ends. For a proof, see e.g. [20].

Lemma 4.1. *Let X be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \to X$ be a continuous bilinear map, that is, there $N > 0$ such that*

$$\|B(x_1, x_2)\|_X \leq N\|x_1\|_X\|x_2\|_X, \text{ for all } x_1,\ x_2 \in X. \tag{4.1}$$

Let $y \in X$, $y \neq 0$ and $\|y\|_X \leq \varepsilon$. If $0 < \varepsilon < \frac{1}{4N}$, then exists a unique solution $x \in \mathcal{E}_{2\varepsilon} = \{x \in X : \|x\|_X \leq 2\varepsilon\}$ for equation $x = y + B(x, x)$. The solution is the limit in X of the iterated sequence $x_1 = y$ and $x_{n+1} = y + B(x_n, x_n)$, $n \geq 1$.

4.1 Estimates for $G(t)$ and $\nabla_x G(t)$ in Morrey spaces

The next lemma provides us estimates for $G(t)$ and $\nabla_x G(t)$ in Morrey spaces, whose proof can be seen e.g. [16].

Lemma 4.2. *Let $1 \leq p_1, p_2 < \infty$ and $0 \leq \lambda_1, \lambda_2 < n$. If $n \geq \frac{n-\lambda_1}{p_1} \geq \frac{n-\lambda_2}{p_2} \geq 0$, there exists a constant $C = C(p_1, p_2, \lambda_1, \lambda_2) > 0$, such that*

$$\|G(t)f\|_{p_2,\lambda_2} \leq Ct^{-\frac{1}{2}\left(\frac{n-\lambda_1}{p_1} - \frac{n-\lambda_2}{p_2}\right)}\|f\|_{p_1,\lambda_1}, \tag{4.2}$$

and

$$\|\nabla_x G(t)f\|_{p_2\lambda_2} \leq Ct^{-\frac{1}{2}\left(1+\frac{n-\lambda_1}{p_1} - \frac{n-\lambda_2}{p_2}\right)}\|f\|_{p_1,\lambda_1}, \tag{4.3}$$

for all $f \in \mathcal{M}_{p_1,\lambda_1}$, and all $t > 0$.

4.2 Bilinear estimates

As already seen in this work, we use the following bilinear operator notation appearing in (2.9)

$$B(u, v)(t) = -\int_0^t \nabla_x G(t - s)(u(\nabla K * v))(s)ds.$$

Lemma 4.3. *Under the hypotheses of the Theorem 3.1 and considering $r > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, there exist constants $M_1, M_2 > 0$ such that*

$$\sup_{t>0} \|B(u, v)(t)\|_{p,\lambda} \leq M_1 \sup_{t>0} t^\eta \|u(t)\|_{q,\lambda} \sup_{t>0} \|v(t)\|_{p,\lambda} \tag{4.4}$$

and

$$\sup_{t>0} t^\eta \|B(u, v)(t)\|_{q,\lambda} \leq M_2 \sup_{t>0} t^\eta \|u(t)\|_{q,\lambda} \sup_{t>0} t^\eta \|v(t)\|_{q,\lambda}, \tag{4.5}$$

for all $u, v \in E_q$.

Proof. By Lemma 4.2, we have

$$\begin{aligned} \|B(u, v)(t)\|_{p,\lambda} &\leq \int_0^t \|\nabla_x G(t - s)(u(\nabla K * v))(s)\|_{p,\lambda}\, ds \\ &\leq C\int_0^t (t - s)^{-\frac{1}{2} - \frac{n-\lambda}{2r} + \frac{n-\lambda}{2p}} \|(u(\nabla K * v))(s)\|_{r,\lambda}\, ds. \end{aligned} \tag{4.6}$$

Therefore, Hölder inequality (2.5) and then Young inequality (2.6) yield

$$\begin{aligned} \|(u(\nabla K * v))(s)\|_{r,\lambda} &\leq \|u(s)\|_{q,\lambda}\, \|(\nabla K * v)(s)\|_{p,\lambda} \\ &\leq \|u(s)\|_{q,\lambda}\, \|\nabla K\|_1\, \|v(s)\|_{p,\lambda}. \end{aligned} \tag{4.7}$$

Inserting (4.7) into (4.6) and since $\frac{n-\lambda}{2p}+\frac{n-\lambda}{2q}=\frac{n-\lambda}{2r}$, we get

$$\begin{aligned}
\|B(u,v)(t)\|_{p,\lambda} &\leq C\,\|\nabla K\|_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}}\|u(s)\|_{q,\lambda}\,\|v(s)\|_{p,\lambda}\,ds \\
&\leq C\,\|\nabla K\|_1 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda} \sup_{t>0}\|v(t)\|_{p,\lambda}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-\eta} ds \\
&= C\,\|\nabla K\|_1 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda} \sup_{t>0}\|v(t)\|_{p,\lambda}\int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-\frac{1}{2}+\frac{n-\lambda}{2q}} ds \\
&= M_1 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda} \sup_{t>0}\|v(t)\|_{p,\lambda}. \qquad (4.8)
\end{aligned}$$

Then, from (4.8) we have

$$\sup_{t>0}\|B(u,v)(t)\|_{p,\lambda}\leq M_1 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda} \sup_{t>0}\|v(t)\|_{p,\lambda}.$$

The estimate (4.5) follows at once from Lemma 4.2 and inequalities (2.5) and (2.6), in fact

$$\begin{aligned}
t^{\eta}\|B(u,v)(t)\|_{q,\lambda} &\leq t^{\eta}\int_0^t \|\nabla_x G(t-s)(u(\nabla K * v))(s)\|_{q,\lambda} ds \\
&\leq Ct^{\eta}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2\frac{q}{2}}+\frac{n-\lambda}{2q}}\|(u(\nabla K * v))(s)\|_{\frac{q}{2},\lambda}\,ds \\
&\leq Ct^{\eta}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}}\|u(s)\|_{q,\lambda}\,\|(\nabla K * v)(s)\|_{q,\lambda}\,ds \\
&\leq C\|\nabla K\|_1 t^{\eta}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}}\|u(s)\|_{q,\lambda}\,\|v(s)\|_{q,\lambda}\,ds \\
&\leq C\|\nabla K\|_1 t^{\eta}\sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda}\sup_{t>0} t^{\eta}\|v(t)\|_{q,\lambda}\int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-2\eta} ds \\
&= C\|\nabla K\|_1 t^{\eta}\sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda}\sup_{t>0} t^{\eta}\|v(t)\|_{q,\lambda}\int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-1+\frac{n-\lambda}{q}} ds \\
&= M_2 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda}\sup_{t>0} t^{\eta}\|v(t)\|_{q,\lambda}. \qquad (4.9)
\end{aligned}$$

By (4.9), we obtain

$$\sup_{t>0} t^{\eta}\|B(u,v)(t)\|_{q,\lambda}\leq M_2 \sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda}\sup_{t>0} t^{\eta}\|v(t)\|_{q,\lambda}.$$

□

4.3 Proof of Theorem 3.1

Part (i) (Existence and uniqueness): Take $X=E_q$ defined by (3.4). We denote $M=\max\{M_1,M_2\}$, $y=G(\cdot)u_0$. For $u,v\in E_q$, recalling (3.5), Lemma 4.3 yields

$$\begin{aligned}
\|B(u,v)\|_{E_q} &= \sup_{t>0}\|B(u,v)(t)\|_{p,\lambda}+\sup_{t>0} t^{\eta}\|B(u,v)(t)\|_{q,\lambda} \\
&\leq M\sup_{t>0} t^{\eta}\|u(t)\|_{q,\lambda}(\sup_{t>0}\|v(t)\|_{p,\lambda}+\sup_{t>0} t^{\eta}\|v(t)\|_{q,\lambda}) \\
&\leq M\|u\|_{E_q}\|v\|_{E_q}. \qquad (4.10)
\end{aligned}$$

Applying the Lemma 4.2 and recalling that $n-\lambda=p$, we obtain

$$\begin{aligned}
\|y\|_{E_q} &= \sup_{t>0}\|G(t)u_0\|_{p,\lambda}+\sup_{t>0} t^{\eta}\|G(t)u_0\|_{q,\lambda} \\
&\leq C_1\|u_0\|_{p,\lambda}+C_2\sup_{t>0} t^{\eta}t^{-\frac{1}{2}(\frac{n-\lambda}{p}-\frac{n-\lambda}{q})}\|u_0\|_{p,\lambda} \\
&\leq C_3\|u_0\|_{p,\lambda}\leq\varepsilon \qquad (4.11)
\end{aligned}$$

provided that $||u_0||_{p,\lambda} \leq \delta = \frac{\varepsilon}{C_3}$. If $0 < \varepsilon < \frac{1}{4M}$ then Lemma 4.1 implies that there is a unique solution $u \in E_q$ of (2.9) such that $\|u\|_{E_q} \leq 2\varepsilon$.

Part (ii)(Continuous dependence on initial data): Let u and v be two solutions as in $Part\ (i)$ with initial data u_0 and v_0, respectively. We have

$$\begin{aligned}
\|u-v\|_{E_q} &= \|G(\cdot)u_0 + B(u-v,u) - G(t)v_0 + B(v,u-v)\|_{E_q} \\
&\leq \|G(\cdot)(u_0-v_0)\|_{E_q} + \|u-v\|_{E_q}(M\|u\|_{E_q} + M\|v\|_{E_q}) \\
&\leq \|G(\cdot)(u_0-v_0)\|_{E_q} + 4M\varepsilon\|u-v\|_{E_q},
\end{aligned}$$

and hence

$$(1-4M\varepsilon)\|u-v\|_{E_q} \leq \|G(\cdot)(u_0-v_0)\|_{E_q}.$$

From (4.11) it follows that $\|G(\cdot)u_0\|_{E_q} \leq C_3\|u_0\|_{p,\lambda}$, then

$$(1-4M\varepsilon)\|u-v\|_{E_q} \leq C_3\|u_0-v_0\|_{p,\lambda}.$$

This complete the proof of continuous dependence.

We will prove that $u(\cdot,t) \rightharpoonup u_0(\cdot)$ as $t \to 0^+$ in $D'(\mathbb{R}^n)$. Firstly we show that $G(t)u_0 \rightharpoonup u_0$.

Let $w \in D(\mathbb{R}^n)$. Since $g(x-y,t) = g(y-x,t)$, then $\langle G(t)u_0, w\rangle = \langle u_0, G(t)w\rangle$. Therefore $G(t)u_0 \rightharpoonup u_0 \Leftrightarrow \langle u_0, G(t)w - w\rangle \to 0$.
Let $R > 0$ such that $\text{supp}(w) \subset B_R(0)$ and let p' such that $\frac{1}{p'} + \frac{1}{p} = 1$. We have

$$\begin{aligned}
R^{-\lambda}\langle u_0, G(t)w - w\rangle &\leq R^{-\lambda}\int_{B_R(0)} |(G(t)w-w)(x)u_0(x)|dx \\
&\leq R^{\frac{-\lambda}{p'}}(\int_{B_R(0)} |(G(t)w-w)(x)|^{p'}dx)^{\frac{1}{p'}} R^{\frac{-\lambda}{p}}(\int_{B_R(0)} |u_0(x)|^p dx)^{\frac{1}{p}} \\
&\leq \|G(t)w-w\|_{p',\lambda}\|u_0\|_{p,\lambda}. \qquad (4.12)
\end{aligned}$$

For (2.3), given p' exists s such that $L^s \subset \mathcal{M}_{p',\lambda}$, then $\|G(t)w - w\|_{p',\lambda} \leq \|G(t)w - w\|_{L^s}$. The family $\{g(x,t)\}_{t>0}$ indexed at t is an approximation of the identity in L^s, ergo $\|G(t)w - w\|_{L^s} \to 0$, when $t \to 0^+$. Therefore $\|G(t)w - w\|_{p',\lambda}\|u_0\|_{p,\lambda} \to 0$ when $t \to 0^+$. Recalling R is fixed, it follows that $G(t)u_0 \rightharpoonup u_0$.
Hence, we need to show that $B(u,u)(t) \rightharpoonup 0$ as $t \to 0^+$ in $D'(\mathbb{R}^n)$. For every $w \in D(\mathbb{R}^n)$, let $R > 0$ such that $\text{supp}(w) \subset B_R(0)$. Then, we have

$$\begin{aligned}
R^{-\lambda}|\langle B(u,u)(t), w\rangle| &\leq \int_0^t R^{-\lambda}\int_{\mathbb{R}^n} |\nabla_x G(t-s)(u(\nabla K * u))(x,s)w(x)|dxds \\
&= \int_0^t R^{-\lambda}\int_{B_R(0)} |\nabla_x G(t-s)(u(\nabla K * u))(x,s)w(x)|dxds. \qquad (4.13)
\end{aligned}$$

Let $l' > 0$ such that $\frac{1}{l'} + \frac{1}{l} = 1$, with $1 < l = \frac{p}{\gamma+1}$ and $0 < \gamma < \frac{p}{q}$. By Holder inequality we obtain

$$\begin{aligned}
&R^{-\lambda}\int_{B_R(0)} |\nabla_x G(t-s)(u(\nabla K * u))(x,s)w(x)|dx \\
&\leq R^{-\lambda}\left(\int_{B_R(0)} |\nabla_x G(t-s)(u(\nabla K * u))(x,s)|^l dx\right)^{\frac{1}{l}}\left(\int_{B_R(0)} |w(x)|^{l'}dx\right)^{\frac{1}{l'}} \\
&\leq \|\nabla_x G(t-s)(u(\nabla K * u))(s)\|_{l,\lambda}\|w\|_{l',\lambda}.
\end{aligned}$$

Let $r > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and since $\gamma < \frac{p}{q}$ we have $1 < r < l$. By (4.3), (2.5) and (2.6) we obtain

$$\|\nabla_x G(t-s)(u(\nabla K * u))(s)\|_{l,\lambda} \leq C(t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2r}+\frac{n-\lambda}{2l}}\|u(s)\|_{q,\lambda}\|\nabla K\|_1\|u(s)\|_{p,\lambda}. \quad (4.14)$$

Hence, by (4.13) and (4.14), we have

$$R^{-\lambda}|\langle B(u,u), w\rangle| \leq LI\|w\|_{l',\lambda},$$

where $L = C\|\nabla K\|_1 \sup_{t>0} t^\eta \|u\|_{q,\lambda} \sup_{t>0} \|u\|_{p,\lambda}$ and

$$\begin{aligned} I &= \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2r}+\frac{n-\lambda}{2l}} s^{-\eta} ds \\ &= t^{-\frac{1}{2}-\frac{n-\lambda}{2r}+\frac{n-\lambda}{2l}} \int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2r}+\frac{n-\lambda}{2l}} (ts)^{-\eta} t ds \\ &= Ct^{\frac{n-\lambda}{2l}-\frac{n-\lambda}{2p}}. \end{aligned}$$

Since $\frac{n-\lambda}{2l} - \frac{n-\lambda}{2p} > 0$ and R is fixed, we concluded that $B(u,u)(t) \rightharpoonup 0$ as $t \to 0^+$ in $D'(\mathbb{R}^n)$.

Part (iii)(Symmetry): Because Lemma 4.1 was used in the proof of $Part\ (i)$, we have that solution u is the limit of the Picard sequence

$$u_1(\cdot,t) = G(t)u_0(\cdot) \text{ and } u_{n+1}(\cdot,t) = u_1(\cdot,t) + B(u_n,u_n)(t), \text{ where } n \in \mathbb{N}.$$

As $g(x,t)$ is even and by hypothesis u_0 is odd (even) thus u_1 is odd (even).

Now, an induction argument demonstrates that u_k is odd (even), for $k \geq 1$. Indeed, by changing variables, a calculation shows that

$$B(u_k,u_k)(-x,t) = -B(u_k,u_k)(x,t)\ (B(u_k,u_k)(-x,t) = B(u_k,u_k)(x,t)),$$

because K is for hypothesis odd (oven).

Therefore, since

$$-u(x,t) = -\lim_{k\to\infty} u_k(x,t) = \lim_{k\to\infty} u_k(-x,t) = u(-x,t)$$

$$(u(x,t) = \lim_{k\to\infty} u_k(x,t) = \lim_{k\to\infty} u_k(-x,t) = u(-x,t)),$$

it follows from the uniqueness of the limit that u is odd (even).

□

4.4 Proof of Theorem 3.3

By hypothesis, we have

$$u(t) = G(t)u_0 + B(u,u)(t) \tag{4.15}$$

and

$$v(t) = G(t)v_0 + B(v,v)(t). \tag{4.16}$$

Subtracting the equations (4.15) and (4.16), taking the norms $t^\eta\|\cdot\|_{q,\lambda}$ and $\|\cdot\|_{p,\lambda}$, we obtain the respective inequalities

$$\begin{aligned} t^\eta\|u(\cdot,t)-v(\cdot,t)\|_{q,\lambda} &\leq t^\eta\|G(t)(u_0-v_0)\|_{q,\lambda} + t^\eta\|B(u-v,u)(t)+B(v,u-v)(t)\|_{q,\lambda} \\ &:= H_0(t)+H_1(t) \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \|u(\cdot,t)-v(\cdot,t)\|_{p,\lambda} &\leq \|G(t)(u_0-v_0)\|_{p,\lambda} + \|B(u-v,u)(t)+B(v,u-v)(t)\|_{p,\lambda} \\ &:= H_2(t)+H_3(t). \end{aligned} \tag{4.18}$$

From (4.9) and (4.8), the terms H_1 and H_3 can be estimated as

$$\begin{aligned} H_1(t) &\leq Ct^\eta\|\nabla K\|_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} \|u(\cdot,s)-v(\cdot,s)\|_{q,\lambda}(\|u(\cdot,s)\|_{q,\lambda}+\|v(\cdot,s)\|_{q,\lambda})ds \\ &\leq 4\varepsilon\|\nabla K\|_1 Ct^\eta \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} \|u(\cdot,s)-v(\cdot,s)\|_{q,\lambda} s^{-\eta} ds \\ &= 4\varepsilon\|\nabla K\|_1 C \int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-2\eta}(ts)^\eta \|u(\cdot,ts)-v(\cdot,ts)\|_{q,\lambda} ds \end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
H_3(t) &\le C\|\nabla K\|_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} \|u(\cdot,s)-v(\cdot,s)\|_{p,\lambda}(\|u(\cdot,s)\|_{q,\lambda}+\|v(\cdot,s)\|_{q,\lambda})ds \\
&\le 4\varepsilon\|\nabla K\|_1 C \int_0^t (t-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} \|u(\cdot,s)-v(\cdot,s)\|_{p,\lambda} s^{-\eta} ds \\
&= 4\varepsilon\|\nabla K\|_1 C \int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} \|u(\cdot,ts)-v(\cdot,ts)\|_{p,\lambda} s^{-\eta} ds, \qquad (4.20)
\end{aligned}$$

respectively, because

$$s^\eta \|u(\cdot,s)\|_{q,\lambda} \le \sup_{s>0} s^\eta \|u(\cdot,s)\|_{q,\lambda} + \sup_{s>0} \|u(\cdot,s)\|_{p,\lambda} = \|u\|_{E_q} \le 2\varepsilon,$$

i.e., $\|u(\cdot,s)\|_{q,\lambda} \le 2\varepsilon s^{-\eta}$. Analogously $\|v(\cdot,s)\|_{q,\lambda} \le 2\varepsilon s^{-\eta}$. Let us set

$$A = \limsup_{t\to\infty} t^\eta \|u(\cdot,t)-v(\cdot,t)\|_{q,\lambda} \text{ and } B = \limsup_{t\to\infty} \|u(\cdot,t)-v(\cdot,t)\|_{p,\lambda}.$$

Computing $\limsup_{t\to\infty}$ in (4.17)–(4.18) and using (3.7), it follows that

$$\begin{aligned}
A &\le \limsup_{t\to\infty} H_0(t) + \limsup_{t\to\infty} H_1(t) \\
&\le 4\varepsilon\|\nabla K\|_1 C \int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-2\eta} ds A = 4\varepsilon M_2 A \le 4\varepsilon M A
\end{aligned}$$

and

$$\begin{aligned}
B &\le \limsup_{t\to\infty} H_2(t) + \limsup_{t\to\infty} H_3(t) \\
&\le 4\varepsilon\|\nabla K\|_1 C \int_0^1 (1-s)^{-\frac{1}{2}-\frac{n-\lambda}{2q}} s^{-\eta} ds B \le 4\varepsilon M_1 B \le 4\varepsilon M B.
\end{aligned}$$

By the conditions of Theorem 3.1, we have that $4\varepsilon M < 1$ and as $A, B \ge 0$, we obtain that $A = B = 0$, i.e, $\lim_{t\to+\infty} \|u(\cdot,t)-v(\cdot,t)\|_{p,\lambda} = \lim_{t\to+\infty} t^\eta \|u(\cdot,t)-v(\cdot,t)\|_{q,\lambda} = 0$.

In order to prove the reciprocal, subtracting the equations (4.15) and (4.16) similarly to the above proof, we get

$$\begin{aligned}
&t^\eta \|G(t)(u_0-v_0)\|_{q,\lambda} + \|G(t)(u_0-v_0)\|_{p,\lambda} \\
&\quad \le t^\eta \|u(\cdot,t)-v(\cdot,t)\|_{q,\lambda} + t^\eta \|B(u-v,u)(t) + B(v,u-v)(t)\|_{q,\lambda} \\
&\quad\quad + \|u(\cdot,t)-v(\cdot,t)\|_{p,\lambda} + \|B(u-v,u)(t) + B(v,u-v)(t)\|_{p,\lambda} \\
&\quad = t^\eta \|u(\cdot,t)-v(\cdot,t)\|_{q,\lambda} + \|u(\cdot,t)-v(\cdot,t)\|_{p,\lambda} + H_1(t) + H_3(t). \qquad (4.21)
\end{aligned}$$

Considering (4.19) and (4.20), we obtain (3.7) after taking $\limsup_{t\to\infty}$ in (4.21) and using (3.8).

□

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Author information

Marta L. Suleiman, Juliana C. Precioso and Andrea C. Prokopczyk, Department of Mathematics, São Paulo State University, São José do Rio Preto, São Paulo 15054-000, Brazil.
E-mail: mls_suleiman@hotmail.com, juliana.precioso@unesp.br, andrea.prokopczyk@unesp.br

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