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ON NONLINEAR NEUMANN INTEGRAL CONDITION FOR A SEMILINEAR HEAT PROBLEM WITH BLOW UP SIMULATION

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Abstract In this paper, we present and discuss a class of semi-linear heat equation with nonlinear nonlocal conditions of second type. The existence and uniqueness of weak solution of the presented problem are investigated in view of the linearisation method. Besides, a well study on the generalized Fujuta problem is also presented. Several graphical comparisons are carried out between the exact and numerical finite-time blow-up solutions.

1 Introduction

The nonlinear diffusion problems are commonly utilized to model many phenomena identified several fields of physics such as electromagnetism, acoustics, electrochemistry, cosmology, biochemistry and dynamics of biological groups [1, 2, 3]. Their complexity in their theoretical and numerical study, which are deemed certainly challenges in mathematics, have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences [4, 5, 6, 7, 8, 9]. The first question to ask in handling such theoretical studies to these problems is whether a nonlinear evolution equation with a given initial data has at least a solution in a certain time, and whether this solution is unique in the considered classes, especially the classes taken under some nonlocal conditions. These studies, which were discussed in [10, 11, 12, 13, 14, 15], can be more generalized through passing to some dynamic notion, especially those notion which are related to the blow-up phenomena. As a matter of fact, this mathematical theoretical view was actively opened by researchers in the 60's mainly after the general blow-up approaches, like Fujita [16], Victtor et al. [17], Chen [18] and many others. Although there is as yet no complete theory developed to deal with the aforesaid generality, there are several detailed studies that have been performed on hierarchy models of increasing complexity.

Numerical methods that can be used to detect the blow-up phenomenon and compute or approximate their blow-up solutions, times and profiles are deemed very rare very collectable. This actually backs to the highly difficult in producing their computational solutions. For instance, the semidiscretization in certain space, which requires a lot of complicated computations, leads typically to initial value problem that consisting of a system of nonlinear ordinary differential equations [17]. In general, the nonlinear diffusion equations subject to some initial and generalized integral conditions are difficult to be studied spatially. Besides, it is also hard to clarify the complete picture of their blow-up phenomena and the existence of their global solutions including some critical situations.

In this work, we will consider the following nonlinear problem:

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t}-a\dfrac{\partial^2 u}{\partial x^2}=f\left(x,t,u,\dfrac{\partial u}{\partial x}\right) & \forall\,(x,t)\in Q, \\ u\,(x,0)=\varphi\,(x) & \forall x\in(0,1)\,, \\ u_x(0,t)=\int_0^1 k_0(x,t)g(u(x,t))dx & \forall t\in(0,T)\,, \\ u_x(1,t)=\int_0^1 k_1(x,t)h(u(x,t))dx & \forall t\in(0,T)\,. \end{array}\right. . \tag{P0}$$

where $Q = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t < T\right\}$, $a > 0$, and $f, g, h, k_0, k_1, \varphi \in L^2(\Omega)$ in which the function f is Lipschitzien, i.e, there exists a positive constant k such that:

$$\|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)\|_{L^2(Q)} \leq k\left(\|u_1 - u_2\|_{L^2(Q)} + \|v_1 - v_2\|_{L^2(Q)}\right), \tag{1.1}$$

for all $(u_1,v_1), (u_2,v_2) \in \left(L^2(Q)\right)^2$. In addition, the functions g and h satisfy respectively the following two inequalities:

$$\|g(x,t,u)\|_{L^2(Q)} \leqslant C_0 \|u\|_{L^2(Q)}, \text{ and } \|h(x,t,u)\|_{L^2(Q)} \leqslant C_1 \|u\|_{L^2(Q)}, \tag{1.2}$$

where C_0 and C_1 are positive constants.

For the reason that this work focuses its attention to the framework of some functional methods, the existence and uniqueness of the solution for the generalization of Fujita problem with nonlocal condition are discussed and explored to the respective linear case. Afterward, the existence and uniqueness are proved for the nonlinear problem. To analyze the problem more generally, the theoretical and numerical investigations via some illustrative examples of the blow-up phenomena are introduced.

2 Existence and uniqueness of strong solution of linear problem

In this section, we will first formulate the main linear problem related to ($P0$), and then explore the existence and uniqueness of that problem. For this purpose, we let

$$Q = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t < T\right\}$$

and consider the following linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial^2 u}{\partial x^2} = f(x,t) & \forall (x,t) \in Q, \\ u(x,0) = \varphi(x) & \forall x \in (0,1), \\ u_x(0,t) = \int_0^1 k_0(x,t)g(u(x,t))dx & \forall t \in (0,T), \\ u_x(1,t) = \int_0^1 k_1(x,t)h(u(x,t))dx & \forall t \in (0,T). \end{cases} . \tag{P1}$$

2.1 Uniqueness of solution of problem ($P1$)

In this subsection, we will use the energy inequality method to obtain a priori estimates for the solution of problem ($P1$). To this aim, we introduce the next theoretical results that related to the uniqueness of solution of such problem.

Theorem 2.1. *For any function $u \in E$, we have the following inequality:*

$$\|u\|_E \leq c\|Lu\|_F, \tag{2.1}$$

where c is a positive constant independent of u.

Proof. Assume that a solution the linear problem exists. Multiplying the equation of the linear problem by u and then integrating the result over Q^τ yield the following assertions:

$$\int_{Q^\tau} \mathcal{L}u \cdot u\, dxdt = \int_{Q^\tau} u_t \cdot u\, dxdt - a\int_{Q^\tau} \Delta u \cdot u.dxdt = \int_{Q^\tau} f(x,t) \cdot u.dxdt, \tag{2.2}$$

where $Q^\tau = \Omega \times (0,\tau)$. Integrating by parts each term of the left-hand side of (2.2) over Q^τ leads to the following equality:

$$\begin{aligned} &\frac{1}{2}\int_0^1 u(x,\tau)^2\, dx + a\int_0^\tau u_x^2\, dt \\ &\quad = a\int_0^\tau u_x(1,t)\,u(1,t)\, dt - a\int_0^\tau u_x(0,t)\,u(0,t)\, dt + \frac{1}{2}\int_0^1 \varphi^2\, dx + \int_{Q^\tau} f \cdot u\, dxdt, \end{aligned} \tag{2.3}$$

where $0 < \tau < T$. The next aim is to derive estimates of the right-hand side of (2.3). In order to achieve this goal, we integrate each term of (2.3) over $(0, T)$ to get:

$$\int_0^T \left(\int_0^1 K_1(x,t)\, h(u(x,t))dx\right)^2 \leq k_1^2 c_1^2 \|u\|_{L^2(Q^\tau)}.$$

$$\int_0^T \left(\int_0^1 K_0(x,t)\, g(u(x,t))dx\right)^2 \leq k_0^2 c_0^2 \|u\|_{L^2(Q^\tau)}.$$

In addition, we have:

$$\int_0^\tau u(1,t)^2 dt \leq 2\|u_x\|^2_{L^2(Q^\tau)} + 2\|u\|^2_{L^2(0,T)}.$$

and

$$\int_0^\tau u(0,t)^2 dt \leq 2\|u_x\|^2_{L^2(Q^\tau)} + 2\|u\|^2_{L^2(0,T)}.$$

Then, after applying the Cauchy-Schwarz inequality to make up for the above equality (2.3) and integrating the resultant inequality, we can have:

$$\|u\|^2_{L^2(\Omega)} + a\|u_x\|^2_{L^2(Q^\tau)} \leq \|f\|^2_{L^2(Q^\tau)} + \|\varphi\|^2_{L^2(\Omega)} + \left(ac_1^2k_1^2 + ac_0^2k_0^2 + a + 1\right)\|u\|^2_{L^2(Q^\tau)}. \quad (2.4)$$

Now, we pose:

$$C' = ac_1^2k_1^2 + ac_0^2k_0^2 + a + 1.$$

Then, by applying Gronwall inequality to (2.4), we obtain:

$$\|u\|_{C(0,T,L^2(\Omega))} + \|u_x\|^2_{L^2(Q^\tau)} \leq \frac{1}{\min\{1,a\}} \exp(C'T)\left(\|f\|^2_{L^2(Q^\tau)} + \|\varphi\|^2_{L^2(\Omega)}\right).$$

Consequently, we obtain the desired inequality in which:

$$C = \frac{1}{\min\{1,a\}} \exp(C'T).$$

Thus, we get:

$$\|u\|^2_{C(0,T,L^2(\Omega))} + \|u_x\|^2_{L^2(Q^\tau)} \leq c^2\left(\|f\|^2_{L^2(Q^\tau)} + \|\varphi\|^2_{L^2(\Omega)}\right). \quad (2.5)$$

Finally, we can obtain the desired inequality so that $c = \sqrt{\dfrac{\exp(mT)}{\min\{1,a\}}}$. □

Corollary 2.2. *If we have the following estimate:*

$$\|u\|_E \leq c\|\mathcal{F}\|_F, \quad (2.6)$$

for any function $u \in D(L)$. Then, the solution of the main linear problem $(P1)$ is unique.

Proof. Let u_1 and u_2 be two solutions of problem $(P1)$. In other words, we have:

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

where L is the linear operator of the problem at hand. Therefore, we can get:

$$L(u_1 - u_2) = 0.$$

According to (2.6), we can obtain:

$$\|u_1 - u_2\|^2_E \leq c\|0\|^2_F = 0,$$

which immediately implies:

$$u_1 = u_2.$$

□

Corollary 2.3. *The solution of the main linear problem $(P1)$, if it exists, depends continuously on $\mathcal{F} \in F$.*

Proof. The proof can be immediately gotten from the above two results. □

2.2 Existence of solution of problem $(P1)$

This part is consecrated to the proof of the existence of the solution of the main linear problem $(P1)$. To this aim, let us consider the following auxiliary problem with homogeneous equation:

$$\mathcal{L}w = \frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} = 0,$$

subject to the initial data:

$$\ell w = w(x,0) = \varphi(x),$$

and the following nonlinear integral conditions of second kind:

$$w_x(0,t) = \int_0^1 K_0(x,t)\, g(w(x,t) + y(x,t))dx,$$

$$w_x(1,t) = \int_0^1 K_1(x,t)\, h(w(x,t) + y(x,t))dx.$$

Let g^* and h^* be two functions defined respectively as:

$$g^*(w) = g(w,y) \text{ and } h^*(w) = h(w,y),$$

so that they satisfy the following inequality:

$$\|g^*(w)\|_{L^2(Q)} \leqslant b_1 \|w\|_{L^2(Q)} + b_2, \text{ and } \|h^*(w)\|_{L^2(Q)} \leqslant b_3 \|w\|_{L^2(Q)} + b_4,$$

where b_1, b_2, b_3 and b_4 are positive constants. Accordingly, the above auxiliary problem with homogeneous equation becomes:

$$\begin{cases} \mathcal{L}w = \dfrac{\partial w}{\partial t} - a\dfrac{\partial^2 w}{\partial x^2} = 0, & x,t \in Q \\ \ell w = w(x,0) = \varphi(x), & x \in (0,1) \\ w_x(0,t) = \int_0^1 K_0(x,t)\, g^*(w(x,t))\, dx, & t \in (0,T) \\ w_x(1,t) = \int_0^1 K_1(x,t)\, h^*(w(x,t))\, dx. & t \in (0,T) \end{cases}. \tag{P2}$$

Note that if u is a solution of the main problem $(P1)$ and w is a solution of problem $(P2)$, then $y = u - w$ satisfies the following problem:

$$\begin{cases} \mathcal{L}y = \dfrac{\partial y}{\partial t} - a\dfrac{\partial^2 y}{\partial x^2} = f(x,t), & x,t \in Q \\ \ell y = y(x,0) = 0, & x \in (0,1) \\ y_x(0,t) = 0, & t \in (0,T) \\ y_x(1,t) = 0, & t \in (0,T) \end{cases}. \tag{P3}$$

Now, to show the existence of solution of the problem $(P2)$, it is enough to transform it to a nonlinear ordinary differential equation. To this aim, we integrate the equation of $(P2)$ over Ω to obtain:

$$\int_0^1 w_t dx - a\int_0^1 w_{xx} dx = 0, \ \forall x \in \Omega.$$

Consequently, we get:

$$\int_0^1 \left[w_t - a(K_1(x,t)\, h^*(w(x,t))\, dx + K_0(x,t)\, g^*(w(x,t)))\right] dx = 0.$$

This implies:

$$\int_0^1 (w_t - F(t, w(x,t)))dx = 0, \tag{2.7}$$

where

$$aK_1(x,t)\, h^*(w(x,t)) - aK_0(x,t) g^*(w(x,t)) = F(t, w(x,t)).$$

So, it is clear that there exists a function ψ satisfying the following assertion:

$$w_t - F(t, w(x,t)) = \psi(x,t), \text{ where } \int_0^1 \psi(x,t)\, dx = 0.$$

Thus, we have:

$$w_t = G(t, w(t)),$$

where

$$G(t, w(t)) = F(t, w(x,t)) + \psi(x,t).$$

One can note that the function G is a Carathéodory mapping, because we have:

- The function G is continuous for w because the function F is continuous for w.
- The function G is measurable for t because F and ψ are measurable for t.
- The function G satisfies the following inequalities:

$$\begin{aligned} \|G(t,w)\| &\le a\|K\|_\infty (\|h^*(w)\| + \|g^*(w)\|) + \|\psi(x,t)\|_{L^2} + b_2 + b_4 \\ &\le b_2 + b_4 + \|\psi(x,t)\|_{L^2} + 2a\|K\|_\infty \|w\|. \end{aligned}$$

Now, by applying the existence and uniqueness theorem, we can assert that $w \subset W^{1,1}$. Besides, by applying the Nemytskii mappings in Lebesgue spaces, we can at the same time assert that w_t in $L^2[0,T]$. Thus, according to these results, we can deduce that problem (P2) admits a unique solution. Hence, it remains to prove the uniqueness of a strong solution for problem ($P3$). In order to accomplish this goal, we present the next theoretical results.

Theorem 2.4. *For any function $y \in E$, we have the inequality:*

$$\|y\|_E \le c\|Ly\|_F, \tag{2.8}$$

where c is a positive constant independent of y.

Proof. Assume that the solution of problem ($P3$) exists. Accordingly, by multiplying the equation of ($P3$) by y and then integrating the result over Q^τ, we get:

$$\int_{Q^\tau} y_t \cdot y - a\int_{Q^\tau} y_{xx} \cdot y = \int_{Q^\tau} f(x,y)\cdot y, \tag{2.9}$$

where $Q^\tau = \Omega \times (0,\tau)$. Furthermore, by integrating by parts each term of the left-hand side of (2.9) over Q^τ, we obtain:

$$\frac{1}{2}\int_0^1 y(x,\tau)^2\, dx + a\int_{Q^\tau} y_x^2\, dt = \int_{Q^\tau} f\cdot y\, dxdt \le \frac{1}{2}\|f\|^2_{L^2(Q)} + \frac{1}{2}\|y\|^2_{L^2(Q)},$$

where $0 < \tau < T$. Now, using Gronwall Lemma leads to the following inequality:

$$\int_0^1 y(x,\tau)^2\, dx + \|y_x\|^2_{L^2(Q^\tau)} \le \frac{\exp(T)}{\min\{1, 2a\}}\|f\|^2_{L^2(Q)}.$$

By passing to the maximum over $(0,T)$, we get:

$$\|y\|^2_{L^\infty(0,T;\, L^2(\Omega))} + \|y_x\|^2_{L^2(Q)} \le c^2\|f\|^2_{L^2(Q)}.$$

Finally, we obtain the desired inequality so that $c = \sqrt{\dfrac{\exp(T)}{\min\{1,2a\}}}$. □

Corollary 2.5. *If we have the following estimate:*

$$\|u\|_E \le C\|\mathcal{F}\|_F,$$

for any function $u \in D(L)$, then the solution of problem ($P3$), if it exists, is unique.

Proof. Let u_1 and u_2 be two solutions of problem $(P3)$. That is,

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0.$$

Since L is a linear operator, we obtain:

$$L(u_1 - u_2) = 0.$$

Consequently, according to (2.8), we can gain the following assertion:

$$\|u_1 - u_2\|_E^2 \leq c\,\|0\|_F^2 = 0.$$

This directly gives:

$$u_1 = u_2,$$

which finishes the proof. □

2.3 Existence of solution of problem $(P3)$

In this subsection, we will explore the existence of solution of problem $(P3)$. For this purpose, we will first state some basic facts related to our task herein for completeness.

Proposition 2.6. *The operator L of E in F has a closure operator.*

Proof. The proof can be carried out in a similar manner of the proof given in [14]. □

Definition 2.7. The solution of the equation:

$$\overline{L}u = \mathcal{F}$$

is said to be a strong generalized solution of problem $(P3)$.

It should be pointed out that Theorem 2.2 is also valid for a strong generalized solution. In other words, we can confirm the following inequality:

$$\|u\|_E \leq K\left\|\overline{L}u\right\|_F \ \forall u \in D(\overline{L}). \tag{2.10}$$

In view of this assertion, we present the next consecutive theoretical results.

Corollary 2.8. *The solution of problem $(P3)$, if it exists, is unique and depends continuously on $\mathcal{F} \in F$.*

Proof. Omitted. □

Corollary 2.9. *The set of values $R(\overline{L})$ of the operator $\overline{L}$ is equal to $\overline{R(L)}$.*

Proof. This corollary can be easily demonstrated from the previous arguments. □

Theorem 2.10. *In light of the above results, the solution of problem $(P3)$ is exist.*

Proof. First of all, we should prove that $R(L)$ is dense in F for every $y \in E$ and for all $\mathcal{F} = (f, \varphi) \in F$. To this aim, we let $\overline{L}$ be the closure of L and $D(\overline{L})$ be the domain of $\overline{L}$. In order to prove the existence of solution of problem $(P3)$, it is enough to prove Ly is surjective. For achieving this objective, we should note that, according to the density of $\overline{L}$, we have $\overline{R(L)} = F$. Therefore, we obtain $R(L)^{\perp} = \{0\}_F$, and so we have:

$$\begin{aligned} R(L)^{\perp} &= \{w \in F, \langle w, \mathcal{L}\rangle_F = 0, \forall \mathcal{F} \in R(L)\} \\ &= \left\{(w, w_0) \in L^2(Q), \langle w, f\rangle_{L^2(Q)} + \langle w_0, \varphi\rangle_{L^2(Q)} = 0, \forall f \in L^2(Q), \forall \varphi \in L^2(Q)\right\}. \end{aligned}$$

and

$$D_0(L) = \{y \in E,\ y(x, 0) = 0\}.$$

Consequently, we have:

$$w_0 = 0.$$

It remains to demonstrate that $w = 0$. In this connection, we have:

$$\langle w, Ly\rangle_{L^2(Q)} = \int_0^1 \int_0^T wLy = 0.$$

Now, we pose $w = y$ to obtain:

$$\int_0^1 \int_0^T y(y_t - a\Delta y) = \int_0^1 \int_0^T y \cdot y_t - a\int_0^1 \int_0^T y \cdot y_{xx} = 0.$$

Therefore, we have:

$$\frac{1}{2}\int_0^1 y^2(x,T) = -a\int_0^T y_x^2 \leq 0.$$

Finally, we obtain:

$$y = 0 \Longrightarrow w = 0,$$

which completes the proof. □

3 The weak solution of problem ($P0$)

This section is devoted to the proof of existence and uniqueness of the nonlinear problem ($P0$). In view of this task, we inted to consider the following auxiliary problem with homogeneous equation:

$$\begin{cases} \dfrac{\partial w}{\partial t} - a\dfrac{\partial^2 w}{\partial x^2} = 0 & \forall\,(x,t) \in Q, \\ w\,(x,0) = \varphi\,(x) & \forall x \in (0,1)\,, \\ w_x(0,t) = \int_0^1 k_0(x,t)g^*(w(x,t))dx & \forall t \in (0,T)\,, \\ w_x(1,t) = \int_0^1 k_1(x,t)h^*(w(x,t))dx & \forall t \in (0,T)\,, \end{cases} . \tag{P'2}$$

such that the two functions g^* and h^* satisfy respectively the following situations:

$$g^*\,(w) = g(w,y) \text{ and } h^*\,(w) = h(w,y),$$

and

$$\|g^*\,(w)\|_{L^2(Q)} \leqslant b_1\,\|w\|_{L^2(Q)} + b_2, \text{ and } \|h^*\,(w)\|_{L^2(Q)} \leqslant b_3\,\|w\|_{L^2(Q)} + b_4,$$

where b_1, b_2, b_3 and b_4 are positive constants. Based on the discussion reported in the previous section, we obviously observe that w is exist and unique. In this regard, if u is a solution of problem ($P0$) and w is a solution of problem ($P'2$), then $y = u - w$ satisfies:

$$\begin{cases} \mathcal{L}y = \dfrac{\partial y}{\partial t} - a\dfrac{\partial^2 y}{\partial x^2} = G\left(x,t,y,\dfrac{\partial y}{\partial x}\right) & x,t \in Q \\ y(x,0) = 0 & x \in (0,1) \\ \dfrac{\partial y}{\partial x}(x,0) = 0 & t \in (0,T) \\ \dfrac{\partial y}{\partial x}(1,t) = 0 & t \in (0,T) \end{cases} . \tag{P'3}$$

where $G\left(x,t,y,\dfrac{\partial y}{\partial x}\right) = f\left(x,t,y+w,\dfrac{\partial y}{\partial x} + \dfrac{\partial w}{\partial x}\right)$. Since the function f is Lipschitzian, then the function G is also Lipschitzian, i.e., there exists a positive constant k such that:

$$\|G\,(x,t,u_1,v_1) - G\,(x,t,u_2,v_2)\|_{L^2(Q)} \leq k\left(\|u_1 - u_2\|_{L^2(Q)} + \|v_1 - v_2\|_{L^2(Q)}\right), \tag{3.1}$$

for any $(u_1,v_1)\,,(u_2,v_2) \in \left(L^2\,(Q)\right)^2$. Now, let $v = v\,(x,t)$ be any function of $L^2(0,T;H^{1}(0,1))$. By multiplying the equation of (P'3) by v and then integrating the result over Q, we obtain:

$$\int\limits_Q \frac{\partial y}{\partial t}v dxdt - a\int\limits_Q \frac{\partial^2 y}{\partial x^2}v dxdt = \int\limits_Q vG\left(x,t,y,\frac{\partial y}{\partial x}\right)dxdt.$$

Consequently, by implementing the integration by parts to the above result and then using the condition given on y, it comes:

$$\int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x}\frac{\partial v}{\partial x} dx dt = \int_Q vG\left(x, t, y, \frac{\partial y}{\partial x}\right) dx dt. \tag{3.2}$$

Based on (3.2), we can gain:

$$A(y, v) = \int_Q vG\left(x, t, y, \frac{\partial y}{\partial x}\right) dx dt, \tag{3.3}$$

where

$$A(y, v) = \int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x}\frac{\partial v}{\partial x} dx dt.$$

In order to proceed with our investigation, we will recall next the definition of the weak solution related to the problem at hand.

Definition 3.1. A function $y \in L^2\left(0, T;\ H^1(0, 1)\right)$ is said to be a weak solution of problem (P'3) if y satisfies (3.3) and $y_x(0, t) = y_x(1, t) = 0$.

In what follow, we aim to build a recurring sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ starting with $y^{(0)} = 0$. This sequence can be defined as follows: Given the element $y^{(n-1)}$, then we have:

$$\begin{cases} \dfrac{\partial y^{(n)}}{\partial t} - a\dfrac{\partial^2 y^{(n)}}{\partial x^2} = G\left(x, t, y^{(n-1)}, \dfrac{\partial y^{(n-1)}}{\partial x}\right) \\ y^{(n)}(x, 0) = 0 \\ \dfrac{\partial y^{(n)}}{\partial x}(0, t) = \dfrac{\partial y^{(n)}}{\partial x}(1, t) = 0 \end{cases}, \tag{P4}$$

for $n = 1, 2, 3, \cdots$. According to the Theorem 2.2, problem (P4) admits a unique solution $y^{(n)}(x, t)$, for each fixed n. Now, suppose $z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t)$. Then, we get consequently a new problem:

$$\begin{cases} \dfrac{\partial z^{(n)}}{\partial t} - a\dfrac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t) \\ z^{(n)}(x.0) = 0 \\ \dfrac{\partial z^{(n)}}{\partial x}(0, t) = \dfrac{\partial z^{(n)}}{\partial x}(1, t) = 0 \end{cases}, \tag{P5}$$

where

$$p^{(n-1)}(x, t) = G\left(x, t, y^{(n)}, \frac{\partial y^{(n)}}{\partial x}\right) - G\left(x, t, y^{(n-1)}, \frac{\partial y^{(n-1)}}{\partial x}\right).$$

Lemma 3.2. *Assume that the condition (3.1) is satisfied, then we have the following priori estimate related to problem ($P5$):*

$$\|z^n\|_{L^2\left(0,T;\ H^1(0,1)\right)} \le c\|z^{n-1}\|_{L^2\left(0,T;\ H^1(0,1)\right)},$$

where

$$c = \sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}}}.$$

Proof. Multiplying the following equation:

$$\frac{\partial z^{(n)}}{\partial t} - a\frac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t), \tag{3.4}$$

by $z^{(n)}(x,t)$ and then integrating the result over Q_τ yield:

$$\int\limits_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}(x,t)\cdot z^{(n)}(x,t)\,dxdt - a\int\limits_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial x^2}(x,t)\cdot z^{(n)}(x,t)\,dxdt = \int\limits_{Q_\tau} p^{(n-1)}(x,t)\cdot z^{(n)}(x,t)\,dxdt.$$

By applying the integration by part to the above result and then by using the given initial and boundary conditions, we obtain:

$$\left|p^{n-1}(x,t)\right|^2 = \left|G\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)\right|^2.$$

Since G is Lipschitzian and that $(a+b)^2 \leq 2\left(a^2+b^2\right)$, then we have:

$$\left|p^{n-1}(x,t)\right|^2 \leq k^2\left(\left|y^{(n)}-y^{(n-1)}\right| + \left|y_x^{(n)}-y_x^{(n-1)}\right|\right)^2 \leq 2k^2\left(\left|z^{(n-1)}\right|^2 + \left|z_x^{(n-1)}\right|^2\right).$$

Consequently, we obtain:

$$\int\limits_{Q} \left|p^{n-1}(x,t)\right|^2 dxdt \leq 2k^2\|z^{n-1}\|^2_{L^2\left(0,T;\,H^1(0,1)\right)}. \tag{3.5}$$

Finally, by applying the Gronwall Lemma and then passing to the maximum on $(0,T)$, we get:

$$\|z^n\|^2_{L^\infty\left(0,T;\,L^2(0,1)\right)} + \|z^n\|^2_{L^2\left(0,T;\,H^1(0,1)\right)} \leq \frac{k^2\exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2},a\right\}}\|z^{n-1}\|^2_{L^2\left(0,T;\,H^1(0,1)\right)}.$$

Accordingly, we can also obtain:

$$\|z^n\|_{L^2\left(0,T;\,H^1(0,1)\right)} \leq c\|z^{n-1}\|_{L^2\left(0,T;\,H^1(0,1)\right)},$$

where

$$c = \sqrt{\frac{k^2\exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2},a\right\}}}.$$

This, however, yields the desired result. □

Theorem 3.3. *If the solution of the problem satisfies (3.1) and*

$$k < \sqrt{\frac{\min\left\{\frac{1}{2},a\right\}}{\exp\left(\frac{T}{2}\right)}},$$

then the problem ($P'3$) admits a weak solution belonging to $L^2\left(0,T;\ L^2(0,1)\right)$.

Proof. It is worth mentioning, according to the criterion of the series' convergence, that the series $\sum\limits_{n=1}^{\infty} z^{(n)}$ is said to be convergent if

$$k < \sqrt{\frac{\min\left\{\frac{1}{2},a\right\}}{\exp\left(\frac{T}{2}\right)}}.$$

Now, since $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$ and $y^{(0)}(x,t) = 0$, we have:

$$\sum_{i=0}^{n-1} z^{(i)} = y^{(n)}.$$

Thus, we get the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ which is defined by:

$$y^{(n)}(x,t) = \sum_{i=0}^{n-1} z^{(i)}.$$

Actually, this sequence converges to an element $y \in L^2\left(0,T;\ H^1(0,1)\right)$. Now, we aim to demonstrate that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of problem $(P4)$. This can be performed by showing that:

$$A(y,v) = \int_Q vG\left(x,t,y,\frac{\partial y}{\partial x}\right) dxdt.$$

To this aim, we consider the weak formulation of the following problem $(P4)$:

$$A\left(y^{(n)},v\right) = \int_Q \frac{\partial y^{(n)}}{\partial t} v dxdt + a\int_Q \frac{\partial y^{(n)}}{\partial x}\frac{\partial v}{\partial x} dxdt.$$

From the linearity of A, we can have:

$$A\left(y^{(n)},v\right) = A\left(y^{(n)}-y,v\right) + A(y,v) \tag{3.6}$$

$$= \int_Q \frac{\partial (y^{(n)}-y)}{\partial t} v dxdt + a\int_Q \frac{\partial (y^{(n)}-y)}{\partial x}\frac{\partial v}{\partial x} dxdt + A(y,v). \tag{3.7}$$

Applying the Cauchy Schwartz inequality to $A\left(y^{(n)}-y,v\right)$ yields:

$$\int_Q x\frac{\partial (y^{(n)}-y)}{\partial t} v dxdt + a\int_Q x\frac{\partial (y^{(n)}-y)}{\partial x}\frac{\partial v}{\partial x} dxdt + b\int_Q xv(y^{(n)}-y)dxdt$$

$$\leq 2\max\{1,a\}\cdot \|v_x\|_{L^2\left(0,T;\ H^1(0,1)\right)} \begin{bmatrix} \|\left(y^{(n)}-y\right)_t\|_{L^2\left(0,T;\ H^1(0,1)\right)} \\ +\|\left(y^{(n)}-y\right)\|_{L^2\left(0,T;\ H^1(0,1)\right)} \end{bmatrix}.$$

On the other hand, since

$$y^{(n)} \longrightarrow y \quad \text{in } L^2\left(0,T;\ H^1(0,1)\right),$$

then we have:

$$\begin{aligned} y^{(n)} &\longrightarrow y \quad \text{in } L^2\left(0,T;\ L^2(0,1)\right), \\ y_t^{(n)} &\longrightarrow y_t \quad \text{in } L^2\left(0,T;\ L^2(0,1)\right), \\ y_x^{(n)} &\longrightarrow y_x \quad \text{in } L^2\left(0,T;\ L^2(0,1)\right). \end{aligned}$$

Consequently, as $n \longrightarrow +\infty$, we can find:

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}-y,v\right) = 0. \tag{3.8}$$

By using (3.8) and passing to the limit in (3.6), we obtain:

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)},v\right) = A(y,v).$$

Thus, we have proved the result. □

In the following content, we will show that the solution of problem $(P'3)$ is unique.

Theorem 3.4. *If the condition reported in (3.1) is satisfied, then the solution of problem $(P'3)$ is unique.*

Proof. Let y_1 and y_2 be two solutions of $(P'3)$ in $L^2\left(0,T\,;H^1\left(0,1\right)\right)$. Then:

$$y = y_1 - y_2,$$

is also a solution of such problem in $L^2\left(0,T\,;H^1\left(0,1\right)\right)$. In other words, we have:

$$\frac{\partial y}{\partial t} - a\frac{\partial^2 y}{\partial x^2} = G\left(x,t,y,\frac{\partial y}{\partial x}\right),$$

$$\begin{aligned} y(x,0) &= 0, \\ \frac{\partial y}{\partial x}(0,t) &= \frac{\partial y}{\partial x}(1,t) = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial y}{\partial t} - a\frac{\partial^2 y}{\partial x^2} &= \Psi(x,t), \qquad \forall\,(x,t)\in Q \\ y\,(x,0) &= 0, \\ \frac{\partial y}{\partial x}(0,t) &= \frac{\partial y}{\partial x}(1,t) = 0 \end{aligned}$$

and

$$\Psi(x,t) = G\left(x,t,y_1,\frac{\partial y_1}{\partial x}\right) - G\left(x,t,y_2,\frac{\partial y_2}{\partial x}\right).$$

With the help of using Lemma 3.2, we can obtain:

$$\|y\|_{L^2(0,T,H^1(0,1))} \le c\|y\|_{L^2(0,T,H^1(0,1))}.$$

This implies:

$$(1-c)\,\|y\|_{L^2(0,T,H^1(0,1))} \le 0.$$

As $c \le 1$, we obtain:

$$\|y\|_{L^2(0,T,H^1(0,1))} = 0,$$

which gives:

$$y_1 = y_2,$$

and hence the uniqueness of the solution is hold. □

4 Finite-time blow-up solution of Fujita problem

In this part, the explosion in a finite-time blow-up solution of the semilinear Fujita problem $(P0)$ will be explored and discussed. In particular, we will take the function f reported in that problem as:

$$f\,(x,t,u,u_x) = u^p, \;\; g(u(x,t)) = h(u(x,t)) = u^r,$$

where p and r are strictly positive integers. Note that if we let $u \in C^{1,2}((0,T)\times(0,1))$ be a positive function with $u_t \gneq 0$, we can then formulate the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial^2 u}{\partial x^2} = u^p & \forall\,(x,t)\in Q \\ u\,(x,0) = \varphi\,(x) & \forall x\in(0,1) \\ u_x(0,t) = \int_0^1 k_0(x,t)u^r dx & \forall t\in(0,T) \\ u_x(1,t) = \int_0^1 k_1(x,t)u^r dx & \forall t\in(0,T)\,. \end{cases} \tag{PB}$$

where k_0 and k_1 are positive and bounded functions and satisfy:

$$\begin{aligned} \alpha_0 &\leq k_0 \leq \beta_0 \\ \alpha_1 &\leq k_1 \leq \beta_1. \end{aligned} \tag{4.1}$$

Now, integrating the equation:

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = u^p$$

over $(0, 1)$, we obtain the following two consecutive statements:

$$\begin{aligned} \int_0^1 \frac{\partial u}{\partial t}dx - a\int_0^1 \frac{\partial^2 u}{\partial x^2}dx &= \int_0^1 u^p dx \\ \left(\int_0^1 u\, dx\right)_t + a\int_0^1 k_0(x,t)u^r dx &= \int_0^1 u^p dx + a\int_0^1 k_1(x,t)u^r dx. \end{aligned}$$

Consequently, using the two inequalities given in (4.1) yields:

$$\left(\int_0^1 u\, dx\right)_t \geq \min\{1, a(\alpha_1 - \beta_0)\}\left[\int_0^1 u^p dx + \int_0^1 u^r dx\right].$$

By applying the Jensen inequality, we can obtain:

$$\left(\int_0^1 u\, dx\right)_t \geq 2\min\{1, a(\alpha_1 - \beta_0)\}\left(\int_0^1 udx\right)^s$$

such that:

$$\begin{cases} s = \max\{r, p\} & if: \quad \int_0^1 udx \leq 1 \\ s = \min\{r, p\} & if: \quad \int_0^1 udx \geq 1 \end{cases}.$$

This immediately implies the following inequality:

$$\left(\int_0^1 u\, dx\right)_t \geq C\left(\int_0^1 udx\right)^s, \qquad \text{where } C = 2\min\{1, a(\alpha_1 - \beta_0)\}.$$

Now, by letting:

$$\Pi(t) = \int_0^1 u(x,t)dx,$$

we can get:

$$\Pi'(t) \geq C\Pi^s(t).$$

This actually motivates to formulate the following Bernoulli equation:

$$\Pi'(t) - C\Pi^s(t) = 0,$$

which has accordingly the following solution:

$$\Pi(t) = \left[\frac{1}{(1-s)\left(Ct + \frac{\Pi^{1-s}(0)}{1-s}\right)}\right]^{\frac{1}{s-1}},$$

for $s \gneq 1$. Now, as $\frac{1}{s-1} \gneq 1$, we get:

$$\Pi \to \infty \text{ if } (1-s)\left(Ct + \frac{\Pi^{1-s}(0)}{1-s}\right) \to 0.$$

Finally, we obtain:

$$T^* = \frac{\Pi^{1-s}(0)}{C\,(s-1)},$$

which terminates our discussion about the finite-time blow-up solution of the considered problem.

5 Numerical simulations

In the following content, we will endeavor to establish an explicit and compact formulas to simulate the blow-up solutions of some Fujita problems. It will be carried out through implementing the forward time centred space scheme.

5.1 The explicit formula

The finite-time derivative reported in $(P0)$ can be approximated by the difference quotient [19, 20]. This can be gained with the help of using the second-order centred approximation for the spatial derivative reported in $(P0)$. This scheme can be written as follows:

$$u_i^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n + \Delta t\,(u_i^n)^p\,, \tag{5.1}$$

where $r = a\Delta t/(\Delta x)^2$. Actually, due to the fact that this procedure is explicit, we still have to determine the two unknowns u_0^{n+1} and u_M^{n+1}. For this purpose, we aim to approximate the integrals reported in problem $(P0)$. But now we will approximate the first derivative reported in such problem with the help of using the forward finite difference scheme of second-order. This, however, can be carried out as follows:

$$u_x(0,t_{n+1}) \simeq \frac{-u_2^{n+1} + 4u_1^{n+1} - 3u_0^{n+1}}{2\Delta x}. \tag{5.2}$$

At the same time, to approximate the first derivative of the problem at hand with the help of using the backward finite difference scheme of second-order, we obtain:

$$u_x(1,t_{n+1}) \simeq \frac{3u_M^{n+1} - 4u_{M-1}^{n+1} + u_{M-2}^{n+1}}{2\Delta x}. \tag{5.3}$$

Now, combining the two equations (5.2) and (5.3) in one formula and then linearizing the terms $\left(u_0^{n+1}\right)^r$ and $\left(u_M^{n+1}\right)^r$ in that formula yield the following equality:

$$\begin{aligned}
&\left[-3-(\Delta x)^2 rk_0(0,t_{n+1})\,(u_0^n)^{r-1}\right]u_0^{n+1} - (\Delta x)^2 rk_0(1,t_{n+1})\,(u_M^n)^{r-1}\,u_M^{n+1}\\
&+2\,(\Delta x)^2\sum_{i=1}^{M-1} k_0(x_i,t_{n+1})\left(u_i^{n+1}\right)^r - (\Delta x)^2 rk_1(0,t_{n+1})\,(u_0^n)^{r-1}\,u_0^{n+1}\\
&+\left[3-(\Delta x)^2 rk_1(1,t_{n+1})\,(u_M^n)^{r-1}\right]u_M^{n+1}\\
=\;& 4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2(1-r)\,k_1(0,t_{n+1})\,(u_0^n)^r + (\Delta x)^2(1-r)\,k_1(1,t_{n+1})\,(u_M^n)^r\\
&+2\,(\Delta x)^2\sum_{i=1}^{M-1} k_1(x_i,t_{n+1})\left(u_i^{n+1}\right)^r.
\end{aligned}$$

By setting:

$$\begin{aligned}
a_1 &= -3-(\Delta x)^2 rk_0(0,t_{n+1})\,(u_0^n)^{r-1}\,,\\
a_2 &= -(\Delta x)^2 rk_0(1,t_{n+1})\,(u_M^n)^{r-1}\,,\\
b_1 &= -(\Delta x)^2 rk_1(0,t_{n+1})\,(u_0^n)^{r-1}\,,\\
b_2 &= 3-(\Delta x)^2 rk_1(1,t_{n+1})\,(u_M^n)^{r-1}\,,\\
c_1 &= -4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2(1-r)\,k_0(0,t_{n+1})\,(u_0^n)^r + (\Delta x)^2(1-r)\,k_0(1,t_{n+1})\,(u_M^n)^r\\
&\quad +2\,(\Delta x)^2\sum_{i=1}^{M-1} k_0(x_i,t_{n+1})\left(u_i^{n+1}\right)^r,
\end{aligned}$$

and

$$c_2 = 4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2(1-r)\,k_1(0,t_{n+1})\,(u_0^n)^r + (\Delta x)^2(1-r)\,k_1(1,t_{n+1})\,(u_M^n)^r$$

$$+2\left(\Delta x\right)^{2}\sum_{i=1}^{M-1}k_{1}(x_{i},t_{n+1})\left(u_{i}^{n+1}\right)^{r},$$

we get:

$$\begin{aligned} a_1 u_0^{n+1} - a_2 u_M^{n+1} &= c_1 \\ b_1 u_0^{n+1} - b_2 u_M^{n+1} &= c_2. \end{aligned}$$

Hence, by using the Crammer method, we can then obtain the two unknowns u_0^{n+1} and u_M^{n+1}, and this completes our sought solution.

5.2 The implicit formula

Herein, we aim to approximate the time derivative in problem $(P0)$ by the forward difference quotient, and then use the econd-order centred approximation for the spatial derivative in the same problem. This, actually, would gives:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}-a\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\left(\Delta x\right)^2}=\left(u_i^n\right)^p.$$

In fact, the above formula can be written as:

$$(1-2r)u_i^{n+1}=ru_{i+1}^{n+1}+ru_{i-1}^{n+1}+\Delta t\left(u_i^n\right)^p+u_i^n, \tag{5.4}$$

where $r = a\Delta t/\left(\Delta x\right)^2$. Clearly, this formula is explicit, which means that we still have to determinate the two unknowns u_0^{n+1} and u_M^{n+1}. To this aim, we approximate the integrals reported in the considered problem numerically by trapezoidal rule. The leads to the following equalities:

$$\begin{aligned} &\left(-3-r\left(\Delta x\right)^2k_0(0,t_{n+1})(u_i^n)^{r-1}\right)u_0^{n+1}-\left(\Delta x\right)^2rk_0(1,t_{n+1})(u_M^n)^{r-1}u_M^{n+1}= \\ &\qquad\left(\Delta x\right)^2\left[k_0(0,t_{n+1})\left[(1-r)(u_0^n)^r\right]+k_0(1,t_{n+1})\left[(1-r)(u_M^n)^r\right]\right. \\ &\quad+2\sum_{i=1}^{M-1}k_0(x_i,t_{n+1})\left[(1-r)(u_i^n)^r+r(u_i^n)^{r-1}u_i^{n+1}\right]\Big]+u_2^{n+1}-4u_1^{n+1}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\left(3-\left(\Delta x\right)^2k_1(1,t_{n+1})r(u_M^n)^{r-1}\right)u_M^{n+1}-\left(\left(\Delta x\right)^2k_1(0,t_{n+1})r(u_i^n)^{r-1}u_0^{n+1}\right)u_0^{n+1}= \\ &\qquad\left(\Delta x\right)^2\left[k_1(0,t_{n+1})\left[(1-r)(u_0^n)^r\right]+k_1(1,t_{n+1})\left[(1-r)(u_M^n)^r\right]\right. \\ &\quad+2\sum_{i=1}^{M-1}k_1(x_i,t_{n+1})\left[(1-r)(u_i^n)^r+r(u_i^n)^{r-1}u_i^{n+1}\right]\Big]+4u_{M-1}^{n+1}-u_{M-2}^{n+1}. \end{aligned} \tag{5.6}$$

Combining (5.5),(5.6), with (5.4) yields an $(M+1)\times(M+1)$ linear system of equations. We write the system in the matrix from We write the system in the matrix form:

$$A^{n+1}U^{n+1}=B^{n+1} \tag{5.7}$$

which can be easily solved in terms of its unknowns.

5.3 Numerical illustration

In order to illustrate the theoretical findings gained from this work, we will use the two discrete finite difference method derived in the above two subsections; the explicit and the compact finite difference schemes. In what follow, we will consider the following three problems:

$$\textbf{problem 1 :}\left\{\begin{array}{ll} u_t=au_{xx}+u^2 & 0<x<1, 0<t<T \\ u_x(0,t)=\int_0^1 k_0(x,t)u^r dx & \forall t\in(0,T) \\ u_x(1,t)=\int_0^1 k_1(x,t)u^r dx & \forall t\in(0,T) \\ u(x,0)=A(x+1)^2 & x\in(0,1) \end{array}\right\}$$

$$\textbf{problem 2 :}\left\{\begin{array}{ll} u_t = au_{xx} + u^3 & 0 < x < 1, 0 < t < T \\ u_x(0,t) = \int_0^1 k_0(x,t)u^r dx & \forall t \in (0,T) \\ u_x(1,t) = \int_0^1 k_1(x,t)u^r dx & \forall t \in (0,T) \\ u(x,0) = A(x+1)^2 & x \in (0,1) \end{array}\right\}$$

$$\textbf{problem 3 :}\left\{\begin{array}{ll} u_t = au_{xx} + u^4 & 0 < x < 1, 0 < t < T \\ u_x(0,t) = \int_0^1 k_0(x,t)u^r dx & \forall t \in (0,T) \\ u_x(1,t) = \int_0^1 k_1(x,t)u^r dx & \forall t \in (0,T) \\ u(x,0) = A(x+1)^2 & x \in (0,1) \end{array}\right\},$$

where $k_1(x,t) = \frac{2}{(A^2)*(x+1)^6)}$, $k_2(x,t) = \frac{4}{(A^2)*(x+1)^6)}$ and $A = 12$. Since the analytical/exact solutions of problems 1, 2 and 3 with the associated initial condition are not known, we can only estimate numerically their blow-up solutions. In the next tables, we present the blow-up time as well as the CPU time in the second unit for different values of the space-step using the explicit and compact backward difference schemes. It should be noted that the numerical blow-up time is computed at the first time that $\|u^n\|_\infty \geq 10^6$, and our numerical experiments are performed using MATLAB together with Intel Core i3 with 2.1 GHz.

h	T^* by explicit scheme	CPU time	T^* by compact scheme	CPU time
$1/40$	0.02021781	0.7639	0.02022113	1.316
$1/80$	0.02021781	4.408	0.02022064	27.229
$1/160$	0.02021875	62.120	0.02021635	344.681
$1/320$	0.02021781	307.456	0.02021456	1842.532

Table 1. Comparison between the explicit and compact schemes in obtaining T^* for Problem 1 with $p = 2$

h	T^* by explicit scheme	CPU time	T^* by compact scheme	CPU time
$1/40$	0.00003347	0.0133	0.00031258	0.1915
$1/80$	0.000032854	0.035	0.00030986	1.932
$1/160$	0.00031258	0.501	0.00030758	3.4815
$1/320$	0.00030761	1.567	0.00030732	63.462

Table 2. Comparison between the explicit and compact schemes in obtaining T^* for Problem 2 with $p = 3$

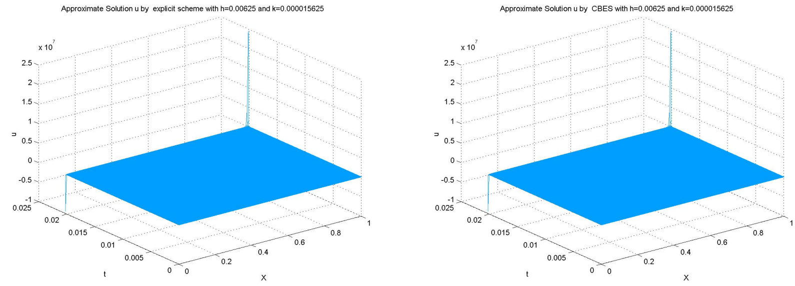
h	T^* by explicit scheme	CPU time	T^* by compact scheme	CPU time
$1/40$	0.0000229157	0.0402	0.00001972346	0.1213
$1/80$	0.0000212346	0.0469	0.00001953125	1.29
$1/160$	0.00001972346	0.0485	0.0000194992	3.315
$1/320$	0.00001953125	0.497	0.0000194931	56.157

Table 3. Comparison between the explicit and compact schemes in obtaining T^* for Problem 3 with $p = 4$

In the same regard, Figures 1, 2 and 3 present respectively the discrete graphs of the numerical solutions of problems 1, 2 and 3 obtained from using explicit and compact schemes, respectively.

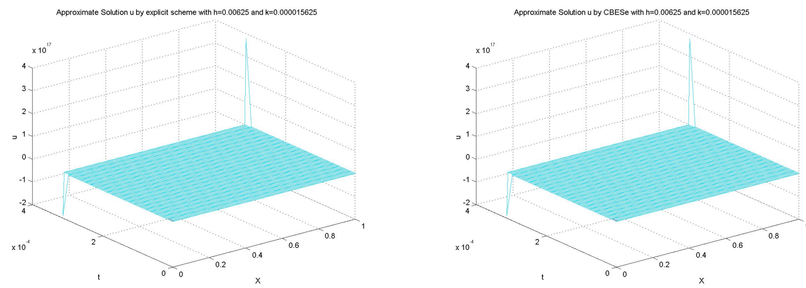
6 Conclusions

In this work, a class of semilinear heat equation with nonlinear nonlocal conditions of second type has been explored and discussed. The existence and uniqueness of weak solution of the



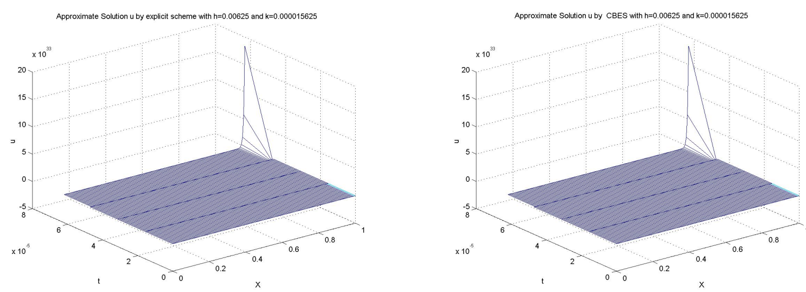
(a) (b)

Figure 1. Numerical solution of problem 1 by (a) explicit and (b) compact schemes.



(a) (b)

Figure 2. Numerical solution of problem 2 by (a) explicit and (b) compact schemes.



(a) (b)

Figure 3. Numerical solution of problem 3 by (a) explicit and (b) compact schemes

presented problem have been investigated in view of the linearisation method. The problem has been furthermore stilled open problem in hight dimension like $n = 2$and $n = 3$.

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