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Graph Based on Linear Inequalities and Uncertain System With Applications

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Abstract. The purpose of this paper is to introduce the notion of (complete) graph based on interval grey numbers as grey (complete) graphs via the concept of greyness of interval grey numbers. To realize the article's aim, we design a grey (complete) graph based on the simple graph (as an underlying graph) and make a relation between grey vertices and grey edges of the grey (complete) graph. The main method in this research is based on value cuts and linear inequalities which are related to grey vertices and grey edges. The result of the research, is constructing some necessary and sufficient conditions in grey vertex (as (non) discrete grey vertices) connectivity of grey complete graph based on interval grey numbers via linear inequalities system. Also, some results were obtained between grey (white-black) vertices and (white-black) grey edges and are established the fundamental conditions for (non) zero white edges. The paper includes implications for the development of simple complete graphs, and for modeling the uncertainty problems by grey vertices, grey edges, and their relations in a grey model as a grey (complete) graph. The new conception of grey complete graphs based on interval grey numbers was broached in this paper for the first time. We find a method that can apply for interval grey numbers in the extension of complete graphs and apply for interval grey numbers in the real world via grey graphs.

1 Introduction

Grey system theory as one of the main subjects for researching uncertainty is one of important systems with partial information known, where today have many important applications. The motivation for introducing grey systems is vagueness in the information contained in a real-world database and this system, applied to describe ambiguity or incompletely in information. Recently some researchers studied this system and investigated its subsystem as the concept of interval grey number. In the point of view of Lin et al. a grey number is a number whose exact value is unknown, but a range within which the value lies is known [11], in the point of view of Yang and John, a grey number is a number with clear upper and lower boundaries, but which has an unknown position within the boundaries [14] and other researchers introduced as an interval grey number such black and white numbers(a number that has neither an upper limit nor lower limit or the upper and the lower limits are all grey numbers is called a black number and if the upper limit is equal to the lower limit, is called a white number) [10]. In addition, graph theory is one of the important tools in modeling to utilize the rich information that lies in complex network structure (complex systems, in general, are made of several interacting elements, and it is rather natural to associate a node to each element and a link to each interaction yielding to a graph) such as food webs, scientific citations, social networks, communication networks, company links in a stock portfolio, the Internet, and the worldwide web [2, 5]. Another application of graph theory is in data structures (the structuring or organizing of data into information so that it's all operations becomes easy, such logical and mathematical model) such as the nonlinear representation of data into memory is possible using graph theory. Arbitrary relationship among data is represented by a graph and its adjacency matrix. Recently, graph theory has combined fuzzy subsets, neutrosophic subsets, and neutro sets and it has many applications in the real world [4, 6, 13]. In this regard, we consider the concept of interval grey numbers and introduce grey (complete) graphs as a weighted graph based on interval grey numbers. Indeed, we are introducing these notations of grey vertices and grey edges based on the concept of interval grey numbers and extend them to a (complete) graph based on the relations between grey vertices and grey edges. The main motivation of this system is obtained from the fuzzy vertex and fuzzy edge but in a different form (interval-valued) fuzzy (complete) graph. There is a fundamental difference between grey numbers and fuzzy subsets, so we found that (complete) graphs based on interval grey numbers are different from (interval-valued) fuzzy (complete) graphs. We try to find some results for equivalence conditions of connected vertices and white edges. The solution of the linear inequalities plays the main role in the relation between grey vertices and grey edges in our work, so we found some algorithms in the concept of graphs based on interval grey numbers.

2 Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Definition 2.1. [3] Let U be an universal set, $\Omega \subseteq \mathbb{R}$ and $a \in U$. A grey number is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. There are several types of grey numbers. A grey number that only takes a finite number or a countable number of potential values is known as discrete and if grey number can potentially take any value within an interval, then it is known as continuous. An interval grey number is a grey number which is expressed mathematically as $(a^{\pm} \in [a^-, a^+], a^- < a^+)$, where $[a^-, a^+] = \{t \mid a^- \leq t \leq a^+\}$, t is an information, $a^{\pm}$ is a grey number, a^- and a^+ are the lower and upper limits of the information. If $a^{\pm} \in [a^-, +\infty)$, is called lower-limit grey number, if $a^{\pm} \in (-\infty, a^+]$, is called upper-limit grey number, if $a^{\pm} \in (-\infty, +\infty)$ is called a black number (it is a number that neither the exact value nor the range is known) and if $a^- = a^+$, it is called a white number (it is an exact value).

Definition 2.2. [7, 8, 9] Let Ω be the universe and $a^{\pm} \in \Omega$ be an interval grey number. Then

(i) the kernel of an interval grey number $a^{\pm}$ is defined as $Ker(a^{\pm}) = \dfrac{a^- + a^+}{2}$.

(ii) the degree of greyness of $a^{\pm}$ is defined as $g^{\circ}(a^{\pm}) = \dfrac{\mu(a^{\pm})}{\mu(\Omega)}$, where $\mu(a^{\pm}) = a^+ - a^-$.

Definition 2.3. [1, 12]

(i) Let $G^* = (V, E)$ be a simple graph, $\sigma^-, \sigma^+ : V \to [0, 1]$ and $\rho^-, \rho^+ : V \times V \to [0, 1]$. Then $G = \big(V, (\sigma^-, \sigma^+), (\rho^-, \rho^+)\big)$ is called an interval-valued fuzzy graph on simple graph $G^* = (V, E)$, if for all $x, y \in V, \sigma^-(x) \leq \sigma^+(x), \mu^-(xy) \leq \sigma^-(x) \wedge \sigma^-(y)$ and $\mu^+(xy) \leq \sigma^+(x) \wedge \sigma^+(y)$.

(ii) A fuzzy graph $G = (V, \sigma, \mu)$ is an algebraic structure of non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V, \mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. It is called σ as fuzzy vertex set and μ as fuzzy edge set of G.

3 Graph based on interval grey number

In this section, the notation of interval grey number-based vertex set or grey vertex and interval grey number-based edge set or grey edges is introduced and the concept of interval grey number-based graph is obtained. From now on, for all $x, y \in \Omega$, we consider $T_{min}(x, y) = \min\{x, y\}$ and $T_{pr}(x, y) = xy$.

Definition 3.1. Let Ω be a universe and $G^{\pm} = (V, E)$ be a simple graph. For all $xy \in E$, define $\mu^-(xy) =$ and $\mu^+(xy) = T_{min}(|Ker(x^{\pm})|, |Ker(y^{\pm})|)$, where

$$\sigma^{\pm} = \{x^{\pm} \in [\sigma^-(x), \sigma^+(x)] \mid x \in V\}, \mu^{\pm} = \{(xy)^{\pm} \in [\mu^-(xy), \mu^+(xy)] \mid xy \in E\}$$

and for all $x, y \in V$ $x^{\pm}, y^{\pm}$ and $(xy)^{\pm}$ are interval grey numbers. An algebraic structure $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is called an interval grey number-based graph on $G^{\pm}$(for simplification it is called by grey graph), if for all $xy \in E$, we have $\mu^-(xy) \leq \mu^+(xy)$ (if $\mu^-(xy) = \mu^+(xy) = 0$, we say that dose not exist an edge between vertices $x^{\pm}, y^{\pm}$). It is called $\sigma^{\pm}$ as interval grey number-based vertices set or grey vertices and $\mu^{\pm}$ as interval grey number-based edges set or grey edges of G.

From now on, for any $t \in \mathbb{R}$, we denote $[t, t] = \{t\}$ and identify it by $\{t\} \sim t$.

In the following, we present some examples interval grey number-based graph, fuzzy graph, interval-valued fuzzy graph and conclude that interval grey number-based graphs are different to fuzzy graphs and interval-valued fuzzy graphs. We marked symbols of grey vertexes and grey edges only in the Figure 10, and all figures in throughout of paper are similar, but we do not mark them, for simplification.

Example 3.2. (i) Let $\Omega = [0, 1000]$. Consider the cycle graph C_4. Then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is an interval grey number-based graph on C_5 in Figure 1. Computations show that, $g^{\circ}(a^{\pm}) = \frac{6}{1000}, g^{\circ}(b^{\pm}) = \frac{5}{1000}, |Ker(a^{\pm})| = \frac{6}{2}, |Ker(b^{\pm})| = \frac{9}{2}$, so $\mu^-(ab) = T_{min}(g^{\circ}(a^{\pm}), g^{\circ}(b^{\pm})) = T_{min}(\frac{6}{1000}, \frac{5}{1000}) = \frac{5}{1000}$, $\mu^+(ab) = T_{min}(|Ker(a^{\pm})|, |Ker(b^{\pm})|) = T_{min}(3, \frac{9}{2}) = 3$ and so $(ab)^{\pm} \in [\mu^-(ab), \mu^+(ab)] = [\frac{5}{1000}, 3]$. In a similar way, other grey vertices and grey edges are computed.

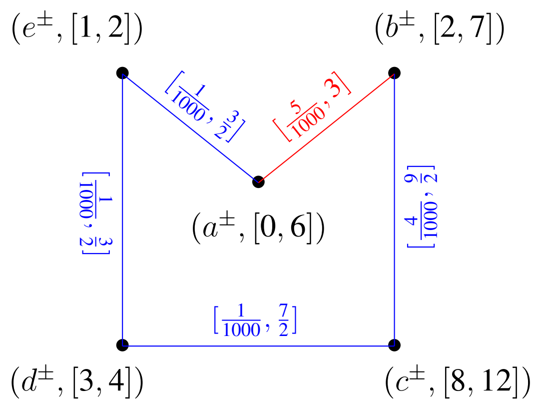

Figure 1. Interval grey number-based graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_5

(ii) Let $\Omega = [0, 1]$ and consider the cycle graph C_4. Then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a fuzzy graph, on C_4 as shown in Figure 2, while is not an interval grey number-based graph on C_4, since

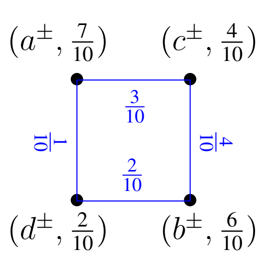

Figure 2. Fuzzy graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_4

$[\mu^-(ab), \mu^+(ab)] = [\mu^-(cb), \mu^+(cb)] = [0, 0.4], [\mu^-(ad), \mu^+(ad)] = [0, 0.2], [\mu^-(db), \mu^+(db)] = [0, 0.2]$.

(iii) Let $\Omega = [-\frac{3}{2}, \frac{1}{3}]$. Consider the cycle graph C_4, then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is an interval grey number-based graph on graph C_4 as depicted in Figure 3, while is not a interval-valued fuzzy graph, since $\mu^+(bc) > T_{min}(\sigma^+(b), \sigma^+(c))$.

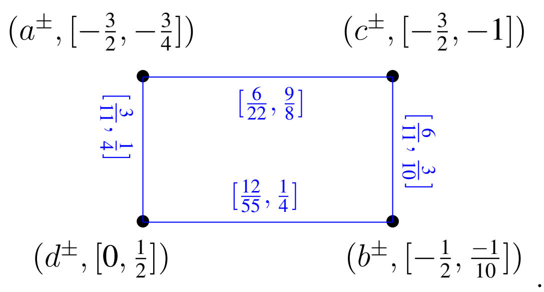

Figure 3. Interval grey number-based graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_4

(iv) Let $\Omega = [0, 1]$. Consider the cycle graph C_3, then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is an interval-valued fuzzy graph on graph C_3 in Figure 4, while it is not an interval grey number-based graph, since $[\mu^-(ab), \mu^+(ab)] = [\frac{3}{10}, \frac{11}{20}], [\mu^-(ac), \mu^+(ac)] = [\frac{9}{22}, \frac{13}{44}]$ and $[\mu^-(cb), \mu^+(cb)] = [\frac{9}{22}, \frac{13}{44}]$.

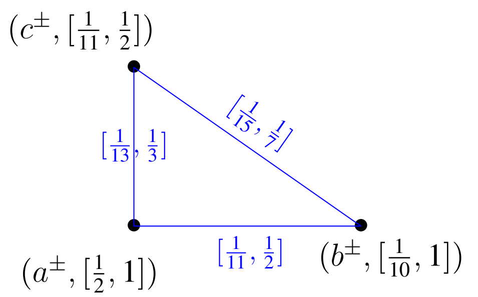

Figure 4. Interval-valued fuzzy graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_3

3.1 Discrete grey vertices in grey graph

In this subsection, we proved some results to obtain the discrete grey vertices in grey graphs.

Theorem 3.3. *Let Ω be a universe, $\mu(\Omega) < 2, G^{\pm} = (V, E)$ be a simple graph and $xy \in E$. Then in $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$:*

(i) *If $x^{\pm} \subseteq y^{\pm}$ and $x^{\pm}$ is 2-polar, then $(xy)^{\pm} \notin \mu^{\pm}$.*

(ii) *If $T_{min}(|x^-|, x^+) = |x^-|$ and $x^- < 0 < x^+ < y^- < \dfrac{y^+}{3}$, then $(xy)^{\pm} \notin \mu^{\pm}$.*

(iii) *If $y^- < x^- < y^+ < 0 < x^+$ and $|x^-| < x^+ < |y^-|$ then $(xy)^{\pm} \notin \mu^{\pm}$.*

(iv) *If $x^+ < -x^-, y^+ < -y^-$, and $x^- < y^- < 0 < x^+ < y^+$ then $(xy)^{\pm} \notin \mu^{\pm}$.*

Proof. (i) Let $xy \in E$. Since $x^{\pm} \subseteq y^{\pm}$ and $x^- < 0 < x^+$, we get that $y^- < x^- < 0 < x^+ < y^+$. It follows that $|Ker(x^{\pm})| = \dfrac{|x^- + x^+|}{2} \leq \dfrac{|x^+| + |x^-|}{2} = \dfrac{(x^+ - x^-)}{2}$, so $\mu(\Omega) < 2$ implies that $|Ker(x^{\pm})| < \dfrac{(x^+ - x^-)}{\mu(\Omega)} = g^{\circ}(x^{\pm})$. In a similar way, one can see that $|Ker(y^{\pm})| < g^{\circ}(y^{\pm})$, hence $T_{min}(|Ker(x^{\pm})|, |Ker(y^{\pm})|) < T_{min}(g^{\circ}(x^{\pm}), g^{\circ}(y^{\pm}))$ and so $(xy)^{\pm} \notin \mu^{\pm}$.

(ii) Let $xy \in E$. Then $T_{min}(|x^-|, x^+) = |x^-|$ implies that $|x^+ + x^-| = x^+ + x^-$. Since $x^- < 0$, we get that $-2x^- > 0$ and so $\dfrac{|x^+ - x^-|}{2} > \dfrac{(x^+ + x^-)}{2}$. Now $\mu(\Omega) < 2$ concludes that $\dfrac{(x^+ - x^-)}{\mu(\Omega)} > \dfrac{(x^+ - x^-)}{2} > \dfrac{(x^+ + x^-)}{2} = \dfrac{|x^+ + x^-|}{2}$ and so $g^{\circ}(x^{\pm}) > |Ker(x^{\pm})|$. In addition, because $x^- < 0 < x^+ < y^-$, we have $x^- + y^+ < y^- + y^+$ and so $x^- + x^+ < y^- + y^+$. It follows that $\dfrac{x^- + x^+}{2} < \dfrac{y^- + y^+}{2}$ and so $\mu^+(xy) = |Ker(x^{\pm})|$. Moreover, since $y^- < \dfrac{y^+}{3}$ and

$x^+ < y^-$, we get that $y^- < \dfrac{y^+ + y^-}{2}$ and $\dfrac{x^+ + x^-}{2} < y^-$. It concludes that $\dfrac{x^+ + x^-}{2} < y^- < \dfrac{y^+ - y^-}{\mu(\Omega)}$ and so $|Ker(x^\pm)| < g^\circ(y^\pm)$. Thus $|Ker(x^\pm)| < g^\circ(y^\pm)$ and $|Ker(x^\pm)| < g^\circ(x^\pm)$ imply $\mu^+(xy) = |Ker(x^\pm)| < T_{min}(g^\circ(x^\pm), g^\circ(y^\pm))$. Therefore, $(xy)^\pm \notin \mu^\pm$.

(iii) Let $xy \in E$. Then

$$|Ker(y^\pm)| = |\frac{y^+ + y^-}{2}| \geq |\frac{|y^-| - |y^+|}{2}| = \frac{|y^-| - |y^+|}{2} > \frac{|x^+| - |x^-|}{2} = \frac{x^+ + x^-}{2} = |Ker(x^\pm)|$$

and so $|Ker(x^\pm)| < |Ker(y^\pm)|$. In addition, $|Ker(x^\pm)| = Ker(x^\pm) = \dfrac{x^+ + x^-}{2} < \dfrac{|y^-| + y^+}{2} = \dfrac{y^+ - y^-}{2}$ implies that $|Ker(x^\pm)| < g^\circ(y^\pm)$. Now, since $|Ker(x^\pm)| = \dfrac{|x^+ + x^-|}{2} \leq \dfrac{|x^+| + |x^-|}{2} = \dfrac{x^+ + x^-}{2} = g^\circ(x^\pm)$, we get that $|Ker(x^\pm)| \leq g^\circ(x^\pm)$. It follows that

$|Ker(x^\pm)| < T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) = \mu^-(xy)$ and $\mu^+(xy) = T_{min}(|Ker(x^\pm)|, |Ker(y^\pm)|) = |Ker(x^\pm)|$. Hence $\mu^+(xy) < \mu^-(xy)$ and so $(xy)^\pm \notin \mu^\pm$.

(iv) By our assumptuion, we get that $(y^- - x^-) > 0, (x^+ - y^-) > 0$ and $(y^+ + x^+) > 0$. Since $(x^- + x^+) < 0$ and $x^+ > 0, x^+ < y^+$, we get that $|x^-| < x^+ < y^+$ and so $(y^+ + x^-) > 0$. In addition, $\mu(\Omega) < 2$ conclude that

$$g^\circ(x^\pm) - |Ker(y^\pm)| = \frac{(x^+ - x^-)}{\mu(\Omega)} - |\frac{y^+ + y^-}{2}| > \frac{(x^+ - x^-)}{2} + \frac{y^+ + y^-}{2}$$
$$= \frac{(x^+ + y^+)}{2} + \frac{y^- - x^-}{2} > 0$$

so $g^\circ(x^\pm) > |Ker(y^\pm)|$. In a similar way $g^\circ(y^\pm) > |Ker(x^\pm)|$ and so $\mu^- > \mu^+$. □

Theorem 3.4. *Let Ω be a universe, $\mu(\Omega) \leq 1, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^\pm$ and $y^\pm$ are 2-polar, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $x^- < 0 < x^+$, we get that $x^+ - x^-| > |x^- + x^-|$. Using $\mu(\Omega) \leq 1$ we obtain $g^\circ(x^\pm) = \dfrac{x^+ - x^-}{\mu(\Omega)} \geq (x^+ - x^-) > |x^- + x^-| > |\dfrac{x^- + x^-}{2}|$ and so $g^\circ(x^\pm) > |Ker(x^\pm)|$. In a similar way $g^\circ(y^\pm) > |Ker(y^\pm)|$ is obtained and so $\mu^-(xy) > \mu^+(xy)$. Therefore, $(xy)^\pm \notin \mu^\pm$. □

Theorem 3.5. *Let Ω be a universe, $k_1, k_2 \in (-\infty, 1), \mu(\Omega) \leq |\dfrac{2k_1 - 2}{k_1 + 1}|, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^-, y^- < 0, x^+ = k_1 x^-, y^+ = k_2 y^-$ and $\max\{\dfrac{k_1 - 1}{k_2 - 1}, |\dfrac{k_1 + 1}{k_2 + 1}|\} < \dfrac{y^-}{x^-}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- < 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1, k_2 < 1$ and $|Ker(x^\pm)| = -|\dfrac{k_1 + 1}{2}|x^-$, and $|Ker(y^\pm)| = -|\dfrac{k_2 + 1}{2}|y^-$. It follows that $|Ker(x^\pm)| < |Ker(y^\pm)|$ if and only if $|\dfrac{k_1 + 1}{k_2 + 1}| < \dfrac{y^-}{x^-}$ and $g^\circ(x^\pm) < g^\circ(y^\pm)$ if and only if $\dfrac{k_1 - 1}{k_2 - 1} < \dfrac{y^-}{x^-}$. In addition $g^\circ(x^\pm) > |Ker(x^\pm)|$ if and only if $\dfrac{k_1 - 1}{\mu(\Omega)} < \dfrac{k_1 + 1}{2}$ if and only if $\mu(\Omega) < \dfrac{2k_1 - 2}{k_1 + 1}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) < g^\circ(y^\pm)$ and $g^\circ(x^\pm) > |Ker(x^\pm)|$ implies that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.6. *Let Ω be a universe, $k_1, k_2 \in (1, +\infty), \mu(\Omega) < \dfrac{2k_1 - 2}{k_1 + 1}, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^-, y^- > 0, x^+ = k_1 x^-, y^+ = k_2 y^-$ and $\max\{\dfrac{k_1 - 1}{k_2 - 1}, |\dfrac{k_1 + 1}{k_2 + 1}|\} < \dfrac{y^-}{x^-}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- > 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1, k_2 > 1$ and $|Ker(x^\pm)| = \dfrac{k_1+1}{2}x^-$, and $|Ker(y^\pm)| = \dfrac{k_2+1}{2}y^-$. It follows that $|Ker(x^\pm)| < |Ker(y^\pm)|$ if and if only if $|\dfrac{k_1+1}{k_2+1}| < \dfrac{y^-}{x^-}$ and $g^\circ(x^\pm) < g^\circ(y^\pm)$ if and if only if $\dfrac{k_1-1}{k_2-1} < \dfrac{y^-}{x^-}$. In addition $g^\circ(x^\pm) > |Ker(x^\pm)|$ if and only if $\dfrac{k_1-1}{\mu(\Omega)} > \dfrac{k_1+1}{2}$ if and only if $\mu(\Omega) < \dfrac{2k_1-2}{k_1+1}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) < g^\circ(y^\pm)$ and $g^\circ(x^\pm) > |Ker(x^\pm)|$ implies that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.7. *Let Ω be a universe, $-1 \neq k_1 \in (-\infty, 1), k_2 \in (1, +\infty), \mu(\Omega) < \dfrac{-2k_1+2}{|k_1+1|}, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^- < 0, y^- > 0, x^+ = k_1x^-, y^+ = k_2y^-$ and $\max\{\dfrac{-k_1+1}{k_2-1}, |\dfrac{k_1+1}{k_2+1}|\} < \dfrac{y^-}{(-x^-)}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^- < 0, y^- > 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1 < 1, k_2 > 1$ and $|Ker(x^\pm)| = \dfrac{|k_1+1|}{2}(-x^-)$, and $|Ker(y^\pm)| = \dfrac{k_2+1}{2}y^-$. It follows that $|Ker(x^\pm)| < |Ker(y^\pm)|$ if and if only if $|\dfrac{k_1+1}{k_2+1}| < \dfrac{y^-}{(-x^-)}$ and $g^\circ(x^\pm) < g^\circ(y^\pm)$ if and only if $\dfrac{-k_1+1}{k_2-1} < \dfrac{y^-}{(-x^-)}$. In addition $g^\circ(x^\pm) > |Ker(x^\pm)|$ if and only if $\dfrac{k_1-1}{\mu(\Omega)}(x^-) > \dfrac{|k_1+1|}{2}(-x^-)$ if and only if $\mu(\Omega) < \dfrac{-2k_1+2}{|k_1+1|}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) < g^\circ(y^\pm)$ and $g^\circ(x^\pm) > |Ker(x^\pm)|$ imply that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.8. *Let Ω be a universe, $k_1, k_2 \in (-\infty, 1) \setminus \{-1\}, \mu(\Omega) < \dfrac{(-2k_2+2)y^-}{(|k_1+1|)x^-}, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^-, y^- < 0, x^+ = k_1x^-, y^+ = k_2y^-$ and $\dfrac{|k_1+1|}{|k_2+1|} < \dfrac{y^-}{x^-} < \dfrac{k_1-1}{k_2-1}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- < 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1, k_2 < 1$ and $|Ker(x^\pm)| = \dfrac{|k_1+1|}{2}(-x^-)$, and $|Ker(y^\pm)| = \dfrac{|k_2+1|}{2}(-y^-)$. It follows that $|Ker(x^\pm)| < |Ker(y^\pm)|$ if and if only if $\dfrac{|k_1+1|}{|k_2+1|} < \dfrac{y^-}{x^-}$ and $g^\circ(x^\pm) > g^\circ(y^\pm)$ if and if only if $\dfrac{k_1-1}{k_2-1} > \dfrac{y^-}{x^-}$. In addition $g^\circ(y^\pm) > |Ker(x^\pm)|$ if and only if $\dfrac{k_2-1}{\mu(\Omega)}(y^-) > \dfrac{|k_1+1|}{2}(-x^-)$ if and only if $\mu(\Omega) < \dfrac{(-2k_2+2)y^-}{(|k_1+1|)x^-}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) > g^\circ(y^\pm)$ and $g^\circ(y^\pm) > |Ker(x^\pm)|$ imply that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.9. *Let Ω be a universe, $k_2 < k_1 \in (1, \infty), \mu(\Omega) < \dfrac{(2k_1-2)y^-}{(k_1+1)x^-}, G^\pm = (V, E)$ be a simple graph and $xy \in E$. If $x^-, y^- > 0, x^+ = k_1x^-, y^+ = k_2y^-$ and $\dfrac{k_1+1}{k_2+1} < \dfrac{y^-}{x^-} < \dfrac{k_1-1}{k_2-1}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- > 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1, k_2 > 1$ and $|Ker(x^\pm)| = \dfrac{k_1+1}{2}(x^-)$, and $|Ker(y^\pm)| = \dfrac{k_2+1}{2}(y^-)$. It follows that $|Ker(x^\pm)| < |Ker(y^\pm)|$ if and if only if $\dfrac{k_1+1}{k_2+1} < \dfrac{y^-}{x^-}$ and $g^\circ(x^\pm) > g^\circ(y^\pm)$ if and if only if $\dfrac{k_1-1}{k_2-1} > \dfrac{y^-}{x^-}$. In addition $g^\circ(y^\pm) > |Ker(x^\pm)|$ if and only if $\dfrac{k_1-1}{\mu(\Omega)}(y^-) > \dfrac{k_1+1}{2}(x^-)$ if and only if $\mu(\Omega) <$

$\dfrac{(2k_1-2)y^-}{(k_1+1)x^-}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) > g^\circ(y^\pm)$ and $g^\circ(y^\pm) > |Ker(x^\pm)|$ imply that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.10. *Let Ω be a universe, $k_1 < k_2 \in (1,\infty), \mu(\Omega) < \dfrac{(2k_1-2)x^-}{(k_2+1)y^-}, G^\pm = (V,E)$ be a simple graph and $xy \in E$. If $x^-, y^- > 0, x^+ = k_1x^-, y^+ = k_2y^-$ and $\dfrac{k_1-1}{k_2-1} < \dfrac{y^-}{x^-} < \dfrac{k_1+1}{k_2+1}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- > 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1, k_2 > 1$ and $|Ker(x^\pm)| = \dfrac{k_1+1}{2}(x^-)$, and $|Ker(y^\pm)| = \dfrac{k_2+1}{2}(y^-)$. It follows that $|Ker(y^\pm)| < |Ker(x^\pm)|$ if and if only if $\dfrac{k_1+1}{k_2+1} > \dfrac{y^-}{x^-}$ and $g^\circ(x^\pm) < g^\circ(y^\pm)$ if and if only if $\dfrac{k_1-1}{k_2-1} < \dfrac{y^-}{x^-}$. In addition $g^\circ(x^\pm) > |Ker(y^\pm)|$ if and only if $\mu(\Omega) < \dfrac{(2k_1-2)x^-}{(k_2+1)y^-}$. Thus $|Ker(x^\pm)| < |Ker(y^\pm)|, g^\circ(x^\pm) > g^\circ(y^\pm)$ and $g^\circ(y^\pm) > |Ker(x^\pm)|$ imply that $\mu^-(xy) > \mu^+(xy)$. □

Theorem 3.11. *Let Ω be a universe, $k_1 \in [\frac{1}{2}, 1), k_2 \in (1,\infty), \mu(\Omega) < \dfrac{(2k_1-2)x^-}{(k_2+1)y^-}, G^\pm = (V,E)$ be a simple graph and $xy \in E$. If $x^- < 0, y^- > 0, x^+ = k_1x^-, y^+ = k_2y^-$ and $\dfrac{1-k_1}{k_2-1} < \dfrac{y^-}{|x^-|} < \dfrac{k_1+1}{k_2+1}$, then $(xy)^\pm \notin \mu^\pm$.*

Proof. Let $xy \in E$. Since $\mu(\Omega) > 0, x^-, y^- > 0$ and $g^\circ(x^\pm), g^\circ(y^\pm) \geq 0$, we get that $k_1 < 1, k_2 > 1$ and $|Ker(x^\pm)| = \dfrac{k_1+1}{2}(|x^-|)$, and $|Ker(y^\pm)| = \dfrac{k_2+1}{2}(y^-)$. Since $\dfrac{1-k_1}{k_2-1} < \dfrac{y^-}{|x^-|}$, we get $g^\circ(x^\pm) < g^\circ(y^\pm)$ and since $\dfrac{y^-}{|x^-|} < \dfrac{k_1+1}{k_2+1}$, we obtain that $|Ker(y^\pm)| < |Ker(x^\pm)|$. Now, $\mu(\Omega) < \dfrac{(2k_1-2)x^-}{(k_2+1)y^-}$, concludes that $g^\circ(x^\pm) > |Ker(y^\pm)|$ and so $(xy)^\pm \notin \mu^\pm$. □

Corollary 3.12. *Let Ω be a universe, $\mu(\Omega) = 2, G^\pm = (V,E)$ be a simple graph and $xy \in E$, $x^- < 0$ and $x^+ = y^+ - (x^- + y^-)$. If $(xy)^\pm$ is not a white edge, then $(xy)^\pm = \emptyset$.*

Proof. Since $\mu(\Omega) = 2$ and $x^- < 0$, we get that $g^\circ(x^\pm) > g^\circ(y^\pm)$. If $|Ker(x^\pm)| > |Ker(y^\pm)|$, then by replacing $x^+ = y^+ - (x^- + y^-)$, we get that $|y^+ + y^-| < |y^+ - y^-| = (y^+ - y^-)$. Suppose $(xy)^\pm$ is not a white number and $(xy)^\pm \neq \emptyset$, then $g^\circ(y^\pm) < |Ker(y^\pm)|$ and so $\dfrac{(y^+ - y^-)}{2} < \dfrac{|y^+ + y^-|}{2}$, which is a contradiction. But if $|Ker(x^\pm)| < |Ker(y^\pm)|$, then by replacing $x^+ = y^+ - (x^- + y^-)$, we get that $|y^+ + y^-| > (y^+ - y^-)$. Assume that $(xy)^\pm$ is not a white number and $(xy)^\pm \neq \emptyset$, then $g^\circ(y^\pm) < |Ker(x^\pm)|$. It follows that $|y^+ - y^-| > (y^+ - y^-)$, which is a contradiction. □

Example 3.13. Consider $\Omega = [-1, 1]$ and the interval grey number-based graph $G = (\Omega, \sigma^\pm, \mu^\pm)$ on graph $G^\pm$ in Figure 5. Then $G = (\Omega, \sigma^\pm, \mu^\pm)$ has white edge and discrete grey vertices.

4 Complete Grey Graph

In this section, introduced the notation of grey number-based vertex set or grey vertex and grey number-based edge set or grey edges and obtained the concept of grey number-based graph.

From now on, for all $x, y \in [0, 1]$ we consider $T_{min}(x, y) = \min\{x, y\}$.

Definition 4.1. Let $G = (\Omega, \sigma^\pm, \mu^\pm)$ be a grey graph on $G^\pm$, $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Omega$. Define $G^{(\alpha,\beta)} = (\Omega, (\sigma^\pm)^\alpha, (\mu^\pm)^\beta)$, where $(\sigma^\pm)^\alpha = \{x \in V \mid \sigma^-(x) \geq \alpha_1, \sigma^+(x) \geq \alpha_2\}$ and $(\mu^\pm)^\beta = \{xy \in E \mid \mu^-(xy) \geq \beta_1, \mu^+(xy) \geq \beta_2\}$.

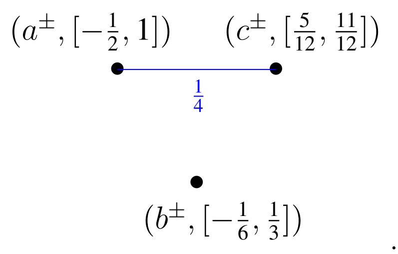

Figure 5. Non-connected Interval grey number-based graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on $G^{\pm}$

Example 4.2. Let $\Omega = [0, 1000]$. Consider the path graph P_5. Then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey number-based graph on P_5 in Figure 6, where $V = \{e, a, b, c, d\}$. If $\alpha = (0, 2), \beta = (\dfrac{1}{1000}, \dfrac{3}{2})$, then $(\sigma^{\pm})^{\alpha} = V, (\mu^{\pm})^{\beta} = V \times V$ and so $G^{(\alpha,\beta)} \cong K_5$ is a complete graph.

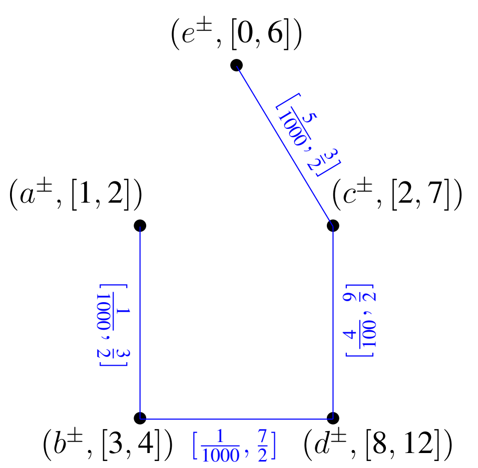

Figure 6. Path Grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on P_5

Let Ω be a universe and $x^{\pm} \subseteq \Omega$. Then $x^{\pm}$ is called 1-polar, if $T_{pr}(x^-, x^+) > 0$, where is showed in Figures 7 and 8 as $positive$-polar and $negative$-polar, respectively. Also $x^{\pm}$ is called

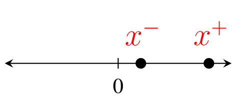

Figure 7. Position of grey number $x^{\pm}$ on real line($positive$-polar).

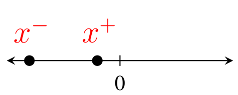

Figure 8. Position of grey number $x^{\pm}$ on real line($negative$-polar).

2-polar, if $T_{pr}(x^-, x^+) < 0$, where is showed in Figure 9.

Definition 4.3. Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm} = (V, E)$. Then G is called complete grey graph, if for all $x, y \in V$, we have $\mu^-(xy) \leq \mu^+(xy)$.

Example 4.4. Let $\Omega = [0, 1000]$. Consider the cycle graph C_4. Then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a complete grey number-based graph on C_4 in Figure 10.

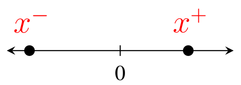

Figure 9. Position of grey number $x^{\pm}$ on real line(2-polar).

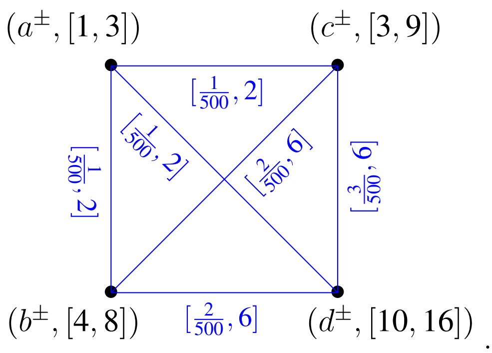

Figure 10. Complete Grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on K_4

Theorem 4.5. *Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$. Then G is a complete grey graph, if and only if one of the following conditions hold:*

(i) $(x^+ \leq x^- + (y^+ - y^-), -(|y^+ + y^-| + x^-) \leq x^+ \leq |y^+ + y^-| - x^-$ *and* $\mu(\Omega) \geq \dfrac{2\mu(x^{\pm})}{Ker(x^{\pm})}$,

(ii) $(y^+ \geq y^- + (x^+ - x^-), -(|x^+ + x^-| + y^-) \leq y^+ \leq |x^+ + x^-| - y^-$ *and* $\mu(\Omega) \geq \dfrac{2\mu(x^{\pm})}{Ker(y^{\pm})}$,

(iii) $(y^+ \leq y^- + (x^+ - x^-), -(|y^+ + y^-| + y^-) \leq x^+ \leq |y^+ + y^-| - x^-$ *and* $\mu(\Omega) \geq \dfrac{2\mu(y^{\pm})}{Ker(x^{\pm})}$,

(iv) $(y^+ \leq y^- + (x^+ - x^-), -(|x^+ + x^-| + y^-) \leq y^+ \leq |x^+ + x^-| - y^-$ *and* $\mu(\Omega) \geq \dfrac{2\mu(y^{\pm})}{Ker(y^{\pm})}$.

Proof. Let $xy \in E$. Then

(i) $\big(g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm}), |Ker(x^{\pm})| \leq |Ker(y^{\pm})|$ and $g^{\circ}(x^{\pm}) \leq |Ker(x^{\pm})|\big)$ if and only if $\big(x^+ - x^- \leq y^+ - y^-, -|y^+ + y^-| \leq x^+ + x^- \leq |y^+ + y^-|$ and $\mu(\Omega)|x^+ + x^-| \geq 2(x^+ - x^-)$.

$(ii), (iii)$ and (iv) are similar to (i). □

Theorem 4.6. *Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$ and $x^{\pm}, y^{\pm}$ be arbitrary grey vertices.*

(i) *If $x^{\pm} \subseteq y^{\pm}$ are positive-polar, then G is a complete grey graph, if and only if $x^+ \leq T_{min}(x^- + y^+ - y^-, y^+ + y^- - x^-)$ and $\mu(\Omega) \geq \dfrac{2(x^+ - x^-)}{x^+ + x^-}$.*

(ii) *If $x^{\pm}$ is negative-polar and $y^{\pm}$ is positive-polar, then G is a complete grey graph, if and only if $-(y^+ + y^- + x^-) \leq x^+ \leq y^+ - y^- + x^-$ and $\mu(\Omega) \geq \dfrac{2(x^- - x^+)}{x^+ + x^-}$.*

(iii) *If $x^{\pm}, y^{\pm}$ are negative-polar and $y^{\pm} \subseteq x^{\pm}$, where $y^- < -y^+$, then G is a complete grey graph, if and only if $y^+ + y^- - x^- \leq x^+ \leq y^+ - y^- + x^-$ and $\mu(\Omega) \geq \dfrac{2(x^- - x^+)}{x^+ + x^-}$.*

(iv) *If $x^{\pm}$ is negative-polar and $y^{\pm}$ is positive-polar, then G is a complete grey graph, if and only if $y^+ - y^- + x^- \leq x^+ \leq y^+ + y^- - x^-$ and $\mu(\Omega) \geq \dfrac{2(y^- - y^+)}{x^+ + x^-}$.*

(v) *If $x^{\pm} \subseteq y^{\pm}$ are positive-polar, then G is a complete grey graph, if and only if $y^+ - y^- + x^- \leq x^+ \leq y^+ + y^- - x^-$ and $\mu(\Omega) \geq \dfrac{2(y^+ - y^-)}{x^+ + x^-}$.*

(vi) *If $x^{\pm}, y^{\pm}$ are negative-polar and $y^{\pm} \subseteq x^{\pm}$, where $y^- < -y^+$, then G is a complete grey graph, if and only if $y^+ - y^- + x^- \le x^+ \le -(y^+ + y^- + x^-)$ and $\mu(\Omega) \ge \dfrac{2(y^- - y^+)}{x^+ + x^-}$.*

Proof. Let $xy \in E$. Then

(i) Since $0 < x^- < x^+$, $0 < y^- < y^+$ and $\mu(\Omega) \ge \dfrac{2(x^- - x^+)}{x^+ + x^-}$, we get that $|Ker(x^{\pm})| \ge g^{\circ}(x^{\pm})$. In addition, $x^+ \le x^- + y^+ - y^-$ and $x^+ \le y^+ + y^- - x^-$ imply that $g^{\circ}(x^{\pm}) \le g^{\circ}(y^{\pm})$ and $|Ker(x^{\pm})| \le |Ker(y^{\pm})|$. Hence by Theorem 4.5, G is a complete grey graph.The converse is similar to.

(ii) Since $0 < x^- < x^+ < y^- < y^+$ and $\mu(\Omega) \ge \dfrac{2(x^+ - x^-)}{x^+ + x^-}$, we get that $|Ker(x^{\pm})| \ge g^{\circ}(x^{\pm})$. In addition, $-|+y^+ + y^-| - x^- \le x^+$ and $x^+ \le y^+ - y^- + x^-$ imply that $|Ker(x^{\pm})| \le |Ker(y^{\pm})|$ and $g^{\circ}(x^{\pm}) \le g^{\circ}(y^{\pm})$. Hence by Theorem 4.5, G is a complete grey graph. The converse is similar to.

$(iii), (iv), (v)$ and (vi) are similar to (i) and (ii). □

Corollary 4.7. *Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$ and $x^{\pm}, y^{\pm}$ be arbitrary grey vertices.*

(i) *If $x^{\pm} \subseteq y^{\pm}$ are positive-polar, then G is a complete grey graph, if and only if $y^+ \le T_{min}(y^- + x^+ - x^-, x^+ + x^- - y^-)$ and $\mu(\Omega) \ge \dfrac{2(y^+ - y^-)}{y^+ + y^-}$.*

(ii) *If $x^{\pm}$ is negative-polar and $y^{\pm}$ is positive-polar, then G is a complete grey graph, if and only if $x^+ - x^- + y^- \le y^+ \le -(x^+ + x^- + y^-)$ and $\mu(\Omega) \ge \dfrac{2(y^+ - y^-)}{y^+ + y^-}$.*

(iii) *If $x^{\pm}, y^{\pm}$ are negative-polar and $y^{\pm} \subseteq x^{\pm}$, where $y^- < -y^+$, then G is a complete grey graph, if and only if $y^- + x^+ - x^- \le y^+ \le -(x^+ + x^- + y^-)$ and $\mu(\Omega) \ge \dfrac{2(x^- - x^+)}{y^+ + y^-}$.*

(iv) *If $x^{\pm}, y^{\pm}$ are negative-polar and $y^{\pm} \subseteq x^{\pm}$, where $y^- < -y^+$, then G is a complete grey graph, if and only if $x^+ + x^- - y^- \le y^+ \le y^- + x^+ - x^-$ and $\mu(\Omega) \ge \dfrac{2(y^- - y^+)}{y^+ + y^-}$.*

(v) *If $x^{\pm} \subseteq y^{\pm}$ are positive-polar, then G is a complete grey graph, if and only if $x^+ - x^- + y^- \le y^+ \le x^+ + x^- - y^-$ and $\mu(\Omega) \ge \dfrac{2(x^+ - x^-)}{y^+ + y^-}$.*

(vi) *If $x^{\pm}$ is negative-polar and $y^{\pm}$ is positive-polar, then G is a complete grey graph, if and only if $x^+ - x^- + y^- \le y^+ \le -(x^+ + x^- + y^-)$ and $\mu(\Omega) \ge \dfrac{2(x^+ - x^+)}{y^+ + y^-}$.*

Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$ and $x^{\pm}, y^{\pm} \in \sigma^{\pm}$. Define $\alpha_1^w = \bigwedge\limits_{x^{\pm} \in \sigma^{\pm}} \sigma^-(x), \alpha_2^w = \bigwedge\limits_{x^{\pm} \in \sigma^{\pm}} \sigma^+(x), \beta_1^w = \bigwedge\limits_{(xy)^{\pm} \in \mu^{\pm}} \mu^-(xy)$ and $\beta_2^w = \bigwedge\limits_{(xy)^{\pm} \in \mu^{\pm}} \mu^+(xy)$. Set $G^{(\alpha^w, \beta^w)} = (\Omega, (\sigma^{\pm})^{\alpha^w}, (\mu^{\pm})^{\beta^w})$, then have te following result.

Theorem 4.8. *Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$. Then $G^{(\alpha^w, \beta^w)} = (\Omega, (\sigma^{\pm})^{\alpha^w}, (\mu^{\pm})^{\beta^w})$ is a complete graph.*

Proof. Immediate by definition. □

Definition 4.9. Let $m, n \in \mathbb{N}, G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$ and $x^{\pm}, y^{\pm} \in \sigma^{\pm}$. Define $\alpha_1 = \bigwedge\limits_{x^{\pm} \in \sigma^{\pm}} (g^{\circ}(x^{\pm})), \alpha_2 = \bigwedge\limits_{x^{\pm} \in \sigma^{\pm}} (|Ker(x)^{\pm}|), \beta_1 = \bigwedge\limits_{(xy)^{\pm} \in \mu^{\pm}} (g^{\circ}((xy)^{\pm}))$ and $\beta_2 = \bigwedge\limits_{(xy)^{\pm} \in \mu^{\pm}} (|Ker(xy)^{\pm}|)$. For any ascending or descending map $\varphi : \Omega \to \mathbb{R}$, consider $G^{(\varphi(\alpha), \varphi(\beta))} = (\Omega, (\sigma^{\pm})^{\varphi(\alpha)}, (\mu^{\pm})^{\varphi(\beta)})$, where $(\sigma^{\pm})^{\varphi(\alpha)} = \{x \in V \mid \sigma^-(x) \ge \varphi(\alpha_1), \sigma^+(x) \ge \varphi(\alpha_2)\}, (\mu^{\pm})^{\varphi(\beta)} = \{xy \in E \mid \mu^-(xy) \ge \varphi(\beta_1), \mu^+(xy) \ge \varphi(\beta_2)\}, \varphi(\alpha) = (\varphi(\alpha_1), \varphi(\alpha_2))$ and $\varphi(\beta) = (\varphi(\beta_1), \varphi(\beta_2))$.

In what follow, we suppose that $G = (\Omega, \sigma^\pm, \mu^\pm)$ has no white vertex and white edge and consider $m' = \min\{\frac{\varphi^{-1}(\sigma^-(x))}{\alpha_1}, \frac{\varphi^{-1}(\sigma^+(x))}{\alpha_2} \mid x \in V\}$ and

$n' = \min\{\frac{\varphi^{-1}(g^\circ(x^\pm))}{\beta_1}, \frac{\varphi^{-1}(|Ker(x)^\pm|)}{\beta_2} \mid x \in V\}$, so we have the following theorem.

Theorem 4.10. *Let $G = (\Omega, \sigma^\pm, \mu^\pm)$ be a grey graph on $G^\pm$. Then*

(i) *if $m \leq m'$ and $n \leq n'$, then $G^{(\varphi(\alpha),\varphi(\beta))}$ is a complete graph,*

(ii) *if $m > m'$ and $n > n'$, then $G^{(\varphi(\alpha),\varphi(\beta))}$ is a null graph.*

Proof. (i) Let $x, y \in V$. Then $\mu^-(xy) \geq \varphi(\beta_1)$ implies that $T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) \geq \varphi(\beta_1)$. Hence $g^\circ(x^\pm) \geq \varphi(\beta_1)$ and $g^\circ(y^\pm) \geq \varphi(\beta_1)$. In a similar way, if $\mu^+(xy) \geq \varphi(\beta_2)$, then $\beta_2 \leq \varphi^{-1}(|Ker(x^\pm)|)$ and $\beta_2 \leq \varphi^{-1}(|Ker(y^\pm)|)$. In addition, $\sigma^-(x) \geq \varphi(\alpha_1)$ and $\sigma^+(x) \geq \varphi(\alpha_2)$, imply that $\alpha_1 \leq \varphi^{-1}(\sigma^-(x))$ and $\alpha_2 \leq \varphi^{-1}(\sigma^-(x))$.

(ii) It is clear by item (i). □

Example 4.11. Let $\Omega = [0, 3]$. Consider the interval grey number-based graph $G = (\Omega, \sigma^\pm, \mu^\pm)$ as shown Figure 11 and $\varphi(x) = e^x$. Computations show that $m' = \frac{22}{3}Ln(\frac{1}{11})$ and $n' = \frac{132}{7}Ln(\frac{3}{22})$ and so for $m \leq m'$ and $n \leq n'$, $G^{(e^\alpha, e^\beta)}$ is a complete graph.

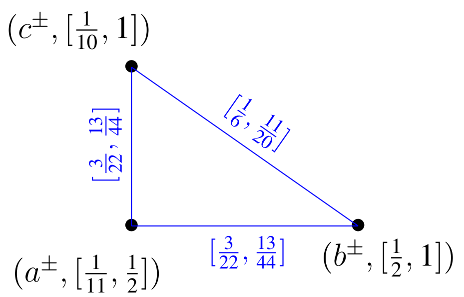

Figure 11. Interval-valued fuzzy graph $G = (\Omega, \sigma^\pm, \mu^\pm)$ on C_3

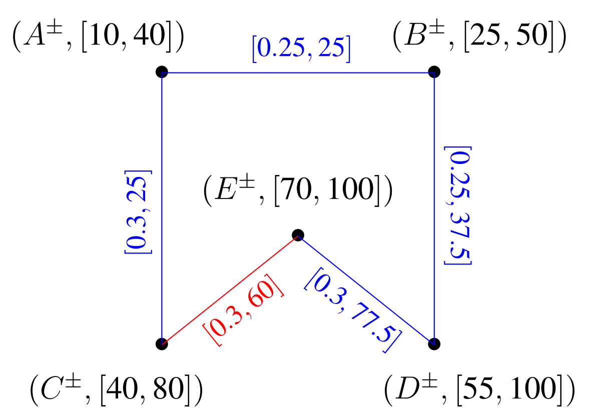

Figure 12. Interval grey number-based graph $G = (\Omega, \sigma^\pm, \mu^\pm)$ on C_5

5 Applications of Interval Grey Number-Based Graphs

In this section, we consider the concepts of interval grey number-based graphs and introduce some applications in real-life related to these concepts.

Transport Complex Network Let Q and T be two cities that we want to travel from city Q to city T and we know that there are some invisible policies on this path, but we do not know its location. Indeed, our objective is to investigate the relationship between the presence of imperceptible police and the attitude of preventing error in driving between cities Q and T with a distance of 100 km. Suppose invisible police A within 10 km to 40 km and B within 25 km to 50 km and C at 40 km to 80 km from town Q and D at 55 km to 100 km and E at 70 to 100 km of the highway. They are moving. Therefore, the presence of any police is imperceptible at distances above gray, because drivers do not know which part of the highway they are on, so we describe it in Table 1. Thus, by 1. Thus, by Figure 12, accordingto the DE edge, the range $[0.3, 77.5]$ of the inhibition rate is from 0.3% to 77.5%. This means that during the above period, drivers must be careful not to be fined by imperceptible police.

Table 1. Transport Complex Network Based On Interval Grey Number.

Invisible Police	A	B	C	D	E
Driving Violations	$[10, 40]$	$[25, 50]$	$[40, 80]$	$[55, 100]$	$[70, 100]$

Attitude between study and result in a test: We want to examine the relationship between study attitudes and test scores in the same topics and conclude about 5 students. We know that the study period is gray for taking a test, because a student may do other things during the study period, so we have a negative period and consider $\Omega = [-2, 2]$. Suppose 5 students {a: Pedro, b: Abdias, c: Dulce, d: Espe, e: Carlo} want to take a physics test. By performing calculations, we examine the relationship between study time and the points earned for them in Table 2. Negative numbers mean the interval that the student is in cyberspace and does not study, for describing, for instance, Pedro spent one hour in cyberspace and two hours of study, so we cisider it by $[-1, 2]$. Given that individuals a, b (edge ab) have the lowest error rate, i.e 0.25. Therefore, their study time and test result are much better than the others as shown in Figure 13.

Table 2. Attitude Via study and Test Based On Interval Grey Number.

Case Study	a	b	c	d	e
Error rate between study and physics test result	[-1, 2]	[1, 2]	[-1, -0.5]	[-2, 0.4]	[0.2, 1]

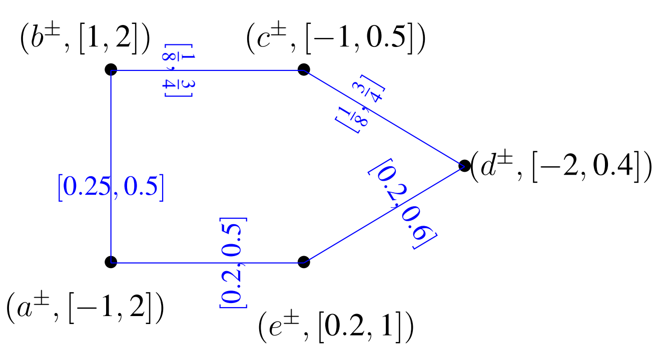

Figure 13. Interval grey number-based graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_5

6 Conclusion

The current paper has introduced a novel concept of grey graphs based on any given simple graph. Indeed, it has presented a new generalization of graphs based on interval grey numbers. This work extended and obtained some properties by solving the linear inequalities based on the concept of kernel and degree of greyness of interval grey numbers. It is showed that the concept of fuzzy graphs and grey graphs are different and since there is a fundamental difference between grey numbers and fuzzy subsets. In, final we presented some application examples in real life for more importance of the concept of grey graphs.

We hope that these results are helpful for further studies in theory of graphs. In our future studies, we hope to obtain more results regarding to hypergraph based on interval grey numbers as a generalization of graphs based on interval grey numbers and obtain some results in this regard to complete grey graphs, traceable grey graph, Hamiltonian grey graphs and Eulerian grey graphs and their applications.

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