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PURE DIRECT PROJECTIVE MODULES

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Abstract In this paper we introduce Pure direct projective module which is a generalization of Direct projective module and dual notion of pure direct injective modules [13]. A module M is pure direct projective if a quotient module determined by a pure submodule of M, is isomorphic to a direct summand of M, then the pure submodule is a direct summand of M. We discuss direct sums and direct summand of pure direct projective modules and characterize semi-simple rings with respect to relative pure direct projective modules. Also, we find connection among pure direct projective modules, pure rickart modules and endoregular modules. In the last part we characterize Von Neumann regular rings with respect to pure direct projective modules.

1 Introduction

Throughout the paper, R will denote an associative ring with an identity element and modules will be unital right R-modules unless otherwise stated. In 1976, Nicholson [15] called an R-module M direct-projective if M/N is isomorphic to some direct summand of M then N is a direct summand of M. Direct-projective modules are also known as D_2-modules. In recent years various generalizations of direct projective modules have been studied like semi-projective modules and simple direct projective modules etc. Motivated by these generalizations and work done in [13] on pure direct injective modules, we introduce the notion of pure direct projective modules which is the dual concept of pure direct injective modules and a generalization of direct projective modules. An R-module M is called pure direct projective if $M/P \cong N \leq^{\oplus} M$ for P pure submodule of M, then $P \leq^{\oplus} M$.

In section 2, we first define pure direct projective module and also give some examples of pure direct projective module and give counter examples such that module is pure direct projective but not direct projective. In remark 2.3 we give the equivalent condition of the pure direct projective module with respect to the pure exact sequence. In Proposition 2.4, we find an equivalent definition of pure direct projective module with respect to its endomorphism ring, which is defined as, R-module M is pure direct projective if and only for $f \in End_R(M)$ if $Img(f)$ is a direct summand of M and $Ker(f)$ is pure in M, then $Ker(f)$ is a direct summand of M. Next, we use the Morita equivalence property, if an R-module M is a pure direct projective, then some module which is Morita equivalent to M, will also be pure direct projective. Further, we introduce RD pure direct projective modules and find its equivalence with respect to pure direct projective modules (see proposition 2.9) and corollary (2.10).

In section 3, we discuss the direct sum and direct summand of pure direct projective modules. In Proposition 3.1, we prove that the direct summand of the pure direct projective module is pure direct projective. Then we infer that sum of two pure direct projective modules need not be a pure direct projective, counter example is also provided, so we find the condition under which the direct sum of pure direct projective modules is pure direct projective. Let M and N be a pure direct projective R-modules, M is called relatively pure direct projective to N, if there exists an R-homomorphism $f \in Hom_R(M, N)$ with $Img(f)$ is a direct summand of N and $Ker(f)$ is a pure submodule of M, then $Ker(f)$ is a direct summand of M. In Proposition 3.4, we prove that if $M = \bigoplus_{i \in \mathbb{N}} M_i$, where each M_i is relatively pure direct projective module with others if and only if M is a pure direct projective module. Also, we find the condition under which semi-simple ring is equivalent to a ring over which all modules are relatively pure direct projective to each other.

In section 4, we characterize some rings with respect to pure direct projective modules. In Proposition 4.1, we prove that for a pure direct projective module M, M is pure rickart module and dual rickart module iff $M \bigoplus M$ has S.S.P. and P.I.P. properties iff M is endoregular module. Further in Proposition 4.2, we prove that over P.D.S. ring, every purely rickart module is pure direct projective. Also, we prove that R is purely semi-simple if and only if every 2-generated R module is pure direct projective (see Proposition 4.3). In Proposition 4.8, we prove that over pure hereditary ring, every pure injective module is a pure direct projective module. In last we find an equivalent condition over right hereditary ring R, every pure direct projective module is direct projective module if and only if the ring R is von-Neumann regular ring. Throughout the paper, basic definitions, symbols and notations that are used to refer [4] and [18].

2 Pure Direct Projective Modules

In this section, we first define pure direct projective modules and give its some examples.

Definition 2.1. An R-module M is said to be pure direct projective module or pure D_2 module if for any pure submodule B of M satisfies $M/B \cong A$ and $A \leq^{\oplus} M$, then B is a direct summand of M.

Since every direct summand of R-module M is a pure submodule, so in general, direct projective module is pure direct projective module. Next we discuss some examples of pure direct projective modules.

Example 2.2. We discuss the following examples of pure direct projective modules:-

(i) Pure direct projective module is a strict generalization of direct projective module. Since every direct summand of right R-module M is also pure submodule of the module M. Hence every direct projective module is pure direct projective module, but its converse need not be true. For an example $\mathbb{Z}$ module $\mathbb{Z} \bigoplus \mathbb{Z}_n$, where $n \in \mathbb{N}$, which is pure direct projective but not direct projective module. Since both $\mathbb{Z}$ and $\mathbb{Z}_n$ has no pure submodules.

(ii) We can similarly construct another example $\mathbb{Q} \bigoplus \mathbb{Q}/\mathbb{Z}$ over the ring $\mathbb{Z}$, this module is a pure direct projective but not a direct projective.

(iii) Every Quasi Dedekind right R-module M is always pure direct projective. Since for every $s \in End_R(M)$ we have $Ker(s) = 0$, so from Proposition 2.4 M is pure direct projective.

(iv) If $M = \bigoplus_{i \in \mathbb{N}} N_i$ and each N_i is uniserial module. Then M is pure direct projective. An R module M is said to be uniserial if submodules of M are totally ordered by inclusion, that is if N and L are two submodules of M then either $N \subseteq L$ or $L \subseteq N$.

(v) If M is a pure simple right R-module then M is pure direct projective module. An R module M is pure simple if only pure submodules of M are trivial submodules.

Remark 2.3. A right R-module M is pure direct projective if, the pure exact sequence

$$0 \longrightarrow Ker(f) \longrightarrow M \longrightarrow M/(Ker(f) \longrightarrow 0$$

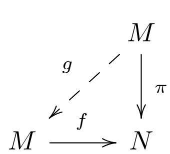
splits for $f \in End_R(M)$ such that $Img(f) \leq^{\oplus} M$.

Proposition 2.4. *The following conditions are equivalent for an R-module M:-*

(1) M is pure direct projective module.

(2) If $Img(f)$ is a direct summand of M with $Ker(f)$ is a pure submodule of M for $f \in End_R(M)$, then $Ker(f)$ is a direct summand of M.

(3) Let N be any direct summand of M and an epimorphism $f : M \longrightarrow N$ such that $Ker(f)$ is pure in M, then there exists a $g \in End_R(M)$ such that following diagram commutes.



equivalently, $f \circ g = \pi$, where π is projection map.

Proof. (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (2) Using the fact, $M/Ker(f) \cong Img(f)$ (By Fundamental theorem of Module Homomorphism) and $Ker(f)$ is pure submodule of M, hence $Ker(f) \leq^{\oplus} M$.

(2) $\Rightarrow$ (1) Consider $M/B \cong N \leq^{\oplus} M$ with B pure submodule of M, let $M/B \cong eM$ for some idempotent $e \in End_R(M)$. Let π be a projection map from M to N and f be a epimorphism from M to N. We set some s as $s = e \circ f^{-1} \circ \pi$, then $Ker(s) \cong B$ and $Img(s) \cong eM$, Since $Ker(s)$ direct summand in M implies B is direct summand of M.

(3) $\Rightarrow$ (1) Let B be any pure submodule of M such that it satisfies $M/B \cong N \leq^{\oplus} M$. Consider $h : M \longrightarrow M/B$, Now since $Ker(h) \cong B$ and pure in M. From hypothesis we will have some $g \in End_R(M)$ such that $h \circ g = \pi$. Hence g is required splitting of h thus $Ker(h)$ is direct summand of M. □

Next we discuss the Morita equivalence with respect to pure direct projective modules. Two rings say R and S are Morita equivalent if their category of modules say R-module M and S-module M are equivalent.

Proposition 2.5. *For two Morita equivalent rings say R and S, let $\phi : Mod - R \longrightarrow Mod - S$ be their Morita equivalence. Then for any Module M which is pure direct projective then $\phi(M)$ is pure direct projective and vice-versa.*

Proof. Let $N \leq M$ be pure submodule then $\phi(N)$ is pure submodule of $\phi(M)$. Since purity is a Morita equivalent property. Hence proof is obvious. □

Proposition 2.6. *Let M be a pure direct projective R-module. If $M = M_1 \bigoplus M_2$ with $f : M_1 \longrightarrow M_2$ be R-homomorphism such that $Img(f)$ is a direct summand of M_2 and $Ker(f)$ is a pure submodule of M_1. Then $Ker(f)$ is a direct summand of M_1.*

Proof. Let $f : M_1 \longrightarrow M_2$ be a module homomorphism. We consider $\pi : M \longrightarrow M_1$ a canonical projection, then $f \circ \pi : M \longrightarrow M_2$. Also $Img(f \circ \pi) = Img(f)$. Therefore $M/Ker(f \circ \pi) \cong Img(f \circ \pi) = Img(f)$. Since $Img(f) \leq^{\oplus} M_2$ and it is a direct summand of module M. By definition of pure direct projective module, and $Ker(f \circ \pi) = M_2 \oplus Ker(f) \leq^{p} M_2 \bigoplus M_1$, as $Ker(f)$ is pure in M_1, hence $Ker(f \circ \pi)$ is pure in M. Therefore $Ker(f \circ \pi) \leq^{\oplus} M$.

This implies, $Ker(f) \leq^{\oplus} M$. Since $Ker(f)$ is a submodule of M_1 implies $Ker(f) \leq^{\oplus} M_1$. □

Corollary 2.7. *If $M = M_1 \bigoplus M_2$ with $f : M_1 \longrightarrow M_2$ be an R-epimorphism such that $Ker(f)$ is pure in M_1, then $Ker(f)$ is a direct summand of M_1.*

Proof. Follows from Proposition 2.6. □

Proposition 2.8. *If R-module $M = M_1 \bigoplus M_2$ is a pure direct projective with M_1 is pure simple and M_2 dual rickart. Then either $Hom_R(M_1, M_2) = 0$ or for each nonzero homomorphism $f : M_1 \to M_2$ then f is a monomorphism, if $Ker(f)$ is pure submodule of M.*

Proof. We consider $Hom_R(M_1, M_2) \neq 0$ as other case is trivial. If $f : M_1 \to M_2$ is non-zero R-homomorphism and M_1 is pure simple, then this implies $Ker(f) \neq 0$ is not pure submodule, implies $Ker(f) \neq 0$ is not direct summand of M_1. Therefore $Ker(f) = 0$ and $f \in Hom_R(M_1, M_2)$ is a monomorphism. □

In [6], RD (relatively divisible) pure submodule P is a pure submodule of M if for every $r \in R$, $rP = rM \cap P$. Similarily, we can define RD direct projective module as, if B is RD pure submodule of M and it satisfies $M/B \cong A \leq^{\oplus} M$, then $B \leq^{\oplus} M$.

Proposition 2.9. *Every pure direct projective module is RD-pure direct projective module.*

Proof. Let P be a pure submodule of a module M. Since every pure submodule is RD-pure and we have M is pure direct projective module. This implies M is RD-pure direct projective.

□

Corollary 2.10. *A flat R-module M is pure direct projective if M is RD-pure direct projective.*

Proof. Follows from the fact that when R-module M is flat module then RD-pure submodule is pure submodule. □

Now we consider when direct projective and pure direct projective modules are equivalent.

Proposition 2.11. *Let M be an R-module and for each $f \in End_R(M)$, $Img(f)$ is a direct summand and $M/Ker(f)$ be pure injective submodule of M. Then M is a pure direct projective if and only if M is a direct projective.*

Proof. Since $M/Ker(f)$ is pure injective. This implies pure exact seq. $0 \longrightarrow M/Ker(f) \longrightarrow M$ splits. Since $M/Ker(f) \cong Img(f)$, is a direct summand of M. Therefore we got our claim. □

Next we find an equivalent condition of pure Rickart ring and pure direct projective ring. A pure direct projective ring is described as cyclic pure direct projective R-module over itself.

Proposition 2.12. *Every right Noetherian ring is a pure direct projective ring if it is a pure Rickart ring.*

Proof. In Noetherian ring, every pure ideal is direct summand. □

3 Direct Sum and Direct summand of Pure-Direct projective modules

In this section we discuss about the direct sum and direct summand of Pure direct projective modules. In the following Proposition we prove that direct summand of pure direct projective module is pure direct projective and also discuss the results related to direct sum of pure direct projective modules.

Proposition 3.1. *For $N \leq^{\oplus} M$, M is a pure direct projective module. Then N is a pure direct projective module.*

Proof. Let N be a direct summand of M and A be a pure submodule of N and satisfy $N/A \cong B \leq^{\oplus} N$ for some $B \leq^{\oplus} M$, we are required to prove $A \leq^{\oplus} N$. Since M is pure direct projective, so there exists some T direct summand of M such that $M/A \cong T \leq^{\oplus} M$, and since $A \leq^{p} N$ this implies $A \leq^{p} M$, therefore A will be a direct summand of M. We already assumed that A is a pure submodule of N hence $A \leq^{\oplus} N$. □

In Proposition 3.1, we see that every direct summand of pure direct projective module is pure direct projective but if we consider direct sum of two pure direct projective modules, then it need not be pure direct projective. For example, consider $\mathbb{Z}$-module $M_1 = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $\mathbb{Z}$-module $M_2 = \prod_{n=1}^{\infty} \mathbb{Z}_2 / \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Here both M_1 and M_2 are pure direct projective but $M_1 \bigoplus M_2$ is not pure direct projective module. Since for $f \in Hom_R(M_1, M_2)$ and $Ker(f)$ is pure submodule of M_1 but $Ker(f)$ is not a direct summand.

Definition 3.2. Let M and N be pure direct projective R-modules, M is called relatively pure direct projective to N, if there exists R-homomorphism $f \in Hom_R(M, N)$ with $Img(f)$ is a direct summand of N and $Ker(f)$ is a pure submodule M, then $Ker(f)$ is a direct summand of M.

Next we will discuss direct sum of pure direct projective module with respect to PIP module. A module is said to have PIP property if intersection of two pure submodules is again pure. Let M be an R-module with the PIP, then for every decomposition $M = K \bigoplus T$ and for every $f \in Hom_R(T, K)$, $Ker(f)$ is a pure submodule in M.

Proposition 3.3. *Let $M = K \bigoplus T$ be a pure direct projective module. If M has the PIP property and T is Dual Rickart module, then K is relatively pure direct projective to T.*

Proof. Follows from the definition. □

In particular, every direct sum of pure direct projective modules is pure direct projective if modules are relatively pure direct projective to each other.

Proposition 3.4. *Let $M = \bigoplus_{i \in \mathbb{N}} M_i$, where each M_i is relatively pure direct projective module with others if and only if M is a pure direct projective module.*

Proof. If M is pure direct projective module then $\bigoplus_{i \in \mathbb{N}} M_i$ is pure direct projective, follows from Proposition 2.6 and Definition 3.2. For the converse part, let for some pure submodule P of M such that $M/P \cong N \leq^{\oplus} M$, then N will be a direct summand of some M_k and P is a pure submodule of some M_t, since M_t is relatively pure direct projective M_k. Hence P is direct summand of M. Also as $k, t \in \mathbb{N}$ are arbitrary. Therefore M is pure direct projective module. □

Proposition 3.5. *Let R be a von-Neumann regular ring. Then the following statements are equivalent:*

(1) R is a semi-simple ring.

(2) All R-modules are relatively pure direct projective to any R-module.

Proof. (1) ⇒ (2) From the definition of semi-simple ring.

(2) ⇒ (1) Let I be an ideal of a ring R. We are required to prove that I is a direct summand of R. From (2) all R-modules are relatively pure direct projective to any R-module. Then the R-module R is relatively Pure direct projective to R/I as an R-module. Then by definition for R-homomorphism $f : R \to R/I$ and $Ker(f) = I$ is pure in R. Then $Ker(f)$ is a direct summand of R. Hence R is a semi-simple ring as I is an arbitrary ideal of R. □

4 Characterization of Pure Direct Projective modules over rings

In this section we characterize pure direct projective modules over some rings. At first, we will consider its endomorphism ring, and develop equivalent conditions for pure direct projective modules and endoregular modules. Since every endoregular module is pure direct projective, but converse needs not to be true. For an example $\mathbb{Z}$ module $\mathbb{Z}_{p^n}$ where p is prime no. and $n \in \mathbb{N}$, which is pure direct projective module but not an endoregular module.

Proposition 4.1. *The following statements are equivalent for pure direct projective R-module M:*

(1) M is pure Rickart and dual Rickart module;

(2) $M \bigoplus M$ has SSP and PIP property;

(3) M is Endoregular module.

Proof. (1) ⇒ (3) To prove M is endoregular module then $Ker(f)$ and $Img(f)$ are direct summands for all $f \in End_R(M)$. Since M is dual Rickart, $Img(f)$ is a direct summand of M, also M is a pure Rickart module, hence $Ker(f)$ is pure for all $f \in End_R(M)$. So we get $Ker(f)$ is a direct summand of M, as M is a pure direct projective module.

(2) ⇒ (3) Since $M \bigoplus M$ has SSP then $Img(g) \leq^{\oplus} M$ for each $g \in End_R(M)$ and due to PIP property $Ker(f)$ is pure for each $f \in End_R(M)$. This implies $Ker(f)$ is a direct summand of M, as M is a pure direct projective module.

(3) ⇒ (1) and (3) ⇒ (2) It is clear from the definition of Endoregular module. □

In next proposition we find when a purely Rickart module is a pure direct projective.

Proposition 4.2. *Every purely Rickart module is pure direct projective module over P.D.S ring.*

Proof. Suppose M be a purely Rickart module over P.D.S ring R, hence for a $P \leq M$ and satisfy $M/P \cong N \leq^{\oplus} M$, then P is pure submodule of M. Since R is P.D.S ring hence P is a direct summand of M this implies M is pure direct projective module. □

Next, we will discuss pure direct projective modules with respect to purely semisimple rings. A ring R is purely semisimple iff every pure submodule of an R module M is a direct summand of M.

Proposition 4.3. *The following statements are equivalent:-*

(1) R is a right purely semisimple ring;

(2) Every finitely generated R-module is a pure direct projective;

(3) Every 2-generated module is a pure direct projective.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$.

$(3) \Rightarrow (1)$ We are required to prove that every pure ideal, say I of R is a direct summand of R. Consider $R \bigoplus R/I$ which is pure direct projective. So from Proposition 2.4, there exists $f : R \longrightarrow R/I$ then $Ker(f) \cong I$, which is pure in R, hence I is direct summand of R. Therefore R is purely semisimple ring.

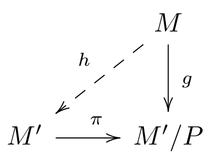
□

Next corollary is based on the fact that every commutative ring is a pure semi-simple if and only if it is Artinian PID.

Corollary 4.4. *A commutative ring R is artinian PID if and only if every fintely generated R-module is pure direct projective.*

Next we characterize for the two modules when one module is pure direct projective with respect to other module and characterize this condition with pure semisimple modules.

Definition 4.5. Let M and M' be R-modules, then M is M' pure direct projective, if for a module homomorphism $g : M \longrightarrow M'/P$ where P is a pure submodule of M' and M'/P is isomorphic to a direct summand of M, $\exists$ a $h \in Hom_R(M, M')$ such that $\pi \circ h = g$, where π is natural epimorphism from M' to M'/P.



This implies the above diagram commutes.

Proposition 4.6. *For each pure submodule P of M, if M/P isomorphic to a direct summand of R-module M and M/P is M-pure direct projective, then R-module M is pure semisimple.*

Proof:- Since M/P is M pure direct projective implies identity mapping from M/P to M/P has a lifting $g : M/P \longrightarrow M$. Hence short exact sequence $0 \longrightarrow P \longrightarrow M \longrightarrow M/P$ splits, implies $P \leq^{\oplus} M$.

Proposition 4.7. *Let $M = M' \bigoplus M''$ be a pure direct projective and M' is a pure simple module. Then M' is M''-pure direct projective module.*

Proof. Follows from above definition. □

Proposition 4.8. *Let a module M be pure injective module over a pure hereditary ring R, then R-module M is pure direct projective module.*

Proof. Since R-module M is pure injective over pure hereditary ring, so every quotient module of pure injective is pure injective. For pure submodule P of M then M/P is pure injective hence $P \leq^{\oplus} M$. Therefore R-module M is pure direct projective. □

Lemma 4.9. *Let T be a projective module and $T \bigoplus N$ be pure direct projective module where N is an R-module. If there exists an epimorphism $f : T \longrightarrow N$ such that $Ker(f)$ is pure, then N is a projective module.*

Proof. Since f is an epimorphism and $Ker(f)$ is pure in T. From Corollary 2.7 $Ker(f)$ will be direct summand of T. Hence epimorphism f splits and N is projective. □

In next proposition we characterize von-Neumann regular rings in terms of pure direct projective modules.

Proposition 4.10. *The following statements are equivalent over right hereditary ring R:-*

(1) R is a von-Neumann regular ring.

(2) Every pure direct projective R-module is a direct projective module.

(3) Every pure projective R-module is a projective.

Proof. $(1) \Rightarrow (2)$ Let R be a von-Neumann regular ring this implies every submodule of M is pure. Now for any submodule P of M, $M/P \cong N \leq^{\oplus} M$, implies $P \leq^{\oplus} M$ as given M is pure direct projective and P is pure submodule. Hence M is a direct projective.

$(2) \Rightarrow (3)$ Let S be projective module and every pure projective module can be written as $S \bigoplus N$ from ([14], Proposition 2.1). If we prove N is projective then we will get our claim. Now since $S \bigoplus N$ is pure projective hence pure direct projective and from (2) it is direct projective hence every epimorphism $f : S \longrightarrow N$ splits hence N is projective.

$(3) \Rightarrow (1)$ Follows from ([14] Proposition 3.7) R is von Neumann regular. □

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