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Approximate numerical solutions of fractional integral equations using Laguerre and Touchard polynomials

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Abstract Two numerical methods based on Laguerre and Touchard polynomials are described in this paper to solve both the fractional integral equations of the first kind and the second kind (FIEs-1K and FIEs-2K, respectively). The fractional integrals are described in the Erdélyi-Kober sense. Both the integral equations are transformed into an algebraic system of linear equations using the Laguerre and Touchard matrices. All the steps of the algorithm are given to find the solutions. Also, the accuracy of the solutions has been demonstrated. Five examples are provided to interpret the methods. The accuracy of solutions is compared for both methods. MATLAB program is used to perform all computations and graphs.

1 Introduction

The subject of fractional calculus has flourished over the last decade due to engineering applications in the fields of feedback control, systems theory, signals processing and many other areas of sciences. Fractional operators have been extensively analyzed, both analytically and numerically.

In the literature, several techniques for solving integral equations have been proposed. Some of these methods are Babenko's method and Abel integrals [14], Adomian decomposition method [16], Fixed point methods [5], the product integration and Haar Wavelet approaches, for an approximate solutions of the fractional Volterra integral equations of the second type [9], Spectral method based on Chebyshev polynomials [7], Galerkin weight residual numerical method with Touchard polynomials as trial functions for Volterra-Fredholm integral equations [23]. Numerical method for the solutions of Volterra–Fredholm fractional integral equations [2]. Also several ways for solving fractional integral equations have been proposed in recent years [1, 3, 4, 15, 17, 19, 20, 21, 26, 27, 30, 32, 36, 37].

Various modifications and generalizations of classical fractional integration operators are known and are widely used both in theory and applications. In this paper, we will focus on generalized Erdélyi-Kober fractional integral operator which is typically used to describe mediums with non-integer mass dimensions, and other applications of fractional integrals of the Erdélyi–Kober type may be found in porous media, viscoelasticity, and electrochemistry [8, 10, 12, 22, 24, 28, 29, 33, 39].

The generalized Erdélyi-Kober fractional integral operator $\mathcal{J}_\beta^{\eta,\delta}$ of order $\delta > 0$ for a real-valued continuous function $u(r)$ is defined as [11]:

$$\mathcal{J}_\beta^{\eta,\delta} u(r) = \frac{r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta\eta}\, u(x)\, d(x^\beta), \tag{1.1}$$

where $\delta > 0, \beta > 0$, $\eta \in \mathbb{R}$ and $\Gamma(\delta) = \int_0^\infty x^{\delta-1} e^{-x}\, dx$.

According to Kiryakova [11], the generalized Erdélyi-Kober fractional integral operator (1.1), possess the advantage that a number of generalized integration and differentiation operators happen to be the particular cases of this operator. Some important special cases of the integral operator $\mathcal{J}_\beta^{\eta,\delta}$ are mentioned below:

(a) For $\eta = 0$, $\delta = n$ ($n \in \mathbb{N}$) and $\beta = 1$, the Eq. (1.1) yields the following ordinary n-fold integration:

$$I^n\{u(r)\} = r^n \mathcal{J}_1^{0,n}\{u(r)\} = \frac{1}{(n-1)!}\int_0^r (r-x)^{n-1} u(x)\, dx.$$

(b) If we set $\eta = 0$ and $\beta = 1$, the operator (1.1) reduces to the Riemann-Liouville fractional integral operators with the following relationship:

$$R^\delta\{u(r)\} = r^\delta \mathcal{J}_1^{0,\delta}\{u(r)\} = \frac{1}{\Gamma(\delta)}\int_0^r (r-x)^{\delta-1} u(x)\, dx.$$

(c) Again, for $\eta = 0$, $\delta = 1$, and $\beta = 1$, the operator (1.1) leads to the Hardy-Littlewood (see Cesaro) integration operator:

$$L_{1,0}\{u(r)\} = \mathcal{J}_1^{0,1}\{u(r)\} = \frac{1}{r}\int_0^r u(x)\, dx.$$

and its generalization for integers $n > m-1$ (when $\eta = n$, $\delta = 1$ and $\beta = 1$), we have

$$L_{m,n}\{u(r)\} = r^{n-m+1}\mathcal{J}_1^{n,1}\{u(r)\} = r^{-m}\int_0^r x^n u(x)\, dx.$$

(d) When $\beta = 1$, $\delta > 0$ and $\eta \in \mathbb{R}$, operator (1.1) reduces to the fractional integral operator, which was originally considered by Kober [13] and Erdélyi [6]:

$$I^{\eta,\delta}\{u(r)\} = \mathcal{J}_1^{\eta,\delta}\{u(r)\} = \frac{r^{-\delta-\eta}}{\Gamma(\delta)}\int_0^r (r-x)^{\delta-1} x^\eta u(x)\, dx.$$

(e) Also for $\beta = 2$, the operator (1.1) yields the Erdélyi-Kober fractional integral operator $I_{\eta,\delta}$ (Sneddon [33]):

$$I_{\eta,\delta} = \mathcal{J}_2^{\eta,\delta}\{u(r)\} = \frac{2r^{-2(\eta+\delta)}}{\Gamma(\delta)}\int_0^r (r^2-x^2)^{\delta-1} x^{2\eta+1} u(x)\, dx.$$

(f) Further, if we set $\eta = -1/2$, $\beta = 2$ and δ is replaced by $\delta + \frac{1}{2}$, the Uspensky integral transform [11] can easily be obtained as follows:

$$P^\delta\{u(r)\} = \frac{1}{2}\mathcal{J}_2^{-\frac{1}{2},\delta+\frac{1}{2}}\{u(r)\} = \frac{1}{\Gamma(\delta+\frac{1}{2})}\int_0^1 (1-x^2)^{\delta-\frac{1}{2}} u(rx)\, dx.$$

For a detailed information about fractional integral operator (1.1) and its more special cases one may refer the book [11].

Sometimes, it is difficult to find the exact solution to these equations explicitly. As a result, we use numerical techniques to get an approximation of the solution. Thus, the current article's goal is to use the Laguerre and Touchard polynomials to obtain approximate numerical solutions for FIEs-1K and FIEs-2K involving the Erdélyi-Kober fractional integral operator. In order to achieve this goal, we develop a numerical scheme to solve the FIEs-1K and FIEs-2K. Furthermore, to illustrate the proposed algorithm, we provide five examples.

This paper is organized as follows: In Section 2 some preliminary information is given. Section 3 describes the matrix representation and approximation function for the Laguerre and Touchard polynomials. In Section 4, we discussed numerical solutions for FIEs-1K and FIEs-2K using the Laguerre and Touchard polynomials. Also, we summarized the algorithm for the solutions in this Section. Accuracy of solutions and convergence rate are discussed in Section 5. Some numerical experiments and figures are shown in Section 6 to demonstrate the applicability and accuracy of the presented methods. Finally, conclusions and future work are given in Section 7.

2 Preliminaries

Some basic knowledge is provided for the readers' convenience.

By following Eq. (1.1) the generalized Erdélyi-Kober fractional integral operator $\mathcal{J}_\beta^{\eta,\delta}$ of order $\delta > 0$ for a real-valued continuous function $u(r)$ can be written as [11]:

$$\mathcal{J}_\beta^{\eta,\delta} u(r) = \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} u(x)\, dx, \tag{2.1}$$

where $\delta > 0, \beta > 0$, $\eta \in \mathbb{R}$ and $\Gamma(\delta) = \int_0^\infty x^{\delta-1} e^{-x}\, dx$.

In light of these considerations, the generalized Erdélyi-Kober fractional integral formulation can be used to define the FIEs-1K and FIEs-2K as follows:

$$\begin{aligned} g(r) &= \mathcal{J}_\beta^{\eta,\delta} u(r) \\ &= \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} u(x)\, dx, \qquad r \in [0, \tau]. \end{aligned} \tag{2.2}$$

$$\begin{aligned} u(r) &= g(r) + \mathcal{J}_\beta^{\eta,\delta} u(r) \\ &= g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} u(x)\, dx, \qquad r \in [0, \tau], \end{aligned} \tag{2.3}$$

where $\delta > 0$, $\beta > 0$, $\eta \in \mathbb{R}$, $\Gamma(\delta) = \int_0^\infty x^{\delta-1} e^{-x}\, dx$ and $g\colon [0, \tau] \longrightarrow \mathbb{R}$ is a known function.

In this paper, our discussion will be focused on the Eq. (2.2) and Eq. (2.3).

2.1 Laguerre Polynomials

Laguerre polynomials consist of a polynomial sequence of binomial type defined on the interval $[0, \infty)$ and of the following form [31, 35]:

$$L_n(r) = \sum_{m=0}^{n} (-1)^m \frac{1}{m!} \binom{n}{m} r^m, \tag{2.4}$$

where m and n are the polynomial index and degree, respectively and $\binom{n}{m}$ is a binomial coefficient.

The first six polynomials of the Laguerre polynomials are given below:

$L_0(r) = 1,$
$L_1(r) = 1 - r,$
$L_2(r) = \frac{1}{2}(2 - 4r + r^2),$
$L_3(r) = \frac{1}{6}(6 - 18r + 9r^2 - r^3),$
$L_4(r) = \frac{1}{24}(24 - 96r + 72r^2 - 16r^3 + r^4),$
$L_5(r) = \frac{1}{120}(120 - 600r + 600r^2 - 200^3 + 25r^4 - r^5).$

2.2 Touchard Polynomials

Touchard polynomials consists of a polynomial sequence of binomial type defined on the interval $[0, 1]$ and of the following form [18, 23, 25, 38]:

$$T_n(r) = \sum_{m=0}^{n} \binom{n}{m} r^m, \tag{2.5}$$

where m and n are the polynomial index and degree, respectively and $\binom{n}{m}$ is a binomial coefficient.

The first six polynomials of the Touchard polynomials are :

$T_0(r) = 1,$
$T_1(r) = 1 + r,$
$T_2(r) = 1 + 2r + r^2,$
$T_3(r) = 1 + 3r + 3r^2 + r^3,$
$T_4(r) = 1 + 4r + 6r^2 + 4r^3 + r^4,$
$T_5(r) = 1 + 5r + 10r^2 + 10r^3 + 5r^4 + r^5.$

3 Approximation Function and Matrix Representation

In this section, we will consider the approximate function by using Laguerre and Touchard polynomials to determine the approximate numerical solutions of Eq. (2.2) and Eq. (2.3).

3.1 For Laguerre Polynomials

For determining the approximate numerical solutions of Eq. (2.2) and Eq. (2.3), assume that the function $J_n(r)$ is approximated by the Laguerre polynomials as folllows,

$$J_n(r) = s_0 \cdot L_0(r) + s_1 \cdot L_1(r) + ... + s_n \cdot L_n(r) = \sum_{m=0}^{n} (s_m \cdot L_m(r)), \;\; 0 \leq r < \infty, \tag{3.1}$$

where the function $\{L_m(r)\}_{m=0}^{n}$ denotes the Laguerre basis polynomials of n-th degree, defined in Eq. (2.4). We have to determine the unknown Laguerre coefficients s_m, $(m = 0, 1, ..., n)$.
Now rewriting Eq. (3.1) as,

$$J_n(r) = \begin{bmatrix} L_0(r) & L_1(r) & \dots & L_n(r) \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix}. \tag{3.2}$$

Again, Eq. (3.2) can be converted as,

$$J_n(r) = [1 \quad r \quad r^2 \quad \dots \quad r^n] \cdot \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \cdots & \gamma_{0n} \\ 0 & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ 0 & 0 & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{nn} \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}, \tag{3.3}$$

where $\{\gamma_{mm}\}_{m=0}^{n}$, $(m = 0, 1, \dots, n)$ are the constants of the power basis, used to obtain the Laguerre polynomials, this matrix is upper triangular and is certainly invertible. Now for $n = 1$, 2 and 3, the operational matrices will be shown in the Eq. (3.4), Eq. (3.5), Eq. (3.6) respectively:

$$J_1(r) = \begin{bmatrix} 1 & r \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \end{bmatrix}, \tag{3.4}$$

$$J_2(r) = [1 \quad r \quad r^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix}, \tag{3.5}$$

$$J_3(r) = [1 \quad r \quad r^2 \quad r^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & -1/6 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}. \tag{3.6}$$

3.2 For Touchard Polynomials

For determining the approximate numerical solutions of Eq. (2.2) and Eq. (2.3), assume that the function $\Theta_n(r)$ is approximated by the Touchard polynomials as follows,

$$\Theta_n(r) = a_0 \cdot T_0(r) + a_1 \cdot T_1(r) + ... + a_n \cdot T_n(r) = \sum_{m=0}^{n} (a_m \cdot T_m(r)), \ 0 \le r \le 1, \tag{3.7}$$

where the function $\{T_m(r)\}_{m=0}^{n}$ denotes the Touchard basis polynomials of n-th degree, defined in Eq. (2.5). We have to determine the unknown Touchard coefficients a_m, $(m = 0, 1, ..., n)$.
Now rewriting Eq. (3.7) as,

$$\Theta_n(r) = \begin{bmatrix} T_0(r) & T_1(r) & \dots & T_n(r) \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{3.8}$$

Again, Eq. (3.8) can be converted as,

$$\Theta_n(r) = [1 \quad r \quad r^2 \quad \dots \quad r^n] \cdot \begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \dots & \sigma_{0n} \\ 0 & \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ 0 & 0 & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \tag{3.9}$$

where $\{\sigma_{mm}\}_{m=0}^{n}$, $(m = 0, 1, \dots, n)$ are the constants of the power basis, used to obtain the Touchard polynomials, this matrix is upper triangular and is certainly invertible. Now for $n = 2$, 3 and 4, the operational matrices will be shown in the Eq. (3.10), Eq. (3.11) and Eq. (3.12) respectively:

$$\Theta_2(r) = [1 \quad r \quad r^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \tag{3.10}$$

$$\Theta_3(r) = [1 \quad r \quad r^2 \quad r^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{3.11}$$

$$\Theta_4(r) = [1 \quad r \quad r^2 \quad r^3 \quad r^4] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{3.12}$$

4 The Numerical Solutions

In this section, we will use the Laguerre and Touchard polynomials to determine the approximate numerical solutions for the FIEs-2K and then consider FIEs-1K.

4.1 Solution of the FIEs-2K via Laguerre Polynomials

Noted that the Eq. (2.3) is in the form

$$\begin{aligned} u(r) &= g(r) + \mathcal{J}_{\beta}^{\eta,\delta} u(r) \\ &= g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} u(x)\, dx, \qquad r \in [0,\tau]. \end{aligned} \tag{4.1}$$

Now to approximate the unknown function in Eq. (4.1), by using Eq. (3.1), let

$$u(r) \cong J_n(r) = \sum_{m=0}^{n} (s_m \cdot L_m(r)). \tag{4.2}$$

Now, substituting the Eq. (4.2) into the Eq. (4.1), we get

$$\sum_{m=0}^{n} (s_m \cdot L_m(r)) = g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\sum_{m=0}^{n} (s_m \cdot L_m(x)\right) dx. \tag{4.3}$$

Therefore, the Eq. (4.3) becomes by using Eq. (3.2) as follows:

$$\begin{aligned} &\begin{bmatrix} L_0(r) & L_1(r) & \dots & L_n(r) \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix} \\ &= g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\begin{bmatrix} L_0(x) & L_1(x) & \dots & L_n(x) \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix} \right) dx. \end{aligned} \tag{4.4}$$

Again, by using Eq. (3.3), the Eq. (4.4) converts to the form:

$$\begin{aligned} &[1 \quad r \quad r^2 \quad \dots \quad r^n] \cdot \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \dots & \gamma_{0n} \\ 0 & \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ 0 & 0 & \gamma_{22} & \dots & \gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{nn} \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \\ &= g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \\ &\quad \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left([1 \quad x \quad x^2 \quad \dots \quad x^n] \cdot \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \dots & \gamma_{0n} \\ 0 & \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ 0 & 0 & \gamma_{22} & \dots & \gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{nn} \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \right) dx. \end{aligned} \tag{4.5}$$

Thus, after finding the integrations of the Eq. (4.5) we have to calculate the values of unknown constants s_m, $(m = 0, 1, 2, \dots, n)$ of the Laguerre polynomials, for this purpose we need $(n+1)$ equations.
Now by choosing r_i, $(i = 0, 1, 2, \dots, n)$ in the interval $[0, \tau]$, a system of $(n+1)$ equations can be obtained. After solving these equations the unknown coefficients $(s_0, s_1, \dots, s_n)$ have uniquely determined. Therefore, by substituting the values of coefficients into the Eq. (3.1) getting the approximate numerical solution to n selecting.

4.2 Solution of the FIEs-1K via Laguerre Polynomials

By following the same techniques that we have used for FIEs-2K, using the equality Eq. (4.2), one can approximate the FIEs-1K provided in Eq. (2.2) . That means from Eq. (2.2) we get,

$$g(r) = \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\sum_{m=0}^{n} (s_m \cdot L_m(x)\right) dx. \tag{4.6}$$

Then, by reorganizing the preceding equation and applying the same concept as before, we may write:

$$g(r) = \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \cdot \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \cdots & \gamma_{0n} \\ 0 & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ 0 & 0 & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{nn} \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \right) dx. \tag{4.7}$$

Therefore, by following the same procedure, one can be obtained easily the approximate numerical solution of FIEs-1K .

Remark 4.1. The same procedure can be applied to the Eq. (2.2) and Eq. (2.3) when using the Touchard polynomials.

4.3 Algorithm for Solutions

In this part, the steps of the algorithm are summarized to find the approximate numerical solutions for FIEs-1K and FIEs-2K when using Laguerre polynomials.

4.3.1 Algorithm for FIEs-2K via Laguerre Polynomial

Step 1: Choose a degree n for the Laguerre polynomials

$$L_n(r) = \sum_{m=0}^{n} (-1)^m \frac{1}{m!} \binom{n}{m} r^m, \tag{4.8}$$

Step 2: Now we have to use Eq. (2.3), (3.3) and (4.2).

Step 3: Substitute Eq. (3.3) into Eq. (2.3).

Step 4: Compute all the integrations obtained in Step 3.

Step 5: Compute $s_0, s_1, s_2, \dots, s_n$, by choosing $r_i \in [0, \tau]$, where $i = 0, 1, 2, \dots, n$.

4.3.2 Algorithm for FIEs-1K via Laguerre Polynomials

The same algorithm can be given for FIEs-1K as follows:

Step 1: Choose a degree n for the Laguerre polynomials in Eq. (4.8).

Step 2: Now we have to use Eq. (2.2), Eq. (3.3) and Eq. (4.2).

Step 3: Substitute Eq. (3.3) into Eq. (2.2).

Step 4: Compute all the integrations obtained in Step 3.

Step 5: Compute $s_0, s_1, s_2, \ldots, s_n$, by choosing $r_i \in [0, \tau]$, where $i = 0, 1, 2, \ldots, n$.

Remark 4.2. The algorithm will be the same for Eq. (2.2) and Eq. (2.3) when using Touchard polynomials.

5 Accuracy of Solutions:

Accuracy of the proposed methods is verified in this section [35].

5.1 For FIEs-2K via Laguerre Polynomials

Since Eq. (4.3) and Eq. (3.1) have the following forms given by Eq. (5.1) and Eq. (5.2) respectively:

$$\sum_{m=0}^{n}(s_m \cdot L_m(r)) = g(r) + \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\sum_{m=0}^{n}(s_m \cdot L_m(x)\right) dx, \tag{5.1}$$

$$J_n(r) = s_0 \cdot L_0(r) + s_1 \cdot L_1(r) + ... + s_n \cdot L_n(r) = \sum_{m=0}^{n}(s_m \cdot L_m(r)), \;\; 0 \le r < \infty, \tag{5.2}$$

and the unknown coefficients $(s_0, s_1, \ldots, s_n)$ are determined by using Eq. (4.5). Also, by using Eq. (4.2), we have:

$$u(r) \cong J_n(r) = \sum_{m=0}^{n}(s_m \cdot L_m(r)), \tag{5.3}$$

thus, Eq. (5.3) is the unique approximate solution for Eq. (5.1), and is substituted into Eq. (5.1).
Now, assume that $r = r_k \in [0, 1], k = 0, 1, 2, \ldots, n$, and then, the error function:

$$\Lambda(r_k) = \left|\sum_{m=0}^{n}(s_m \cdot L_m(r_k)) - g(r_k) - \frac{\beta \cdot r_k^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \int_0^{r_k} \left(r_k^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \sum_{m=0}^{n}(s_m \cdot L_m(x))dx\right| \cong 0,$$

then $\Lambda(r_k) \le \epsilon$, for every $r_k \in [0, 1]$ and $\epsilon > 0$.
Then, at each point r_k, the difference for error function $\Lambda(r_k)$ will be less than any positive integer $\epsilon > 0$.
Thus, the following relation can be used to determine the error function, i.e.,

$$\Lambda_n(r) = \sum_{m=0}^{n}(s_m \cdot L_m(r)) - g(r) - \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} (\sum_{m=0}^{n}(s_m \cdot L_m(x)) \, dx.$$

Hence, $\Lambda_n(r) \le \epsilon$.

5.2 For FIEs-1K via Laguerre Polynomials

Since Eq. (4.6) and Eq. (3.1) have the following forms given by Eq. (5.4) and Eq. (5.5) respectively:

$$g(r) = \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\sum_{m=0}^{n} (s_m \cdot L_m(x)\right) dx, \tag{5.4}$$

$$J_n(r) = s_0 \cdot L_0(r) + s_1 \cdot L_1(r) + ... + s_n \cdot L_n(r) = \sum_{m=0}^{n} (s_m \cdot L_m(r)), \ 0 \le r < \infty, \tag{5.5}$$

and the unknown coefficients $(s_0, s_1, \ldots, s_n)$ are determined by using Eq. (4.7). Also, by using Eq. (4.2), we have:

$$u(r) \cong J_n(r) = \sum_{m=0}^{n} (s_m \cdot L_m(r)), \tag{5.6}$$

thus, Eq. (5.6) is the unique approximate solution for Eq. (5.4), and is substituted into Eq. (5.4).
Now, assume that $r = r_k \in [0, 1], k = 0, 1, 2, \ldots, n$, and then, the error function:

$$\Lambda(r_k) = \left| g(r_k) - \frac{\beta \cdot r_k^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \int_0^{r_k} \left(r_k^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \sum_{m=0}^{n} (s_m \cdot L_m(x))\, dx \right| \cong 0,$$

then $\Lambda(r_k) \le \epsilon$, for every $r_k \in [0, 1]$ and $\epsilon > 0$.
Then, at each point r_k, the difference for error function $\Lambda(r_k)$ will be less than any positive integer $\epsilon > 0$.
Thus, the following relation can be used to determine the error function, i.e.,

$$\Lambda_n(r) = g(r) - \frac{\beta \cdot r^{-\beta(\eta+\delta)}}{\Gamma(\delta)} \cdot \int_0^r \left(r^\beta - x^\beta\right)^{\delta-1} x^{\beta(\eta+1)-1} \left(\sum_{m=0}^{n} (s_m \cdot L_m(x)\right) dx.$$

Hence, $\Lambda_n(r) \le \epsilon$.

5.3 Convergence Rate:

In this section, the error function can be defined by the following form [35, 26]:

$$\|\Lambda_n\| = \left(\int_0^1 \Lambda_n^2(x) dx\right)^{1/2},$$

where $\|\Lambda_n\|$, is an arbitrary vector norm of error function, $\Lambda_n(r) = u(r) - J_n(r)$, where the exact numerical solution is $u(r)$, and the approximate numerical solution is $J_n(r)$.

Remark 5.1. The same procedure is also suitable for Eq. (2.2) and Eq. (2.3) via Touchard polynomials.

6 Numerical Applications and Discussions:

In this section, five numerical examples have been given to illustrate the efficiency and accuracy of the presented techniques used to find the approximate numerical solutions. Examples 6.1, 6.2, 6.3 and 6.4 are related to FIEs-2K and Example 6.5 is related to FIEs-1K. The approximate numerical solutions have been compared for both the methods and the convergence of solutions have been shown in the graphs.

Example 6.1. Consider the following example of FIE-2K :

$$u(r) = \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r + \frac{r^{-\frac{16}{9}}}{3 \cdot \Gamma(1/3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^{-\frac{2}{3}} x\, u(x)\, dx, \qquad r \in [0,1], \tag{6.1}$$

where the exact solution is $u(r) = r$.

Comparing with Eq. (2.3), we get $g(r) = \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r$, $\delta = \frac{1}{3}$, $\beta = \frac{1}{3}$, $\eta = 5$.

Now applying the algorithm for $n = 2$.

Solving via Touchard polynomials :

Substituting Eq. (3.10) into Eq. (6.1), we have,

$$\begin{aligned}
&\begin{bmatrix} 1 & r & r^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \\
&\quad = \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r \\
&\quad + \frac{r^{-\frac{16}{9}}}{3 \cdot \Gamma(1/3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^{-\frac{2}{3}} x \left(\begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) dx.
\end{aligned} \tag{6.2}$$

Which can be written as,

$$\begin{aligned}
&a_0 + a_1(1 + r) + a_2(1 + 2r + r^2) \\
&\quad = \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r \\
&\quad + \frac{r^{-\frac{16}{9}}}{3 \cdot \Gamma(1/3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^{-\frac{2}{3}} x \left(a_0 + a_1(1 + x) + a_2(1 + 2x + x^2)\right) dx.
\end{aligned} \tag{6.3}$$

Therefore from Eq. (6.3), after computing the integrations and by choosing $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$ in the given interval $[0, 1]$, we are getting three equations:
$0.4393a_0 + 0.4906a_1 + 0.5475a_2 = 0.0513$,
$0.4393a_0 + 0.5419a_1 = 0.6670a_2 = 0.1026$,
$0.4393a_0 + 0.5933a_1 + 0.7976a_2 = 0.1540$.
After solving these equations, we get
$a_0 = -1, a_1 = 1, a_2 = 0$.
Now substituting these values in Eq. (3.7) and the approximate solution is:

$$\Theta_2(r) = (-1) \cdot T_0(r) + 1 \cdot T_1(r) + 0 \cdot T_2(r) = r. \tag{6.4}$$

Solving via Laguerre polynomials :

Substituting Eq. (3.5) into Eq. (6.1) we have,

$$\begin{aligned}
&\begin{bmatrix} 1 & r & r^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} \\
&= \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r \\
&+ \frac{r^{-\frac{16}{9}}}{3 \cdot \Gamma(1/3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^{-\frac{2}{3}} x \left(\begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} \right) dx.
\end{aligned} \tag{6.5}$$

This can be written as,

$$\begin{aligned}
&s_0 + s_1(1-r) + s_2(1 - 2r + \frac{1}{2}r^2) \\
&= \left(1 - \frac{531441}{407550} \cdot \frac{1}{\Gamma(1/3)}\right) \cdot r \\
&+ \frac{r^{-\frac{16}{9}}}{3 \cdot \Gamma(1/3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^{-\frac{2}{3}} x \left(s_0 + s_1(1-x) + s_2(1 - 2x + \frac{1}{2}x^2)\right) dx.
\end{aligned} \tag{6.6}$$

Therefore from Eq. (6.6), after computing the integrations and by choosing $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$ in the given interval $[0, 1]$, we are getting three equations:

$0.4393s_0 + 0.3880s_1 + 0.3394s_2 = 0.0513,$
$0.4393s_0 + 0.3367s_1 + 0.2452s_2 = 0.1026,$
$0.4393s_0 + 0.2853s_1 + 0.1565s_2 = 0.1540.$

After solving these equations, we get
$s_0 = 1, s_1 = -1, s_2 = 0$.

Now substituting these values in Eq. (3.1) and the approximate solution is:

$$J_2(r) = 1 \cdot L_0(r) + (-1) \cdot L_1(r) + 0 \cdot L_2(r) = r. \tag{6.7}$$

The comparison of the approximate solutions shows that both solutions are same for both the proposed methods. Also this comparison shows that for $n = 2$ the approximate solution is equal to the exact solution. Figure 1 shows the comparison for $n = 2$ with the exact solution. The error functions are zero in this case for both presented methods.

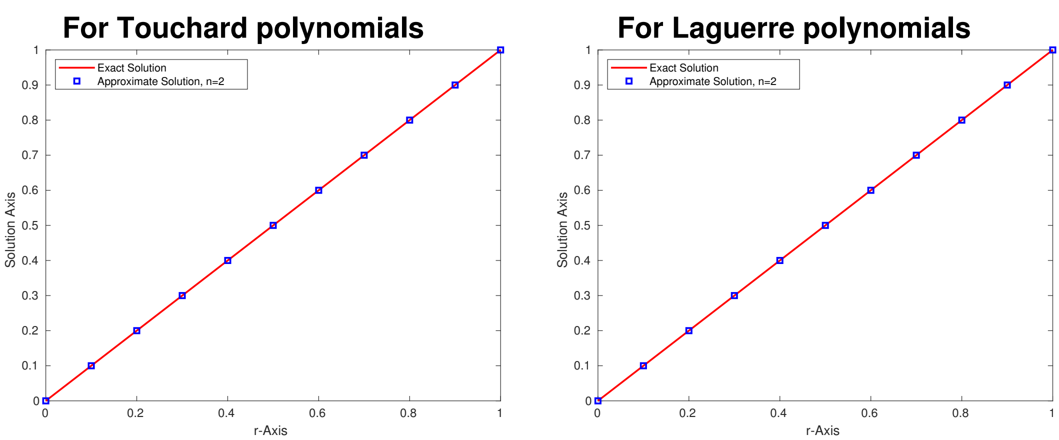

Figure 1. Graphical Representation of Exact Solution and Approx. Solution for Example 6.1

Example 6.2. Consider the following example for FIE-2K:

$$u(r) = g(r) + \frac{2 \cdot r^{-7}}{\Gamma(3)} \cdot \int_0^r \left(r^2 - x^2\right)^2 x^2 \, u(x) \, dx, \qquad r \in [0, 1], \tag{6.8}$$

with

$$g(r) = \left(1 - 720 \cdot r^{-6} + 72 \cdot r^{-4}\right) \sin r - \left(720 \cdot r^{-7} - 312 \cdot r^{-5} + 8 \cdot r^{-3}\right) \cos r + \left(720 \cdot r^{-7} + 48 \cdot r^{-5} + 2 \cdot r^{-3}\right).$$

The exact solution of this problem is $u(r) = \sin r$.

Comparing with Eq. (2.3), we get $\delta = 3$, $\beta = 2$, $\eta = \frac{1}{2}$.

Now, by applying the same algorithm for $n = 2, 3$ and by choosing $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$, $r_3 = 0.4$ in the given interval $[0, 1]$ and computing all steps, we get the approximate solutions as follows:

For Touchard polynomials method :

$u(r) \cong \Theta_2(r) = (-1.1139) \cdot T_0(r) + (1.2095) \cdot T_1(r) - (0.0964) \cdot T_2(r)$,
$u(r) \cong \Theta_3(r) = (-0.8409) \cdot T_0(r) + (0.5205) \cdot T_1(r) + (0.4818) \cdot T_2(r) - (0.1614) \cdot T_3(r)$.

For Laguerre polynomials method :

$u(r) \cong J_2(r) = (0.8175) \cdot L_0(r) - (0.6187) \cdot L_1(r) - (0.1998) \cdot L_2(r)$,
$u(r) \cong J_3(r) = (0.2199) \cdot L_0(r) + (1.2731) \cdot L_1(r) - (2.1963) \cdot L_2(r) + (0.7024) \cdot L_3(r)$.

In this example, the best approximation has been observed for $n = 3$ using Touchard polynomials. Also by using Laguerre polynomials, approximate solution has been observed for $n = 3$. The error function provides the value for Touchard polynomials is 0.0016 and for Laguerre polynomials the value of the error function is 0.0027. Figure 2 shows the comparison for the both presented methods and the figures are seem to be identical.

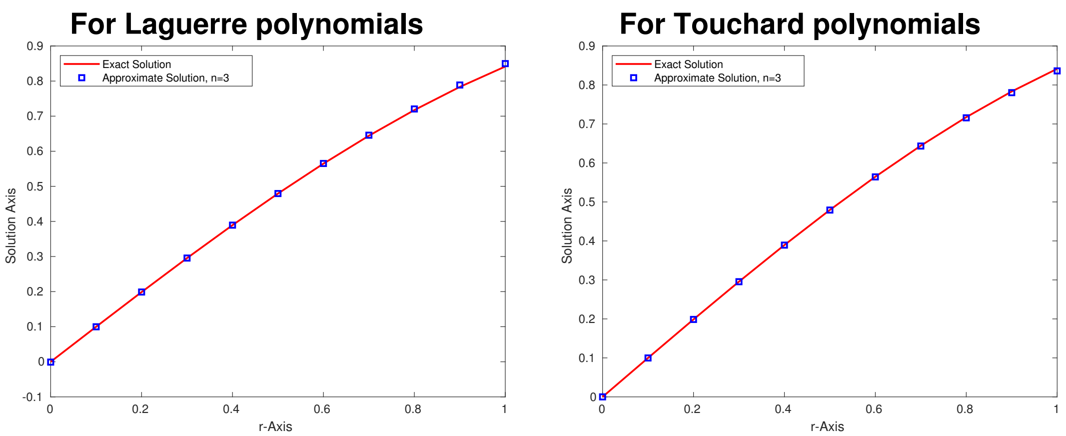

Figure 2. Graphical Representation of Exact Solution and Approx. Solution for Example 6.2

Example 6.3. Consider the following example for FIE-2K :

$$u(r) = \left(\Gamma(1/5) - \frac{78125}{24024}\right) \cdot r + \frac{r^{-\frac{8}{5}}}{2 \cdot \Gamma(1/5)} \cdot \int_0^r \left(r^{1/2} - x^{1/2}\right)^{-\frac{4}{5}} x\, u(x)\, dx, \qquad r \in [0, 1], \tag{6.9}$$

whose exact solution is $u(r) = \Gamma(1/5) \cdot r$.

Comparing with Eq. (2.3), we have $g(r) = \left(\Gamma(1/5) - \frac{78125}{24024}\right) \cdot r$, $\delta = \frac{1}{5}$, $\beta = \frac{1}{2}$, $\eta = 3$.

Now, by applying the same algorithm for $n = 2$ and by choosing $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$ in the given interval $[0, 1]$ and computing all steps, we get the approximate solutions for Touchard polynomials method and Laguerre polynomials method respectively as follows:

$u(r) \cong \Theta_2(r) = (-4.5836) \cdot T_0(r) + (4.5856) \cdot T_1(r) + 0 \cdot T_2(r)$,

$u(r) \cong J_2(r) = (4.7092) \cdot L_0(r) - (4.8541) \cdot L_1(r) + (0.1471) \cdot L_2(r).$

In this example, the best approximation has been observed for $n = 2$ using Touchard polynomials. Also by using Laguerre polynomials, approximate solution has been observed for $n = 2$. The error function gives the value for Touchard polynomials is 0.0016 and for Laguerre polynomials the value of the error function is 0.0175. Figure 3 shows the comparison for the both presented methods and the figures are seem to be identical.

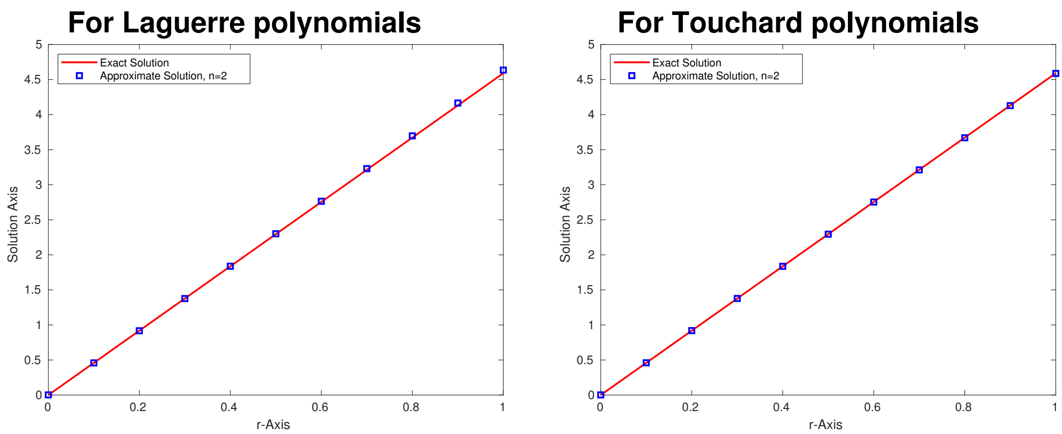

Figure 3. Graphical Representation of Exact Solution and Approx. Solution for Example 6.3

Example 6.4. Consider the following example for FIE-2K :

$$u(r) = \left(1 - \frac{1953125}{638352} \cdot \frac{1}{\Gamma(1/5)}\right) \cdot r^2 + \frac{r^{-\frac{8}{5}}}{2 \cdot \Gamma(1/5)} \cdot \int_0^r \left(r^{1/2} - x^{1/2}\right)^{-\frac{4}{5}} x\, u(x)\, dx, \qquad r \in [0,1], \tag{6.10}$$

whose exact solution is $u(r) = r^2$

Comparing with Eq. (2.3), we have $g(r) = \left(1 - \frac{1953125}{638352} \cdot \frac{1}{\Gamma(1/5)}\right) \cdot r^2$, $\delta = \frac{1}{5}$, $\beta = \frac{1}{2}$, $\eta = 3$.

Now, by applying the same algorithm for $n = 2$ and by choosing $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$ in the given interval $[0, 1]$ and computing all steps, we get the approximate solutions for Touchard polynomials method and Laguerre polynomials method respectively as follows:
$u(r) \cong \Theta_2(r) = (1.0203) \cdot T_0(r) - (2.0355) \cdot T_1(r) + (1.0152) \cdot T_2(r),$
$u(r) \cong J_2(r) = (2.1282) \cdot L_0(r) - (4.2855) \cdot L_1(r) + (2.1602) \cdot L_2(r).$

In this example, the best approximation has been observed for $n = 2$ using Touchard polynomials. Also by using Laguerre polynomials, approximate solution has been observed for $n = 2$. The error function gives the value for Touchard polynomials is 0.0040 and for Laguerre polynomials the value of the error function is 0.0188. Figure 4 shows the comparison for the both presented methods. It can be observed from the figure and error function is that, the approximate numerical solutions for Touchard polynomials method is highly accurate than Laguerre polynomials method for the same degree of polynomials.

Example 6.5. Consider the following example for FIE-1K :

$$\left(\frac{1}{495 \cdot \Gamma(3)}\right) \cdot r = \frac{r^{-\frac{8}{3}}}{3 \cdot \Gamma(3)} \cdot \int_0^r \left(r^{1/3} - x^{1/3}\right)^2 x\, u(x)\, dx, \qquad r \in [0,1], \tag{6.11}$$

whose exact solution is $u(r) = r$.

Comparing with Eq. (2.2), we have $g(r) = \left(\frac{1}{495 \cdot \Gamma(3)}\right) \cdot r$, $\delta = 3$, $\beta = \frac{1}{3}$, $\eta = 5$.

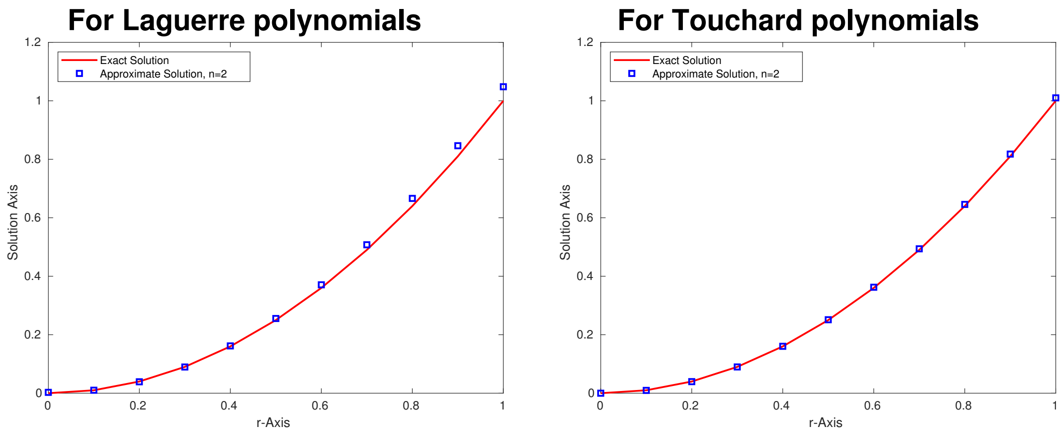

Figure 4. Graphical Representation of Exact Solution and Approx. Solution for Example 6.4

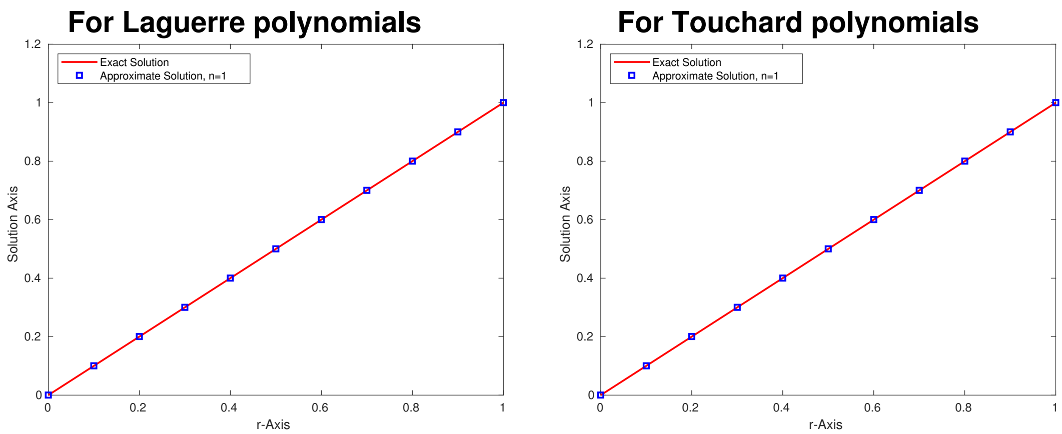

Figure 5. Graphical Representation of Exact Solution and Approx. Solution for Example 6.5

Now, by applying the same algorithm for $n = 1$ and by choosing $r_0 = 0.1$, $r_1 = 0.001$ in the given interval $[0, 1]$ and computing all steps, we get the approximate solutions for Touchard polynomials method and Laguerre polynomials method respectively as follows:

$$u(r) \cong \Theta_1(r) = (-0.9997) \cdot T_0(r) + (1.0000) \cdot T_1(r),$$
$$u(r) \cong J_1(r) = (1.0003) \cdot L_0(r) - (1.0000) \cdot L_1(r).$$

In this example, the best approximation has been observed for $n = 1$ using Touchard polynomials. Also by using Laguerre polynomials, approximate solution has been observed for $n = 1$. The error functions give the same value for both the presented methods and the value of the error function is $3.0000e-04$. Figure 5 shows the comparison for the both presented methods and the figures are seem to be identical.

7 Conclusions and Future Work

In this study, two effective methods based on the Laguerre polynomials and Touchard polynomials have been used to get approximate numerical solutions for both FIEs-1K and FIEs-2K. Fractional integrals are stated in the sense of generalized Erdélyi–Kober fractional integral. We have provided five examples to compare the accuracy of solutions for both presented methods. In Example 6.1, the approximate solutions were exactly the same as exact solution for both methods. It was observed that, for Example 6.5 the value of the error function is same for both methods. Example 6.2 to Example 6.4 have shown that Touchard polynomials method is better than Laguerre polynomials method as the error is less for Touchard polynomials method than Laguerre polynomials method. Also, the comparisons of approximate solutions with the exact solution were shown by the relevant figures, and all figures seemed to be identical for both the presented methods. Moreover, we realized that when we increase the degree of these polynomials, there is less error. This method is simple and effective.

In the future, one can be investigate by using these two methods for the approximate solutions of Volterra-Fredholm type fractional integral equations. Also, one can apply to examine the effectiveness of these proposed methods for the solutions of generalized fractional integral operator, i.e., (k, s)-fractional integral operator [34].

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