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Generalized relative order (α, β) oriented some growth analysis of composite *p*-adic entire functions

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Abstract In this paper we wish to establish some results relating to the growths of composition of two *p*-adic entire functions with their corresponding left and right factors on the basis of their generalized relative order (α, β) and generalized relative lower order (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1 Introduction, Definitions and Notations

Let us consider an algebraically closed field $\mathbb{K}$ of characteristic zero complete with respect to a *p*-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$) . For any $\xi \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x - \xi| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \xi| < R\}$ are denoted by $d(\xi, R)$ and $d(\xi, R^-)$ respectively. Also $C(\xi, r)$ denotes the circle $\{x \in \mathbb{K} : |x - \xi| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [21, 22, 23, 25]. During the last several years the ideas of *p*-adic analysis have been studied from different aspects and many important results were gained (see [2] to [6], [9] to [20]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r, therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the (p, q)-th order and (p, q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

Definition 1.1. [11] Let $f \in \mathcal{A}(\mathbb{K})$ and p, q are any two positive integers. Then the (p, q)-th order $\varrho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of f are respectively defined as:

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}.$$

Definition 1.1 avoids the restriction $p \geq q$ of the original definition of (p, q)-th order (respectively (p, q)-th lower order) of entire functions introduced by Juneja et al. [24] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\varrho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\varrho^{(2,1)}(f) = \varrho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\varrho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [17].

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$, i.e., α is a slowly increasing function. Clearly $L^0\subset L$.

The concept of generalized order (α,β) of entire function in complex context was introduced by Sheremeta [26] where $\alpha,\beta\in L$. In complex context, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications of generalized order (α,β) of entire function in complex context, Biswas et al. [7, 11] rewrite the definition of generalized order (α,β) of an entire function considering $\alpha,\beta\in L^0$. For details about generalized order (α,β) and generalized lower order (α,β), one may see [7, 11]. Considering the ideas developed by Biswas et al. [7, 11], one can define the generalized order (α,β) and generalized lower order (α,β) of an entire function $f\in\mathcal{A}(\mathbb{K})$ respectively in the following way:

Definition 1.2. [5] Let $f\in\mathcal{A}(\mathbb{K})$ and $\alpha,\ \beta\in L^0$. The generalized order (α,β) and generalized lower order (α,β) of f denoted by $\varrho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(|f|(r))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(|f|(r))}{\beta(r)}.$$

If $\alpha(r)=\log^{[p]}r$ and $\beta(r)=\log^{[q]}r$, then Definition 1.1 is a special case of Definition 1.2.

The notion of relative order was first introduced by Bernal [1]. In order to make some progresses in the study of *p*-adic analysis, Biswas [10] introduced the definitions of relative order $\varrho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f\in\mathcal{A}(\mathbb{K})$ with respect to another entire function $g\in\mathcal{A}(\mathbb{K})$ in the following way:

$$\varrho_g(f)=\limsup_{r\to+\infty}\frac{\log\widehat{|g|}(|f|(r))}{\log r}\text{ and }\lambda_g(f)=\liminf_{r\to+\infty}\frac{\log\widehat{|g|}(|f|(r))}{\log r}.$$

In the case of relative order, it therefore seems reasonable to define suitably the generalized relative order (α,β) of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function $f\in\mathcal{A}(\mathbb{K})$ with respect to another entire function $g\in\mathcal{A}(\mathbb{K})$ denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively, in the follows way:

Definition 1.3. [4] Let $f,\ g\in\mathcal{A}(\mathbb{K})$ and $\alpha,\ \beta\in L^0$. The generalized relative order (α,β) and generalized relative lower order (α,β) of f with respect to g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f]_g=\limsup_{r\to+\infty}\frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]_g=\liminf_{r\to+\infty}\frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)}.$$

The main aim of this paper is to establish some newly developed results related to the growth rates of composition of two *p*-adic entire functions on the basis of generalized relative order (α,β) and generalized relative lower order (α,β) where $\alpha,\ \beta\in L^0$. Further we assume that throughout the present paper α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 always denote the functions belonging to L^0.

2 Lemma

In this section we present the following lemma which can be found in [16] or [17] and will be needed in the sequel.

Lemma 2.1. *Let $f,g\in\mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f(g)|(r)=|f|(|g|(r)).$$

3 Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let* f, g, h, $k \in \mathcal{A}(\mathbb{K})$ *be such that* $\varrho_{(\alpha_1,\beta_1)}[f(g)]_h < \infty$ *and* $\lambda_{(\alpha_3,\beta_3)}[g]_k > 0$. *Then*

$$\lim_{r\to+\infty} \frac{\{\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r))))\}^2}{\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(\log r)))) \cdot \alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(r))))} = 0.$$

Proof. For arbitrary positive ε we have for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r)))) \le (\varrho_{(\alpha_1,\beta_1)}[f(g)]_h + \varepsilon)\log r. \tag{3.1}$$

Again for all sufficiently large values of r we get

$$\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(\log r))) \ge (\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon)\log r. \tag{3.2}$$

Similarly for all sufficiently large values of r we have

$$\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(r)))) \ge (\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) r. \tag{3.3}$$

From (3.1) and (3.2) we have for all sufficiently large values of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r))))}{\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(\log r)))} \le \frac{(\varrho_{(\alpha_1,\beta_1)}[f(g)]_h + \varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon)\log r}.$$

As $\varepsilon(>0)$ is arbitrary we obtain from above that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r))))}{\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(\log r)))} \le \frac{\varrho_{(\alpha_1,\beta_1)}[f(g)]_h}{\lambda_{(\alpha_3,\beta_3)}[g]_k}. \tag{3.4}$$

Again from (3.1) and (3.3) we get for all sufficiently large values of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r))))}{\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(r))))} \le \frac{(\varrho_{(\alpha_1,\beta_1)}[f(g)]_h + \varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) r}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\lim_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_1^{-1}(\log r))))}{\alpha_3(\widehat{|k|}(|g|(\beta_3^{-1}(r))))} = 0. \tag{3.5}$$

Thus the theorem follows from (3.4) and (3.5). □

Theorem 3.2. *Let* f, g, h, k, l, $m \in \mathcal{A}(\mathbb{K})$ *be such that* $\varrho_{(\alpha_1,\beta_1)}[f]_l < +\infty$, $\lambda_{(\alpha_3,\beta_3)}[h]_m > 0$, $\lambda_{(\alpha_4,\beta_4)}[k] > 0$ *and* $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$. *Also let* C *and* D *be any two positive constants.*
(i) Any one of the following four conditions are assumed to be satisfied:
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*
(b) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*;*
(c) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*
(d) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*; then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\widehat{|m|}(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r))))} = \infty.$$

(ii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*;*
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*; then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\widehat{|m|}(|h(k)|(\beta_4^{-1}(\log r)))))))}{\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r))))} = \infty.$$

(iii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_3(r) = D\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L^0$*;*
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L^0$*; then*

$$\lim_{r\to+\infty} \frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))} = \infty.$$

(iv) If $\alpha_2(\beta_1^{-1}(r)) \in L_1$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))} = \infty.$$

Proof. Since $|\widehat{h}|(r)$ is an increasing function of r, it follows from Lemma 2.1 and for all sufficiently large values r that

$$\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)\beta_1(|g|(\beta_2^{-1}(\log r))). \tag{3.6}$$

Case I. Let $\beta_1(r) = C(\exp(\alpha_2(r)))$. Then we have from (3.6) for all sufficiently large values of r that

$$\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant C(\varrho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.7}$$

Case II. Let $\exp(\alpha_2(r)) > \beta_1(r)$. Then we have from (3.6) for all sufficiently large values of r that

$$\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.8}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then we get from(3.6) for all sufficiently large values of r that

$$\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))))) \leqslant r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.9}$$

Further in view of Lemma 2.1 for all sufficiently large values r that

$$\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)\beta_3(|\widehat{k}|(\beta_4^{-1}(\log r))). \tag{3.10}$$

Case IV. Let $\beta_3(r) = D\exp(\alpha_4(r))$ Then from (3.10) it follows for all sufficiently large values of r that

$$\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))) \geq D(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.11}$$

Case V. Let $\beta_3(r) > \exp(\alpha_4(r))$. Now from (3.10) it follows for all sufficiently large values of r that

$$\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))) > (\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.12}$$

Case VI. Let $\alpha_4(\beta_3^{-1}(r)) \in L^0$. Then from (3.10) we obtain for all sufficiently large values of r that

$$\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))))) \geqslant r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.13}$$

Since $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$ we can choose $\varepsilon(> 0)$ in such a way that

$$\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_4,\beta_4)}[k] - \varepsilon. \tag{3.14}$$

Now combining (3.7) of Case I and (3.11) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))} \geq \frac{D(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(\varrho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (3.14) and above we obtain that

$$\liminf_{r\to+\infty} \frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))} = \infty. \tag{3.15}$$

Similarly combining (3.7) of Case I and (3.12) of Case V we get that

$$\liminf_{r\to+\infty}\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))}=\infty. \tag{3.16}$$

Analogously combining (3.8) of Case II and (3.11) of Case IV, we obtain that

$$\lim_{r\to+\infty}\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))}=\infty. \tag{3.17}$$

Likewise combining (3.8) of Case II and (3.12) of Case V it follows that

$$\lim_{r\to+\infty}\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))}=\infty. \tag{3.18}$$

Hence the first part of the theorem follows from (3.15), (3.16), (3.17) and (3.18).

Again combining (3.7) of Case I and (3.13) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))}\geq\frac{r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(\varrho_{(\alpha_1,\beta_1)}[f]_l+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}$$

So from (3.14) and above we obtain that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))))))}{\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r))))}=\infty.$$

Similarly combining (3.8) of Case II and (3.13) of Case VI we also get same conclusion. Therefore the second part of the theorem is established.

Again combining (3.9) of Case III and (3.11) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))}\geq\frac{D(\lambda_{(\alpha_3,\beta_3)}[h]_m-\varepsilon)r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}. \tag{3.19}$$

Now in view of (3.14) we obtain from (3.19) that

$$\lim_{r\to+\infty}\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))}=\infty. \tag{3.20}$$

Similarly combining (3.9) of Case III and (3.12) of Case V we get that

$$\lim_{r\to+\infty}\frac{\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))}=\infty. \tag{3.21}$$

Hence the third part of the theorem follows from (3.20) and (3.21).

Further combining (3.9) of Case III and (3.13) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))}\geq\frac{r^{(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Now in view of (3.14) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(|\widehat{m}|(|h(k)|(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(|\widehat{l}|(|f(g)|(\beta_2^{-1}(\log r)))))))}=\infty.$$

This proves the fourth part of the theorem.

Thus the theorem follows. □

Theorem 3.3. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \le \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $\varrho_{(\alpha_2,\beta_2)}[g] > 0$. If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \ge \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definition of $\varrho_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r)))) \le (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)r. \tag{3.22}$$

Further in view of Lemma 2.1, it follows for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r)))) \ge (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(|g|(\beta_2^{-1}(r))).$$

Since $\alpha_2(\beta_1^{-1}(r)) \in L^0$, we obtain from above for a sequence of values of r tending to infinity that

$$\begin{aligned} \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r)))))) &\ge (1+o(1))\alpha_2(|g|(\beta_2^{-1}(r))) \\ i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r)))))) &\ge (1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g] - \varepsilon)r. \end{aligned}$$

Now combining (3.22) and above we get that

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \ge \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}.$$

Hence the theorem follows. □

Theorem 3.4. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \le \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \ge \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}.$$

Theorem 3.5. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \ge \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

The proofs of Theorem 3.4 and Theorem 3.5 would run parallel to that of Theorem 3.3. We omit the details.

Theorem 3.6. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \le \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $\varrho_{(\alpha_2,\beta_2)}[g] < +\infty$. If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \le \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r)))) \ge (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r. \tag{3.23}$$

Further in view of Lemma 2.1, we obtain for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(|g|(r)). \tag{3.24}$$

Since $\alpha_2(\beta_1^{-1}(r)) \in L^0$, we obtain from above for all sufficiently large values of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r)))))) \leq (1+o(1))\alpha_2(|g|(\beta_2^{-1}(r)))$$
$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r)))))) \leq (1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)r.$$

Now combining (3.23) and above we get that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \leq \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Hence the theorem follows. □

Theorem 3.7. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ *and* $\lambda_{(\alpha_2,\beta_2)}[g] < +\infty$. *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \leq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Theorem 3.8. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ *and* $\varrho_{(\alpha_2,\beta_2)}[g] < +\infty$. *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(r))))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(r))))} \leq \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}.$$

The proofs of Theorem 3.7 and Theorem 3.8 would run parallel to that of Theorem 3.6. We omit the details.

Theorem 3.9. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *be such that* $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h$. *Also let* C *be any positive constant.*
(i) *Any one of the following two conditions are assumed to be satisfied:*
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$;
(b) $\exp(\alpha_2(r)) > \beta_1(r)$; *then*

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))\cdot\beta_1(|g|(\beta_2^{-1}(\log r)))} = 0.$$

(ii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))\cdot\beta_1(|g|(\beta_2^{-1}(\log r)))} = 0.$$

Proof. From the definition of generalized relative lower order (α_1,β_1) of f with respect to h, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))) \geq r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}. \tag{3.25}$$

As $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h$ we can choose $\varepsilon(>0)$ in such a way that

$$\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon. \tag{3.26}$$

Now in view of (3.7) of Case I and (3.25) we have for all large positive numbers of r,

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} \leq \frac{C(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}.$$

In view of (3.26) we get from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} = 0. \tag{3.27}$$

Again in view of (3.8) of Case II and (3.25) it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} \le \frac{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon) r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)}}.$$

Now in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} = 0. \tag{3.28}$$

Further in view of (3.9) of Case III and (3.25) it follows for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} \le \frac{r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)}}.$$

So in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} = 0. \tag{3.29}$$

Now in view of (3.6) we get that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\beta_1(|g|(\beta_2^{-1}(\log r)))} \le \varrho_{(\alpha_1,\beta_1)}[f]_h. \tag{3.30}$$

From (3.27) and (3.30) we obtain for all sufficiently large values of r that

$$\begin{aligned}
&\limsup_{r\to+\infty} \frac{\{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))) \cdot \beta_1(|g|(\beta_2^{-1}(\log r)))} \\
= &\lim_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} \cdot \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\beta_1(|g|(\beta_2^{-1}(\log r)))} \\
\le &\ 0 \cdot \varrho_{(\alpha_1,\beta_1)}[f]_h = 0.
\end{aligned} \tag{3.31}$$

Similarly from (3.28) and (3.30) we obtain that

$$\limsup_{r\to+\infty} \frac{\{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))) \cdot \beta_1(|g|(\beta_2^{-1}(\log r)))} = 0.$$

Therefore the first part of the theorem follows from (3.31) and above.
Again from (3.29) and (3.30) we get for all large values of r that

$$\begin{aligned}
&\limsup_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r))))))) \cdot \alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))) \cdot \beta_1(|g|(\beta_2^{-1}(\log r)))} \\
= &\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))} \cdot \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\beta_1(|g|(\beta_2^{-1}(\log r)))} \\
\le &\ 0 \cdot \varrho_{(\alpha_1,\beta_1)}[f]_h = 0.
\end{aligned}$$

$$i.e., \lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|l|}(|f(g)|(\beta_2^{-1}(\log r))))))) \cdot \alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))) \cdot \beta_1(|g|(\beta_2^{-1}(\log r)))} = 0.$$

Thus the second part of the theorem is established. □

Theorem 3.10. *Let f, g, h, l, $k \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\varrho_{(\alpha_3,\beta_3)}[f(g)]_l < \infty$. Then*

$$\limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \cdot \alpha_3(\widehat{|l|}(|f(g)|(\beta_3^{-1}(r)))}{\beta_1(|g|(\beta_2^{-1}(\log r))) \cdot \alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(r)))} \leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]_l \cdot \varrho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

Proof. For all sufficiently large values of r we have

$$\alpha_3(\widehat{|l|}(|f(g)|(\beta_3^{-1}(r))) \leq (\varrho_{(\alpha_3,\beta_3)}[f(g)]_l + \varepsilon)r. \tag{3.32}$$

Again for all sufficiently large values of r it follows that

$$\alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(r))) \geq (\lambda_{(\alpha_2,\beta_2)}[g]_k - \varepsilon)r. \tag{3.33}$$

Now combining (3.32) and (3.33) we have for all sufficiently large values of r that

$$\frac{\alpha_3(\widehat{|l|}(|f(g)|(\beta_3^{-1}(r)))}{\alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(r)))} \leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]_l + \varepsilon}{\lambda_{(\alpha_2,\beta_2)}[g]_k - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_3(\widehat{|l|}(|f(g)|(\beta_3^{-1}(r)))}{\alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(r)))} \leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]_l}{\lambda_{(\alpha_2,\beta_2)}[g]_k}. \tag{3.34}$$

Now from(3.30) and (3.34) we obtain that

$$\begin{aligned}
&\limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \cdot \alpha_3(\widehat{|l|}(|f(g)|(\beta_3^{-1}(r)))}{\beta_1(|g|(\beta_2^{-1}(\log r))) \cdot \alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(r)))} \\
\leq\ & \limsup_{r\to+\infty}\frac{\alpha_1(|f(g)|(\beta_2^{-1}(\log r)))}{\beta_1(|g|(\beta_2^{-1}(\log r)))} \cdot \limsup_{r\to+\infty}\frac{\alpha_3(|f(g)|(\beta_3^{-1}(r)))}{\alpha_2(|g|(\beta_2^{-1}(r)))} \\
\leq\ & \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]_l \cdot \varrho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.
\end{aligned}$$

Hence the theorem follows. □

Theorem 3.11. *Let f, g, h, $k \in \mathcal{A}(\mathbb{K})$ be such that $\varrho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\lambda_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha_3(\widehat{|k|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\eta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(\widehat{|k|}(|f(g)|(r))) \leq \eta \cdot \alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_3(r))))). \tag{3.35}$$

Again from the definition of $\varrho_{(\alpha_1,\beta_1)}[f]_h$, it follows for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_3(r))))) \leq (\varrho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r). \tag{3.36}$$

Thus from (3.35) and (3.36), we have for a sequence of values of r tending to infinity that

$$\begin{aligned}
&\alpha_3(\widehat{|k|}(|f(g)|(r))) \leq \eta(\varrho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r) \\
i.e.,\ &\frac{\alpha_3(\widehat{|k|}(|f(g)|(r)))}{\beta_3(r)} \leq \frac{\eta(\varrho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r)}{\beta_3(r)} \\
i.e.,\ &\liminf_{r+\infty}\frac{\alpha_3(\widehat{|k|}(|f(g)|(r)))}{\beta_3(r)} = \lambda_{(\alpha_3,\beta_3)}[f(g)]_k < \infty.
\end{aligned}$$

This is a contradiction.

Thus the theorem follows. □

Remark 3.12. Theorem 3.11 is also valid with "limit superior" instead of "limit" if $\lambda_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$ is replaced by $\varrho_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$ and the other conditions remain the same.

Analogously one may also state the following theorem without its proof as it may be carried out in the line of Theorem 3.11.

Theorem 3.13. *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$ *be such that* $\varrho_{(\alpha_1,\beta_1)}[g]_h < \infty$ *and* $\varrho_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$. *Then*

$$\limsup_{r\to+\infty} \frac{\alpha_3(\widehat{|k|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|g|(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Remark 3.14. Theorem 3.13 is also valid with "limit" instead of "limit superior" if $\varrho_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$ is replaced by $\lambda_{(\alpha_3,\beta_3)}[f(g)]_k = \infty$ and the other conditions remain the same.

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