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# On Endomorphism of $FI$-Semi-Injective Modules

S. Chase and M. K. Patel

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**Abstract:** In this article, we study the properties of endomorphism ring of $FI-$semi injective modules. As a result, we observed that, finite-dimensional $FI-$Semi injective module has a semi local endomorphism ring and endomorphism ring of an epi-retractable $R-$module $M$, whose submodules are $FI-M-$principally injective, is right principally projective. Further we prove that, extending $FI-$semi-injective module $M$ is co-Hopfian if and only if it satisfies the cancellation property.

## 1 Introduction

Structures and various properties of principally injective rings and principally injective modules has been studied in detail by many authors ([11], [14]), in the past. It may be recalled, for a ring $R$, if each homomorphism from a principal right ideal of $R$ to $R$ is given by a multiplication by an element of $R$ on the left, then $R$ is called principally injective. Along the same line, a module $M$ is called principally injective if each homomorphism $f : aR \to M, a \in R$, extends to $R$. This notion was extended to modules by Sanh et.al.[15], and they came up with $M-$principal injectivity for a given right $R-$module $M$ as a generalization to the idea of principal injectivity. It may be recalled that for a right $R-$module $N$, if every homomorphism from an $M-$cyclic submodule $s(M)$ of $M$ to $N$ can be extended to a homomorphism from $M$ to $N$, then it is called $M-$principally injective. If $M$ is $M-$principally injective, it is called semi injective (see [13], [15], [18]). In our study, we shall consider those $M-$cyclic submodule $s(M)$ of $M$ that are fully invariant. This will further extend the concept to $FI-$semi-injective modules from semi-injective modules. Similarly, the concept of $FI-M-$principally injective ($FI-$semi-injective) module are introduced as a proper generalization of $M-$principally injective (semi-injective) module respectively. Thus the class of $FI-$semi-injective module is larger than that of the semi-injective modules, and the following implication is obtained:

Injective ⇒ Quasi-injective ⇒ Semi-injective ⇒ $FI-$semi-injective module

In the following discussion, $R$ will denote an associative ring with unity and the $R-$modules are unitary right $R$-modules. We shall use the notation $N \subseteq M$ to mean that $N$ is a submodule of $M$. For $N \subseteq M$, if $N \cap L = 0$ implies $L = 0$, then $N$ is known as an essential submodule of $M$, denoted by $N \subseteq_e M$. For a nonzero module $M$ if every submodule of $M$, other than the zero submodule, is essential in $M$, then it is called uniform. If there exists an epimorphism from $M^{(I)}$ to $N$, where $I$ is any index set, then $N$ is called $M-$generated. $N$ is called finitely $M$-generated in the case when $I$ is finite. More particularly, $N$ is called an $M$-cyclic submodule of $M$ if the index set $I$ is a singleton set, equivalently if $N \cong M/L$ for some submodule $L$ of $M$. If $s(K) \subseteq K$ for all $s \in End(M_R)$, then $K$ is called fully invariant. A module is termed as duo if all of its submodules are fully invariant. A ring is called a duo ring if the ring, considered as a module over itself, is a duo module. A module which generates all its submodules is called a self generator module. When submodules of a module are totally ordered by inclusion, then the module is called a unisesrial module. A module is called Hopfian (resp., co-Hopfian), if every surjective (resp., injective) endomorphism of the module is an automorphism. If a module is such that every injective endomorphism is essential, then the module is called weakly co-Hopfian.

From [10], the following conditions may be recalled for an $R-$module $M$:

$(C_1)$ In a direct summand of $M$, every submodule of $M$ is essential or large.

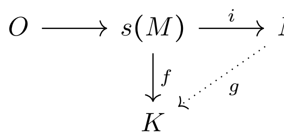
$(C_2)$ A submodule of $M$ is a direct summand of $M$ whenever it is isomorphic to a direct summand of $M$.

$(C_3)$ Direct sum of two zero intersection direct summands is a direct summand of $M$.

An $R-$module $M$ is called extending (or CS) if and only if every proper direct summand of $M$ is a closed submodule or it satisfies $(C_1)$, continuous if it satisfies $(C_1)$ with $(C_2)$ and quasi-continuous if it satisfies $(C_1)$ with $(C_3)$. An $R$-module $M$ satisfies $(C_4)$ if, whenever $H$ and $K$ are submodules of $M$ with $M = H \oplus K$ and $f : H \to K$ is an $R$-homomorphism with $kerf \subseteq^{\oplus} H$, then $Imf \subseteq^{\oplus} K$. If an $R$-module $M$ satisfies $(C_4)$, it is called a $C_4$-module. An extending $C_4$-module is called a pseudo-continuous module.

## 2 *FI*−*M*− Principal Injectivity

If there is an extended homomorphism from $M$ to $K$ for every homomorphism from $s(M)$ to $K$, $s \in End(M_R)$ and $s(M)$ is an $M-$cyclic submodule of $M$ that is fully invariant, then $K$ is termed fully invariant$-M-$principally injective (briefly, $FI - M-$principally injective or $FI - M-$p-injective).



That is to say that, $K$ is $FI - M-$p-injective, whenever $s(M)$ is an $M-$cyclic submodule of $M$ which is also fully invariant, every homomorphism $f : s(M) \to K$, can be splitted as $f = g \circ i$, where the homomorphism from $M$ to $K$ is denoted by $g$ and the inclusion map to $M$ from $s(M)$ is denoted by $i$. If $K$ is $FI - R-$principally injective then $K$ is called $FI-$principally injective. As examples, we may take $\mathbb{Z}_4$ and $\mathbb{Z}_6$ as modules over $\mathbb{Z}$. It can be verified that $\mathbb{Z}_4$ is $FI - \mathbb{Z}_6-$p-injective and $\mathbb{Z}_6$ is $FI - \mathbb{Z}_4-$p-injective. If $M$ is $FI - M-$p-injective, then $M$ is called $FI-$semi-injective. For a ring $R$, if $R_R$ is $FI - R-$p-injective then $R$ is called right $FI-$self-p-injective. As examples, we can see that $\mathbb{Z}_4$ and $\mathbb{Z}_6$, considered as modules over $\mathbb{Z}$, are $FI-$semi-injective modules, whereas $\mathbb{Z}$ is not $FI-$semi-injective over itself. In fact, examples of $FI-$ semi-injective modules consists of modules that are simple, semisimple, semi-injective, quasi-injective, $FI-$self-p-injective rings and direct summands of these.

**Proposition 2.1.** *A uniserial duo $FI$-semi injective module is co-Hopfian.*

**Proof**: Suppose $M$ is a uniserial duo $FI-$semi injective module. Let us consider an injective endomorphism $f$ on $M$. As $M$ is duo $FI$-semi-injective, $f$ splits i.e., $f(M) \subset^{\oplus} M$. Since $M$ is uniserial, $M$ is indecomposable and hence $f(M) = M$, therefore $f$ is automorphism and hence $M$ is co-Hopfian.

**Corollary 2.2.** *A uniserial duo $FI$-semi-injective module is weakly co-Hopfian.*

**Proposition 2.3.** *For a duo $FI$-semi-injective module $M$, $M$ is non-co-Hopfian if and only if there exists a decomposition $M = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for all $r \in \mathbb{Z}^+$, where $N_r \cong M$ and $M_i \neq 0$ for $i = 1, 2, 3, \ldots, r$.*

**Proof**: Firstly, consider that $M$ is non-co-Hopfian then we have a homomorphism $f : M \to M$ which is a monomorphism but not an isomorphism. Let $N_1 = f(M), N_1 \neq M$ and $g : N_1 \to M$ be an isomorphism. Then by $FI$-semi-injectivity of $M$, $g$ can be extended to a homomorphism $h : M \to M$ such that $h|_{N_1} = g$. Therefore $M = N_1 \oplus Ker(h) = N_1 \oplus M_1$where $M_1 = Ker(h) \neq 0$.

Again, since $N_1$ is non-co-Hopfian, by similar argument we get $N_1 = N_2 \oplus M_2$ with $N_2 \cong N_1$ and $M_2 \neq 0$, thus $M = N_2 \oplus (M_1 \oplus M_2)$. Now using the principal of mathematical induction and definition of co-Hopfian module, we reach the desired result $M = N_r \oplus (\oplus_{i=1}^{r} M_i)$ for $r \in \mathbb{Z}^+$, where $N_r \cong M$ and $M_i \neq 0$ for $i = 1, 2, \ldots, r$. Conversely, assume that $M = N_r \oplus (\oplus_{i=1}^{r} M_i)$ where $N_r \cong M$ and $M_i \neq 0$ then $M$ is non-co-Hopfian as it is not directly finite.

**Proposition 2.4.** *If $M$ is extending $FI$-semi-injective module, then $M$ is co-Hopfian if and only if it satisfies the cancellation property.*

**Proof**: Assume that $M$ is co-Hopfian, so is directly finite (Proposition 1.33 [12]). Then by Corollary 1.19 [12] $M$ satisfies cancellation property. Conversely, let $M$ be non-co-Hopfian, then by Proposition 2.4, there exists decomposition of $M = M \oplus 0 = N_1 \oplus M_1$ where $N_1 \cong M$ and $M_1 \neq 0$. By assumption $M$ has cancellation property, so we get $M_1 = 0$ which is a contradiction to our assumption $M_1 \neq 0$. Hence, assumption that $M$ is non-co-Hopfian is wrong.

**Proposition 2.5.** *Any $FI$-semi-injective module satisfies $(C_4)$ condition.*

**Proof**: By Proposition 1.17 [12] FI-semi-injective module satisfies $(C_2)$ and $(C_3)$ conditions and we know $(C_2) \Rightarrow (C_3) \Rightarrow (C_4)$, thus we get the desired result.

**Proposition 2.6.** *Following statements are equivalent for extending FI-semi-injective module:*

i. *$M$ is a clean module.*

ii. *$M$ has finite exchange property.*

iii. *$M$ has full exchange property*

**Proof**: Proof follows in the light of Theorem 4.3 of [3] and remark 1.18 of [12].

The concept of epi-retractable module came into being in 2009 through Ghorbani and Vedadi gave [6]. If every submodule of $M$ is $M-$cyclic, then the $R-$module $M$ is called epi-retractable .

**Lemma 2.7.** *For an $FI$-semi injective and epi-retractable module $M$ and $S = End(M_R)$, $S = \{f \in S | f(M) \subseteq^{\oplus} M\}$ if and only if every submodule of $M$ is $FI - M-$Principally injective.*

**Proof**: Let $f \in S$, then $f(M) \subseteq^{\oplus} M$ and so it is $FI - M-$principally injective. Conversely, Let $A \subseteq M$, $s \in S$ and let $g : s(M) \to A$ be any homomorphism. By assumption, $s(M) \subseteq^{\oplus} M$. Set $f = g\pi$, where $\pi$ is the natural projection from $M$ to $s(M)$. Then $f|_{s(M)} = g$. Hence, $A$ is $FI - M-$principally injective.

**Proposition 2.8.** *The endomorphism ring of a finite-dimensional $FI-$semi injective module is a semi local ring.*

**Proof**: Let the finite-dimensional $FI-$semi injective module be $M$. If $M$ is finite-dimensional, from a result by Camps and Dicks in [4] it suffices to show that every monomorphism $f$ on $M$ is an isomorphism. By hypothesis $M$ is $FI-$semi-injective,so $M$ satisfies $(C_2)$ hence $M \cong f(M)$. It follows that $f(M) \subseteq^{\oplus} M$. Also, since $M$ is finite dimensional, $f(M) \subseteq_e M$. Hence, $f(M) = M$, showing that $f$ is an isomorphism.

If the right annihilator of every element of a ring $R$ is generated by an idempotent, then the ring $R$ is called right principally projective

**Proposition 2.9.** *If $M$ is an epi-retractable $R$-module whose submodules are $FI-M-$principally injective, then its endomorphism ring $S$ is right principally projective.*

**Proof**: A submodule of an $FI-$semi-injective is a direct summand of $M$, since it is an $M$-cyclic submodule of $M$. If $f(M)$ is an $M-$cyclic submodule of $M$ for some $f \in S$, there exist an idempotent $e \in S$ with $Im(f) = f(M) = e(M)$. For the $S-$homomorphism $g : S \to fS,\ g(s) = sf$, we have $Ker(g) = (1-e)S$. For this $(1-e)f(M) = (1-e)e(M) = 0$ and so $(1-e)S \subseteq Ker(g)$. Next, $a \in Ker(g)$ implies that $ae(M) = af(M) = 0$ implies that $ae = 0$ this shows that $a \in (1-e)S$.

Let $J(M)$ denote the Jacobson radical of $M$, which is nothing but the intersection of all maximal submodules of $M_R$. If $M$ has no maximal submodule, then we define $J(M) = M$. Let $W(S) = \{w \in S | Ker(w) \subset_e M\}$ where $S = End(M_R)$. It is known that $W(S)$ is a right, as well as a left ideal in $S$ [10]. It can be seen that $W(S) \subseteq \{w \in S | 1 - sw$ is a monomorphism for all $s \in S\}$ since $Ker(w) \cap Ker(1 - sw) = 0$ for all $s \in S$.

**Lemma 2.10.** *For two $R-$modules $M$ and $N$, $N$ is $FI-M-$principally injective if and only if for each fully invariant endomorphism $s$, $Hom_R(M, N)s = \{f \in Hom_R(M, N) | f(Ker(s)) = 0\}$.*

**Proof** : Proof is similar to the proof of Proposition 1.9 [12].

**Proposition 2.11.** *Let $M$ be $FI-$semi-injective then $J(S) = \{s \in S | 1 - ts$ is a monomorphism for all $t \in S\}$, where $S = End(M_R)$.*

**Proof**: Let $s \in J(S)$ and $t \in S$, $g(1-ts) = 1_M$ for some $g \in S$. Hence, $Ker(1-ts) = 0$, implying that $J(S) \subseteq \{s \in S | 1 - ts$ is a monomorphism for all $t \in S\}$. Conversely, if $Ker(1 - ts) = 0$, then $l_S(Ker(1 - ts)) = S$. From the above Lemma, we then have $S = S(1 - ts)$ implying that $1_M = g(1 - ts)$ for some $g \in S$. Hence, $s \in J(S)$.

**Proposition 2.12.** *If $S$ is local, where $S = End(M_R)$ and $M$ is $FI-$ semi injective, then $J(S) = \{s \in S | s$ is fully invariant endomorphism and $Ker(s) \neq 0\}$*

**Proof**: If $S$ is local, $Ss \neq S$ for any fully invariant endomorphism $s \in J(S)$. If $s$ is a monomorphism, then $\alpha : s(M) \to M$ given by $\alpha(s(M)) = m,\ m \in M$, is an $R-$homomorphism. An extension of $\alpha$ to $M$ exist, say $\beta \in S$, since $M$ is $FI-$semi-injective. But this implies that $\beta s = 1_M$ so $Ss = S$, which is a contradiction, showing that $J(S) \subseteq \{s \in S | s$ is fully invariant endomorphism and $Ker(s) \neq 0\}$. The other inclusion is clear.

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## Author information

S. Chase and M. K. Patel, Department of Mathematics, National Institute of Technology Nagaland, Dimapur-797103, Nagaland, India.
E-mail: sedevikho.chase@gmail.com, mkpitb@gmail.com

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