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Computational Methods for Solving System of Linear Fredholm Integral Equations

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Abstract In this paper, three numerical methods, namely the Legendre wavelet method, Taylor series expansion method and the improved block-Pulse functions method are implemented to solve linear systems of Fredholm integral equations. To test the validity of these methods, two numerical examples with known exact solution are presented. Numerical results show that the convergence and accuracy of these methods are in good agreement with analytical solution. However, according to comparison of these methods, we conclude that the Legendre wavelet method provides more accurate results.

1 Introduction

System of integral equations have motivated huge amount of research in recent years. They arise in many physical phenomena like heat conduction, convection and radiation, wind ripple in the desert, nano-hydrodynamics, population growth model, glass-forming process and oceangraphy [1, 2, 3]. Several numerical methods for solving systems of Fredholm integral equations have been developed by many researchers over the years. Babolian et al. [4] applied the Adomian decomposition method to solve linear and nonlinear systems of Fredholm integral equations. The homotopy perturbation method and the modified homotopy method have been used by Javidi and Golbabai [5, 6] to solve nonlinear Fredholm integral equations. Jaferian et al. [7, 8] implemented the Bernestain collocation method for solving linear systems of Fredholm integral equations. A direct method to compute numerical solutions of linear systems as Fredholm integral equations has been proposed by Babolian et al. [9, 10]. This method uses vector forms of triangular functions to convert the system of integral equations to a system of algebraic equations that can be solved by sampling of functions and multiplication and addition of matrices. Maleknejad et al. [11] have used rationalized Haar functions method and Block-Pluse functions to solve systems of Fredholm integral equations. Moreover, Ren et al. [12] have developed a simple Taylor series expansion method to solve integral equations of the second kind. In addition, Hamaydi and Qatanani [13] used the Taylor expansion and the variational methods to give approximate solution of Voltera integral equation of the second kind. Issa et al. [14] applied the collocation method using sinc functions and Chebyshev wavelet method to solve linear systems of Voltera integro-differential equations. On the other hand, Jarar'a [15] has implemented various computational methods for solving system of linear Fredholm integral equations. Other numerical methods for solving systems of integral and integro-differential equations are (power) functions and Chebyshev polynomials [16], single term Walsh series [17], Chebyshev collocation [18], differential transform [19], power series [20], and finite difference approximation [21]. In this article, we propose three numerical methods, namely, Legendre wavelet method, Taylor series expansion method and improved block-pluse function method to approximat the solution of a system of linear Fredholm integral equations given by

$$u_i(x) = f_i(x) + \int_a^b \left(\sum_{j=1}^{N} k_{ij}(x,t) u_j(t) \right) dt, \quad 1 \le i \le N \tag{1.1}$$

The kernels $k_{ij}(x,t)$ and the function $f_i(x)$ are given real valued functions and the unknown functions $u_i(x)$ are to be determined. A comparison between these methods is carried out by solving some numerical examples. This paper is organized as follows: In section 2 the Legendre wavelet method is presented. The Taylor series expansion method and the improved block-pluse method are addressed in sections 3 and 4 respectively. In section 5, the proposed methods are implemented using two numerical examples with known analytical solution. Conclusions are followed in section 6.

2 The Legendre Wavelets Method

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter p and the translation parameter q vary continuously we have the following family of continuous wavelets as [41]:

$$\psi_{p,q}(t) = |p|^{-\frac{1}{2}}\psi\Big(\frac{t-q}{p}\Big), \quad p,q \in \mathbb{R}, p \neq 0$$

If we restrict the parameters p and q to discrete values:

$$p = p_0^{-k}, \quad q = nq_0p_0^{-k}$$

where $p_0 > 1, q_0 > 1$ and n, k can assume any positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |p_0|^{-\frac{k}{2}}\psi\big(p^k t - nq_0\big), \quad n,k \in \mathbb{N}$$

where $\psi_{k,n}(t)$ form wavelet basis for $L^2(\mathbb{R})$ with:

$$L^2(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{C}, \int_{-\infty}^{\infty} |g(t)|^2 dt < \infty \right\}$$

In particular, when $p_0 = 2, q_0 = 1$, then $\psi_{k,n}(t)$ form an orthogonal basis [17]. Legendre wavelets $\psi_{k,n}(t) = \psi(k, \hat{n}, m, t)$ have four arguments;

$$\hat{n} = 2n - 1, \quad n = 1, 2, \cdots, 2^{k-1}$$

k can assume any positive integer, m is the order of Legendre polynomial and t is the normalized time. They are defined on the interval $[0, 1)$ as:

$$\psi_{k,n}(t) = \psi(k, \hat{n}, m, t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} S_m(2^k t - 2n + 1) & \dfrac{2n-2}{2^k} \leq t \leq \dfrac{2n}{2^k} \\ 0 & otherwise \end{cases}$$

where $m = 0, 1, 2, \ldots, M-1$ and $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $p = 2^{-k}$ and the translation parameter is $q = \hat{n}2^{-k}$. Here $S_m(t)$ are the well known Legendre polynomials of order m, which are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$ and satisfy the following recurrence formula:

$$\begin{aligned} S_0(t) &= 1 \\ S_1(t) &= t \\ S_{m+1}(t) &= \left(\frac{2m+1}{m+1}\right) t S_m(t) - \left(\frac{m}{m+1}\right) S_{m-1}(t), \quad where \quad m = 1, 2, 3, \ldots \end{aligned}$$

A function $f(t)$ defined on $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{n,m}\psi_{n,m}(t), \tag{2.1}$$

where $b_{n,m} = \left(f(t), \psi_{n,m}(t) \right)$, in which $(.,.)$ denotes the inner product. If the infinite series in equation (2.1) is truncated, then it can be written as:

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} b_{n,m} \psi_{n,m}(t) = B^T \Psi(t)$$

where B and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$B = \left[b_{10}, b_{11}, \ldots, b_{1M-1}, b_{20}, b_{21}, \ldots, b_{2M-1}, \ldots, b_{2^{k-1}0}, b_{2^{k-1}1}, \ldots, b_{2^{k-1}M-1} \right]^T$$

$$\Psi(t) = \left[\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{k-1}0}, \psi_{2^{k-1}1}, \ldots, \psi_{2^{k-1}M-1} \right]^T$$

Likewise, a function $h(x,t) \in L^2\big([0,1] \times [0,1]\big)$ can be approximated as:

$$h(x,t) = \Psi^T(x) H \Psi(t)$$

where H is $2^{k-1}M \times 2^{k-1}M$ matrix with

$$h_{ij} = \left(\psi_i(x), \Big(h(x,t), \psi_j(t) \Big) \right), i,j = 1,2,\ldots,2^{k-1}M$$

We consider the system of linear Fredholm integral equations of the form

$$V(x) = F(x) + \lambda \int_0^1 H(x,t)V(t)dt, \quad 0 \leq x \leq 1 \tag{2.2}$$

with

$$V(x) = \Big[v_1(x), v_2(x), \ldots, v_s(x) \Big]^T$$

$$F(x) = \Big[f_1(x), f_2(x), \ldots, f_s(x) \Big]^T$$

$$H(x,t) = \Big[h_{ij}(x,t) \Big], \quad i,j = 1,2,\ldots,s$$

where the functions H and F are given, and V is the vector function of the solution of system (2.2).
Consider the i^{th} equation of system (2.2)

$$v_i(x) = f_i(x) + \int_0^1 \sum_{j=1}^{s} h_{ij}(x,t) v_j(t) dt, \quad i = 1,2,\ldots,s, \tag{2.3}$$

where $f_i(x) \in L^2[0,1), h(x,t) \in L^2\big([0,1) \times [0,1)\big)$, and $v_i(x)$ is unknown function.
We approximate $f_i(x), v_i(x)$ and $h_{ij}(x,t)$ as follows:

$$f_i(x) \simeq F_i^T \Psi(x) \tag{2.4}$$

$$v_i(x) \simeq B_i^T \Psi(x) \tag{2.5}$$

$$h_{ij}(x,t) \simeq \Psi^T(x) H_{ij} \Psi(t) \tag{2.6}$$

By substituting the approximations (2.4 -2.6) into system (2.3), we obtain [22, 23, 24]:

$$\begin{aligned}\Psi^T(x)B_i &= \Psi^T(x)F_i + \int_0^1 \sum_{j=1}^{s} \Psi^T(x)H_{ij}\Psi(t)\Psi^T(t)B_j dt \\ &= \Psi^T(x)F_i + \Psi^T(x)\sum_{j=1}^{s} H_{ij}\left(\int_0^1 \Psi(t)\Psi^T(t)dt\right)B_j \\ &= \Psi^T(x)F_i + \Psi^T(x)\sum_{j=1}^{s} H_{ij}B_j\end{aligned}$$

This can be written as:

$$\Psi^T(x)B_i = \Psi^T(x)\left(F_i + \sum_{j=1}^{s} H_{ij}B_j\right) \tag{2.7}$$

Multiplying both sides of system (2.7) by $\Psi(x)$, and integrating with respect to x, we get the following linear system:

$$B_i = F_i + \sum_{j=1}^{s} H_{ij}B_j$$

By solving this linear system we obtain the vector B_i, so

$$v_i(x) \simeq B_i^T\Psi(x), \quad i = 1, 2, \ldots, s.$$

3 Taylor-Series Expansion Method

An alternative method proposed for solving system of linear Fredholm equations is the Taylor-series expansion method. This method transforms the system of integral equations into a system of linear ordinary differential equations [25, 26]. Consider the p^{th} equation of system (2.2):

$$v_p(x) = f_p(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}(x,t)v_q(t)dt, \quad p = 1, 2, \ldots, s, \tag{3.1}$$

a Taylor-series expansion can be made for the solution v_q in the integral equation (3.1):

$$v_q(t) = v_q(x) + v_q'(x)(t-x) + \ldots + \frac{1}{m!}v_q^{(m)}(x)(t-x)^m + E(t) \tag{3.2}$$

where $E(t)$ denotes the error between v_q and its Taylor-series expansion (3.2):

$$E(t) = \frac{1}{(m+1)!}v_j^{(m+1)}(x)(t-x)^{m+1} + \ldots$$

If we use the first m term of the Taylor-series expansion (3.2) and neglect $\int_0^1 \sum_{q=1}^{s} h_{pq}(x,t)E(t)dt$ then by substituting equation (3.2) into equation (3.1) we get:

$$\begin{aligned}v_p(t) &\simeq f_p(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}(x,t)\sum_{a=0}^{m}\frac{1}{a!}v_q^a(x)(t-x)^a dt \\ &\simeq f_p(x) + \sum_{q=1}^{s}\sum_{a=0}^{m}\frac{1}{a!}v_q^a(x)\int_0^1 h_{pq}(x,t)(t-x)^a dt\end{aligned}$$

or,

$$v_p(t) - \sum_{q=1}^{s}\sum_{a=0}^{m} \frac{1}{a!} v_q^a(x) \left(\int_0^1 h_{pq}(x,t)(t-x)^a dt \right) \simeq f_p(x), \quad p = 1,2,\ldots,s \tag{3.3}$$

However, this system requires the manufacture of an appropriate number of boundary conditions which can be obtained by differentiating both sides of equation (3.1) to get for $0 < x < 1$ and $p = 1,2,\ldots,s$

$$\begin{aligned} v_p'(x) &= f_p'(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}'(x,t) v_q(t) dt \\ v_p''(x) &= f_p''(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}''(x,t) v_q(t) dt \\ &\vdots \\ v_p^{(m)}(x) &= f_p^{(m)}(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}^{(m)}(x,t) v_q(t) dt \end{aligned} \tag{3.4}$$

Replacing $v_q(t)$ by $v_q(x)$ gives,

$$\begin{aligned} v_p'(x) - \left[\int_0^1 \sum_{q=1}^{s} h_{pq}'(x,t) dt \right] v_q(t) &\simeq f_p'(x) \\ v_p''(x) - \left[\int_0^1 \sum_{q=1}^{s} h_{pq}''(x,t) dt \right] v_q(t) &\simeq f_p''(x) \\ &\vdots \\ v_p^{(m)}(x) - \left[\int_0^1 \sum_{q=1}^{s} h_{pq}^{(m)}(x,t) dt \right] v_q(t) &\simeq f_p^{(m)}(x) \end{aligned} \tag{3.5}$$

Now system (3.3) combined with system (3.5) become m^{th} order linear system of algebraic equations that can be solved analytically or numerically. For example, if we use this method with $s = 2$ and $m = 1$ for solving system of linear Fredholm integral equations, we have:

$$\begin{aligned} v_1(x) &= f_1(x) + \int_0^1 \Big[h_{11}(x,t)v_1(t) + h_{12}(x,t)v_2(t) \Big] dt \\ v_2(x) &= f_2(x) + \int_0^1 \Big[h_{21}(x,t)v_1(t) + h_{22}(x,t)v_2(t) \Big] dt \end{aligned} \tag{3.6}$$

a Taylor series expansion can be made for $v_1(t)$ and $v_2(t)$ as follows:

$$\begin{aligned} v_1(t) &= v_1(x) + (t-x)v_1'(x) + \frac{(t-x)^2}{2!} v_1''(x) + E_1(t) \\ v_2(t) &= v_2(x) + (t-x)v_2'(x) + \frac{(t-x)^2}{2!} v_2''(x) + E_2(t) \end{aligned}$$

Substituting $v_1(t)$ and $v_2(t)$ into equation (3.6) gives:

$$\begin{aligned}
v_1(x) &\simeq f_1(x) + \int_0^1 h_{11}(x,t)\Big[v_1(x) + (t-x)v_1'(x) + \frac{(t-x)^2}{2!}v_1''(x)\Big]dt \\
&\quad + \int_0^1 h_{12}(x,t)\Big[v_2(x) + (t-x)v_2'(x) + \frac{(t-x)^2}{2!}v_2''(x)\Big]dt \\
v_2(x) &\simeq f_2(x) + \int_0^1 h_{21}(x,t)\Big[v_1(x) + (t-x)v_1'(x) + \frac{(t-x)^2}{2!}v_1''(x)\Big]dt \\
&\quad + \int_0^1 h_{22}(x,t)\Big[v_2(x) + (t-x)v_2'(x) + \frac{(t-x)^2}{2!}v_2''(x)\Big]dt
\end{aligned} \tag{3.7}$$

Equation (3.7) is a system of ordinary differential equations that can be solved after producing boundary conditions.To achieve this, we differentiate both sides of equation (3.6) to get:

$$\begin{aligned}
v_1'(x) &= f_1'(x) + \int_0^1 h_{11}'(x,t)v_1(t) + h_{12}'(x,t)v_2(t)dt \\
v_2'(x) &= f_2'(x) + \int_0^1 h_{21}'(x,t)v_1(t) + h_{22}'(x,t)v_2(t)dt
\end{aligned} \tag{3.8}$$

Substituting equation (3.7) into equation (3.8) gives:

$$\begin{aligned}
v_1'(x) - \left[v_1(x)\int_0^1 h_{11}'(x,t)dt + v_2(x)\int_0^1 h_{12}'(x,t)dt\right] &\simeq f_1'(x) \\
v_2'(x) - \left[v_1(x)\int_0^1 h_{21}'(x,t)dt + v_2(x)\int_0^1 h_{22}'(x,t)dt\right] &\simeq f_2'(x)
\end{aligned} \tag{3.9}$$

Now equations (3.7) combined with equations (3.9) become a linear two order system of algebraic equations that can be solved easily.
Another technique [25] proposed is based on replacing each $v_q(t)$ in equation (3.3) by the right side of equation (3.2) to obtain:

$$\begin{aligned}
v_p'(x) &= f_p'(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}'(x,t)\left[\sum_{a=0}^{m}\frac{1}{a!}(t-x)^a v_q^{(a)}(x)\right]dt \\
v_p''(x) &= f_p''(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}''(x,t)\left[\sum_{a=0}^{m}\frac{1}{a!}(t-x)^a v_q^{(a)}(x)\right]dt \\
&\vdots \\
v_p^{(m)}(x) &= f_p^{(m)}(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}^{(m)}(x,t)\left[\sum_{a=0}^{m}\frac{1}{a!}(t-x)^a v_q^{(a)}(x)\right]dt
\end{aligned} \tag{3.10}$$

or,

$$v_p^{(l)}(x) - \sum_{a=0}^{m}\sum_{q=1}^{s}\frac{1}{a!}\left[\int_0^1 h_{pq}^{(l)}(x,t)(t-x)^a dt\right]v_q^{(a)}(x) \simeq f_p^{(l)}(x), \quad l = 1, 2, \ldots, m. \tag{3.11}$$

Now equation (3.3) combined with equation (3.11) become a linear system of $(m+1)s$ ordinary differential equations. If $s=2, m=1$, we can write this system of the form:

$$\begin{pmatrix} 1-\int\limits_0^1 h_{11}(x,t)dt & -\int\limits_0^1 h_{11}(x,t)(t-x)dt & -\int\limits_0^1 h_{12}(x,t)dt & -\int\limits_0^1 h_{12}(x,t)(t-x)dt \\ -\int\limits_0^1 h_{21}(x,t)dt & -\int\limits_0^1 h_{21}(x,t)(t-x)dt & 1-\int\limits_0^1 h_{22}(x,t)dt & -\int\limits_0^1 h_{22}(x,t)(t-x)dt \\ -\int\limits_0^1 h'_{11}(x,t)dt & 1-\int\limits_0^1 h'_{11}(x,t)(t-x)dt & -\int\limits_0^1 h'_{12}(x,t)dt & -\int\limits_0^1 h'_{12}(x,t)(t-x)dt \\ -\int\limits_0^1 h'_{21}(x,t)dt & -\int\limits_0^1 h'_{21}(x,t)(t-x)dt & -\int\limits_0^1 h'_{22}(x,t)dt & 1-\int\limits_0^1 h'_{22}(x,t)(t-x)dt \end{pmatrix} \begin{pmatrix} v_1(x) \\ v'_1(x) \\ v_2(x) \\ v'_2(x) \end{pmatrix} = \begin{pmatrix} f \\ f \\ f \\ f \end{pmatrix}$$

Solving this algebraic system will give the functions $v_1(x)$ and $v_2(x)$.

4 Improved Block-Pulse Function Method

Definition 4.1. [27, 28]: A set of block-pulse functions $b_k(t), k=1,2,\ldots,K$ on the interval $[0,T)$ are described as

$$b_k(t) = \begin{cases} 1 & , t_{k-1} < t < t_k \\ 0 & , otherwise \end{cases}$$

where $t_0 = 0, t_k = T$ and $[t_{k-1}, t_k) \subset [0,T), k = 1,2,\ldots,K$

The variable interval block-pulse is $(m+1)$ set of functions described over the interval $[0,T)$ as follows:

$$\beta_0(t) = \begin{cases} 1, & t \in [0, \frac{1}{2m}) \\ 0, & otherwise \end{cases}$$

$$\beta_i(t) = \begin{cases} 1, & t \in \left[\frac{i-\frac{1}{2}}{m}, \frac{i+\frac{1}{2}}{m}\right), i = 1,2,\ldots,m-1 \\ 0, & otherwise \end{cases}$$

$$\beta_m(t) = \begin{cases} 1, & t \in \left[1-\frac{1}{2m} 1\right) \\ 0, & otherwise \end{cases}$$

where m is a positive constant that represents the number of sub-intervals to be decided for the accuracy needed for solving the problem. The IBFs are disjointed. Therefore, we obtain

$$\beta_i(t)\beta_j(t) = \begin{cases} \beta_i(t) & , i = j \\ 0 & , otherwise \end{cases}$$

where $i, j = 0,1,2,\ldots,m$ and are orthogonal on the interval $[0,1)$:

$$\int\limits_0^1 \beta_i(t)\beta_j(t)dt = \begin{cases} \frac{1}{2m} & , i = j \in 0, m \\ \frac{1}{m} & , i = j \in 1,2,\ldots,m-1 \\ 0 & , otherwise \end{cases}$$

where $t \in [0,1)$.

The first $(m+1)$ terms of the IBFs can be written in vector form:

$$\beta_m(t) = \left[\beta_0(t), \beta_1(t), \ldots, \beta_m(t)\right]^T, t \in [0,1) \tag{4.1}$$

Suppose that $f(t)$ is a continuous function, where $f(t) \in L^2[0,1)$ and may be expanded by the

IBFs as follows:

$$\begin{aligned} f(t) &\simeq f_m(t) \\ &= \sum_{i=0}^{m} f_i \beta_i(t) \\ &= F_m^T \beta_m(t) \\ &= \beta_m^T(t) F_m \end{aligned}$$

where F_m is an $(m+1) \times 1$ vector given by

$$F_m(t) = \Big[F_0, F_1, \dots, F_m\Big]^T$$

and $\beta_m(t)$ is defined in equation (4.1), and f_i are the improved block-pluse coefficients which are given as:

$$f_i = \begin{cases} 2m \int\limits_0^{\frac{1}{2m}} f(t)dt & , i = 0 \\ m \int\limits_0^{\frac{i+\frac{l}{2}}{m}} f(t)dt & , 1 \le i \le m-1 \\ 2m \int\limits_{1-\frac{l}{2m}}^{1} f(t)dt & , i = m \end{cases}$$

Similarly, a function of two variables, $h(x,t) \in L^2\Big([0,1) \times [0,1)\Big)$ can be approximated by IBFs as follows:

$$h(x,t) \simeq h_m(x,t) = \beta_m^T(t) h_m \beta_m^T(s)$$

To this end, we consider the p^{th} equation of system (2.2), that is:

$$v_p(x) = f_p(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}(x,t) v_q(t) dt, \quad p = 1, 2, \dots, s \tag{4.2}$$

To solve system (4.2) by IBPF method, firstly; we approximate the functions $v_p(x), f_p(x)$ and $h_{pq}(x,t), p, q = 1, 2, \dots, s$ by IBPFs as follows:

$$\begin{aligned} v_p(x) &\simeq \beta^T(x) V_p^* \\ f_p(x) &\simeq \beta^T(x) F_p^* \\ h_{pq}(x,t) &\simeq \beta^T(x) H_{pq}^* \beta(t) \end{aligned} \tag{4.3}$$

where F_p^*, V_p^* and H_{pq}^* are IBPF coefficients of $f_p(x), v_p(x)$ and $h_{pq}(x,t)$ respectively. Substituting equation (4.3) into equation (2.2), we get [32]:

$$\begin{aligned} \beta^T(x) V_p^* &= \beta^T(x) F_p^* + \sum_{q=1}^{s} \int_0^1 \beta^T(x) H_{pq}^* \beta(t) \beta^T(x) V_p^* dt \\ &= \beta^T(x) F_p^* + \sum_{q=1}^{s} \beta^T(x) H_{pq}^* \left(\int_0^1 \beta(t) \beta^T(x) dt \right) V_p^*, \quad p = 1, 2, \dots, s \end{aligned}$$

In virtue of equation (4.1) we get:

$$\beta^T(x) V_p^* = \beta^T(x) F_p^* + \sum_{q=1}^{s} \beta^T(x) H_{pq}^* P V_p^*, \quad p = 1, 2, \dots, s$$

Set $M_{pq}^* = \Big[H_{pq}^* \quad P\Big]$, we have:

$$V^* = F^* + M^* V \tag{4.4}$$

where,

$$V^* = \left[V_1^*, V_1^*, \ldots, V_s^*\right]$$

$$F^* = \left[F_1^*, F_2^*, \ldots, F_s^*\right]$$

$$M^* = \left[M_{pq}^*\right], \quad p, q = 1, 2, \ldots, s$$

Solving linear system (4.4) gives the vector V^* and then $V_p^*, \quad p, q = 1, 2, \ldots, s$. Finally, we can find the solution as:

$$v_p(x) \simeq \beta^T(x) V_p^*, \quad p, q = 1, 2, \ldots, s$$

.

5 Numerical Examples and Results

In this section, some numerical examples are presented to show the validity of the proposed methods. In addition, the numerical results are compared with exact solution.

Example 5.1. Consider the system of linear Fredholm integral equations:

$$\begin{aligned} v_1(x) &= \frac{x}{18} + \frac{17}{36} + \int_0^1 \frac{x+t}{3}\Big(v_1(t) + v_2(t)\Big)dt \\ v_2(x) &= x^2 - \frac{19}{12}x + 1 + \int_0^1 xt\Big(v_1(t) + v_2(t)\Big)dt \end{aligned} \tag{5.1}$$

The exact solution of system (5.1) is $v_1(x) = x + 1$ and $v_2(x) = x^2 + 1$.

We start by implementing Algorithm (5.2) to solve system (5.1) using the Legendre wavelet method with $M = 4$. We obtain the following equations:

$$\begin{aligned} \psi_{10}(x) &= 1 \\ \psi_{11}(x) &= \sqrt{3}(-1 + 2x) \\ \psi_{12}(x) &= \sqrt{5}(6x^2 - 6x + 1) \\ \psi_{13}(x) &= \sqrt{7}(-1 + 12x - 30x^2 + 20x^3) \end{aligned} \tag{5.2}$$

Rewrite equations (5.2) in vector notation, we have

$$\psi(x) = \begin{bmatrix} 1 & \sqrt{3}(-1+2x) & \sqrt{5}(6x^2 - 6x + 1) & \sqrt{7}(-1 + 12x - 30x^2 + 20x^3) \end{bmatrix}^T$$

Put:

$$\begin{aligned} v_1(x) &= \psi^T(x)\beta_1 \\ v_2(x) &= \psi^T(x)\beta_2 \\ f_1(x) &= \psi^T(x)F_1 \\ f_2(x) &= \psi^T(x)F_2 \\ h_{11}(x,t) &= \psi^T(x)H_{11}\psi(t) \\ h_{12}(x,t) &= \psi^T(x)H_{12}\psi(t) \\ h_{21}(x,t) &= \psi^T(x)H_{21}\psi(t) \\ h_{22}(x,t) &= \psi^T(x)H_{22}\psi(t) \end{aligned}$$

where:

$$\psi^T(x) = \Big[\psi_{10}(x) \quad \psi_{11}(x) \quad \psi_{12}(x) \quad \psi_{13}(x)\Big]$$

$$\beta_1(x) = \Big[\beta_{10}(x) \quad \beta_{11}(x) \quad \beta_{12}(x) \quad \beta_{13}(x)\Big]^T$$

$$\beta_2(x) = \Big[\beta_{20}(x) \quad \beta_{21}(x) \quad \beta_{22}(x) \quad \beta_{23}(x)\Big]^T$$

$$F_1(x) = \Big[f_{10}(x) \quad f_{11}(x) \quad f_{12}(x) \quad f_{13}(x)\Big]^T$$

$$F_2(x) = \Big[f_{20}(x) \quad f_{21}(x) \quad f_{22}(x) \quad f_{23}(x)\Big]^T$$

and

$$\begin{aligned} H_{pq} &= \Big(\psi_p(x), \Big(h_{pq}(x,t), \psi_q\Big)\Big) \\ &= \int_0^1 \psi_p(x)\Big(\int_0^1 h_{pq}(x,t), \psi_q\Big)dt\Big)dx, \quad p,q = 1,2 \end{aligned}$$

Using these matrices to solve the linear algebraic system:

$$\beta_1 = \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{3}}{18} & 0 & 0 \\ \frac{\sqrt{3}}{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \beta_1 + \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{3}}{18} & 0 & 0 \\ \frac{\sqrt{3}}{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \beta_2 + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{108} \\ 0 \\ 0 \end{bmatrix}$$

$$\beta_2 = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{12} & 0 & 0 \\ \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \beta_1 + \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{12} & 0 & 0 \\ \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \beta_2 + \begin{bmatrix} \frac{13}{24} \\ \frac{-7\sqrt{3}}{72} \\ \frac{\sqrt{5}}{30} \\ 0 \end{bmatrix}$$

We get β_1 and β_2:

$$\beta_1 = \begin{bmatrix} \frac{3}{2} \\ \frac{\sqrt{3}}{6} \\ 0 \\ 0 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} \frac{4}{3} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{5}}{30} \\ 0 \end{bmatrix}$$

So,

$$v_1(x) = \frac{3}{2}\psi_{10} + \frac{\sqrt{3}}{6}\psi_{11} + 0\psi_{12} + 0\psi_{13}$$

$$v_2(x) = \frac{4}{3}\psi_{10} + \frac{\sqrt{3}}{6}\psi_{11} + \frac{\sqrt{5}}{30}\psi_{12} + 0\psi_{23}$$

Substituting for ψ_{01}, ψ_{11} and ψ_{12} gives $v_1(x) = x + 1$ and $v_2(x) = x^2 + 1$ which is the exact solution.

Algorithm 5.2 (h!). (LWM) [1] Input:

- s, M, K
- The functions of the integral system $f_p(x)$ for $p = 1, 2, \ldots, s$

- The Kernel functions $h_{pq}(x)$ for $p, q = 1, 2, \ldots, s$

Define: The Legendre polynomials $S_m(x)$ as:

$$S_0(x) = 1, S_1(x) = x, S_{m+1}(x) = \frac{2m+1}{m+1} x S_m(x) - \frac{m}{m+1} S_{m-1}(x)$$

for $m = 1, 2, \ldots$. Compute $S_m(x)$ for $m = 0, 1, 2, \ldots, M-1$ Define the Legendre wavelets $\psi_{n,m}(x)$ as:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} S_m(2^k x - 2n + 1) & \frac{2n-2}{2^k} \leq t \leq \frac{2n}{2^k} \\ 0 & o.w \end{cases}$$

Compute $\psi_{n,m}(x)$ for $n = 1, 2, \ldots, 2^{k-1}, \quad m = 0, 1, 2, \ldots, M-1$
Define the vector

$$\psi(x) = \Big[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1M-1}(x), \psi_{20}(x), \psi_{21}(x), \ldots, \psi_{2M-1}(x), \psi_{2^{k-1}0}(x), \psi_{2^{k-1}1}(x), \ldots, \psi_{2^{k-1}M-1}(x$$

Calculate

$$F_p = \int_0^1 f_p(x) \psi(x) dx$$

for $p = 1, 2, \ldots, s$ Compute the matrices H_{pq} where

$$H_{pq} = \int_0^1 \psi(x) \left(\int_0^1 h_{pq}(x,t) \phi^T(t) dt \right) dx$$

for $p, q = 1, 2, \ldots, s$ Solve the linear system

$$\beta_1 = F_1 + \sum_{q=1}^{s} H_{1q} \beta_q$$

$$\beta_2 = F_2 + \sum_{q=1}^{s} H_{2q} \beta_q$$

$$\vdots$$

$$\beta_s = F_s + \sum_{q=1}^{s} H_{sq} \beta_q$$

to get β_p for $p = 1, 2, \ldots, s$ Compute the vectors $V_p(x) = B_p^T \psi(x), \quad p = 1, 2, \ldots, s$. Set $v_{p(approx)}(x)$ for $p = 1, 2, \ldots, s$ Input $v_{p(exact)}(x)$ for $p = 1, 2, \ldots, s$ Plot $v_{p(approx)}(x), v_{p(exact)}(x)$ Define $error = |v_{p(exact)}(x) - v_{p(approx)}(x)|$ Plot error.

Next, we implement Algorithm (5.3) to solve system (5.1) using the Taylor-series expansion method. Table (1) contains the exact and numerical solutions with $m = 3$ and Table (2) contains the resulting error. Figures (1) and (2) show a comparison between the exact and numerical solutions for system (5.1). The maximum error corresponding to $v_1(x)$ and $v_2(x)$ is $E_1 \simeq 8.57 \times 10^{-3}$ and $E_2 \simeq 4.43 \times 10^{-3}$ respectively.

Algorithm 5.3 (h!). (TSEM): [1] Input:

- s, m
- $f_p(x)$ for $p = 1, 2, \ldots, s$
- $h_{pq}(x,t)$ for $p, q = 1, 2, \ldots, s$

Expand $v_q(t)$ for $q = 1, 2, \ldots, s$, by Taylor series to m terms at $t_0 = x$ as:

$$v_q(t) = v_q(x) + v_q'(x)(t-x) + \ldots + \frac{1}{m!} v_q^{(m)}(x)(t-x)^m, q = 1, 2, \ldots, s$$

Substitute the expansion of $v_q(t)$ in the Fredholm integral system:

$$v_p(x) = f_p(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}(x,t) \left(v_q(x) + v_q'(x)(t-x) + \ldots + \frac{1}{m!} v_q^{(m)}(x)(t-x)^m \right) dt, \quad p = 1, 2, \ldots, s \tag{5.3}$$

Differentiate the Fredholm integral system, before substitute the expansion, with respect to x for m times:

$$\begin{aligned} v_p'(x) &= f_p'(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}'(x,t) v_q(x) dt, \quad p = 1, 2, \ldots, s \\ v_p''(x) &= f_p''(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}''(x,t) v_q(x) dt, \quad p = 1, 2, \ldots, s \\ &\vdots \\ v_p^{(m)}(x) &= f_p^{(m)}(x) + \int_0^1 \sum_{q=1}^{s} h_{pq}^{(m)}(x,t) v_q(x) dt, \quad p = 1, 2, \ldots, s \end{aligned} \tag{5.4}$$

Combined system (5.3) with system (5.3) will get system of ordinary differential system which can solved to get $v_p(x)$ for $p = 1, 2, \ldots, s$. Set $v_{p(approx)}(x)$ for $p = 1, 2, \ldots, s$ Input $v_{p(exact)}(x)$ for $p = 1, 2, \ldots, s$ Plot $v_{p(approx)}(x), v_{p(exact)}(x)$ Define $error = |v_{p(exact)}(x) - v_{p(approx)}(x)|$ Plot error.

x	Exact solution $v_1(x) = x + 1$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^2 + 1$	Numerical solution $v_{2approx}$
0	1.0	1.0000000000000	1.0	1.000000000000
0.1	1.1	1.084538095238095	1.01	1.008461904761905
0.2	1.2	1.193066666666667	1.04	1.039433333333333
0.3	1.3	1.266528571428571	1.09	1.081042857142857
0.4	1.4	1.395866666666666	1.16	1.148914076190818
0.5	1.5	1.472023809523809	1.25	1.238690476190479
0.6	1.6	1.585942857142856	1.36	1.353485714285714
0.7	1.7	1.708566666666666	1.49	1.4944333333333333
0.8	1.8	1.840838095238095	1.64	1.663161904761904
0.9	1.9	1.983699999999999	1.81	1.861299999999999
1	2.0	2.138095238095238	2.0	2.090476190476191

Table 1. The exact and numerical solutions of applying Algorithm (5.3) for system (5.1) with $m = 3$.

The Numerical Realization of System (5.1) using the Improved Block-Pulse Function Method.

We will use Algorithm (5.4) with $m = 16$ to solve system (5.1) using Improved Block-Pulse function method with Matlab software. Table (3) contains both the exact and numerical solutions

x	Absolute error $\lvert v_1 - v_{1approx} \rvert$	Absolute error $\lvert v_2 - v_{2approx} \rvert$
0.0	0.0	0.0
0.1	$1.5461904761905 \times 10^{-2}$	$1.538095238095 \times 10^{-3}$
0.2	$2.693333333333 \times 10^{-2}$	$5.0666666666667 \times 10^{-3}$
0.3	$3.3471428571429 \times 10^{-2}$	$8.957142857143 \times 10^{-3}$
0.4	$3.413333333334 \times 10^{-2}$	$1.1580952380953 \times 10^{-2}$
0.5	$2.7976190476191 \times 10^{-2}$	$1.1309523809521 \times 10^{-2}$
0.6	$1.4057142857144 \times 10^{-2}$	$6.5142857114289 \times 10^{-3}$
0.7	$8.566666666666 \times 10^{-3}$	$4.4333333333333 \times 10^{-3}$
0.8	$4.0838095238095 \times 10^{-2}$	$2.3161904761904 \times 10^{-2}$
0.9	$8.369999999999 \times 10^{-2}$	$5.1299999999999 \times 10^{-2}$
1.0	$1.38095238095238 \times 10^{-1}$	$9.0476190476191 \times 10^{-2}$

Table 2. The resulting error for the numerical solution using Algorithm (5.3) for system (5.1) with $m = 3$.

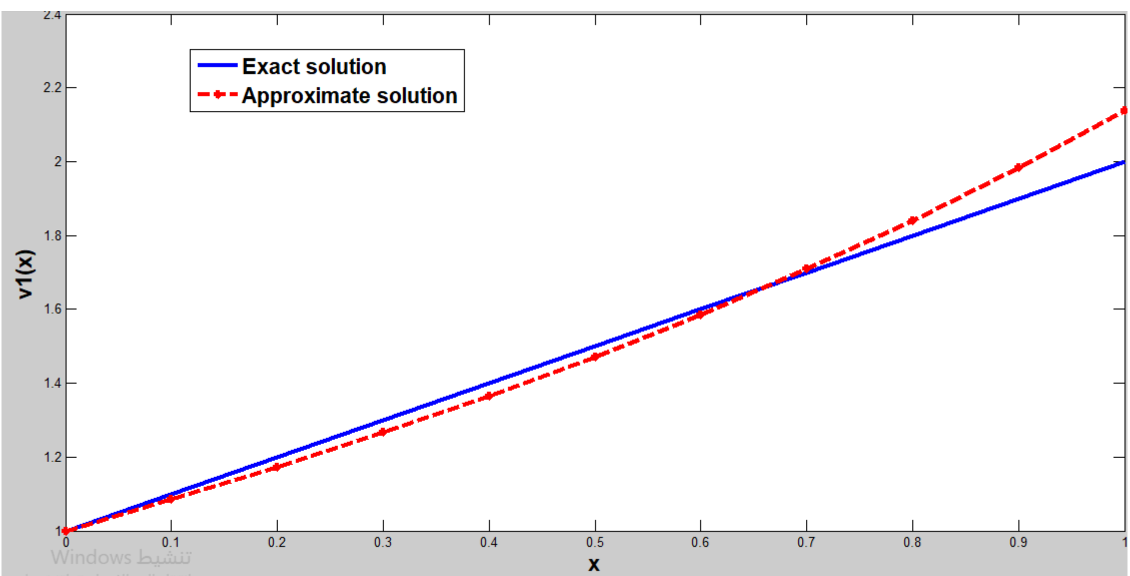

Figure 1. The exact and approximate solutions of $v_1(x)$ using Algorithm (5.3) for system (5.1) with $m = 3$.

with $m = 16$ and Table (4) contains the resulting error. Figures (3) and (4) show a comparison between the exact and numerical solutions for system (5.1). The maximum error corresponding to $v_{1(approx)}$ and $v_{2(approx)}$ is $E_1 \simeq 3 \times 10^{-2}$ and $E_2 \simeq 4.5 \times 10^{-2}$ respectively.

Algorithm 5.4 (h!). (IBPFM): [1] Input:

- s, m
- $f_p(x)$ for $p = 1, 2, \ldots, s$

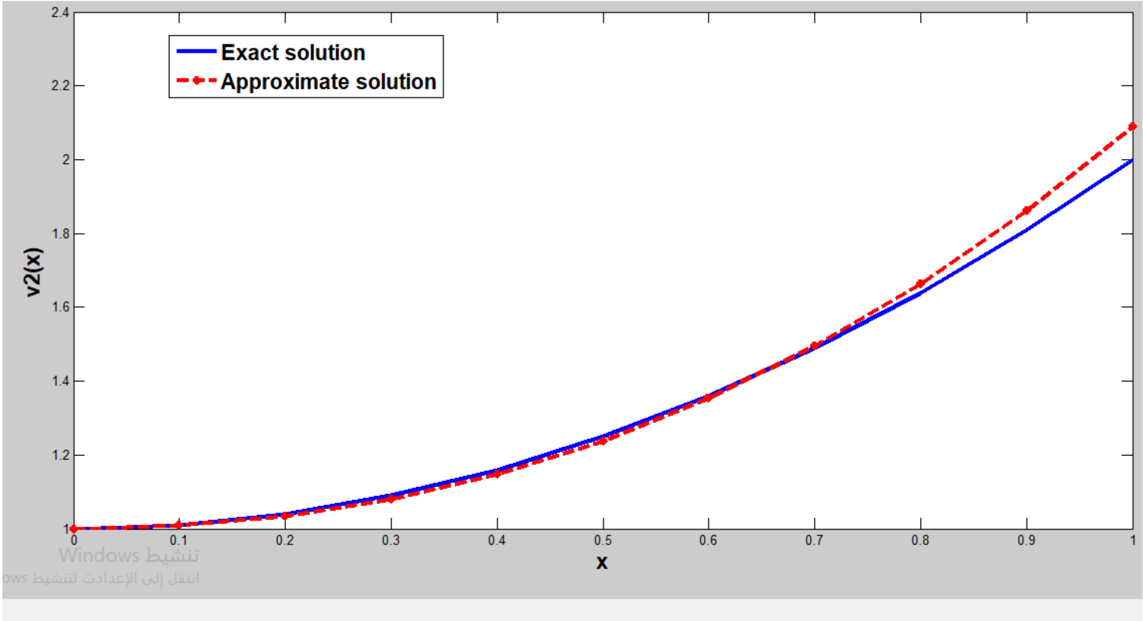

Figure 2. The exact and approximate solutions of $v_2(x)$ using Algorithm (5.3) for system (5.1) with $m = 3$.

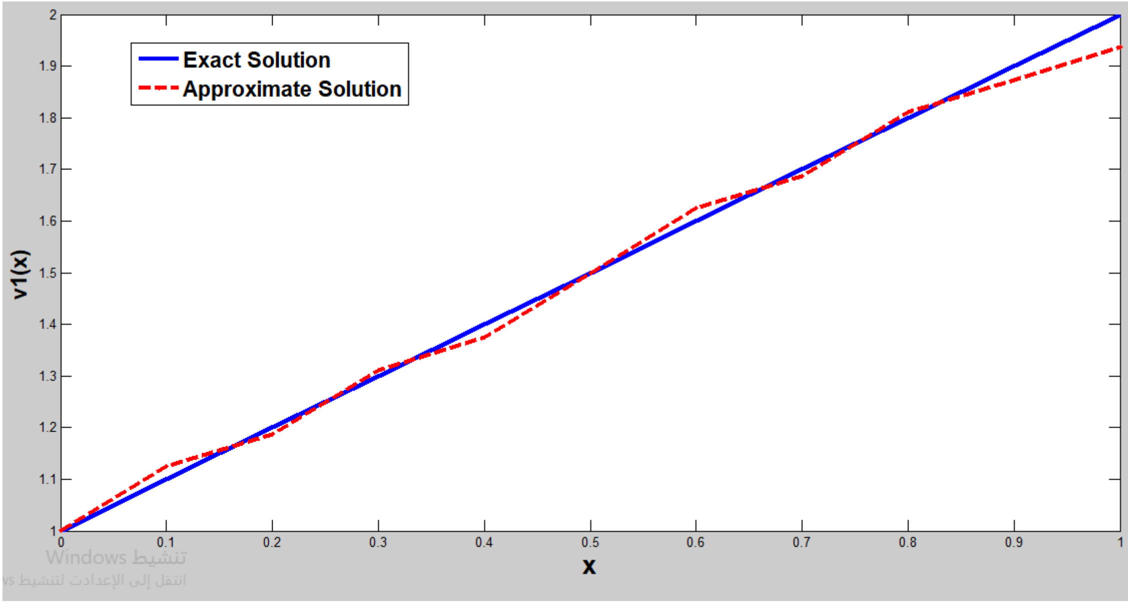

Figure 3. The exact and approximate solutions of $v_1(x)$ using Algorithm (5.3) for system (5.1) with $m = 3$.

- $h_{pq}(x,t)$ for $p, q = 1, 2, \ldots, s$

Compute the value $l = \frac{1}{m}$ Define the Block-Pulse function $\beta_i(x)$ Compute $\beta_i(x)$ for $i = 1, 2, , \ldots, m$ Define the vector $\beta(x) = \Big[\beta_1(x), \beta_2(x), \ldots, \beta_m(x)\Big]^T$ Calculate $f_{pi} = \begin{cases} 2m \int\limits_0^{\frac{l}{2}} f_p(t)dt & , i = \\ m \int\limits_{(i-1)l+\frac{l}{2}}^{il+\frac{l}{2}} f_p(t)dt & , i = \\ 2m \int\limits_{1-\frac{l}{2}}^{1} f_p(t)dt & , i = \end{cases}$

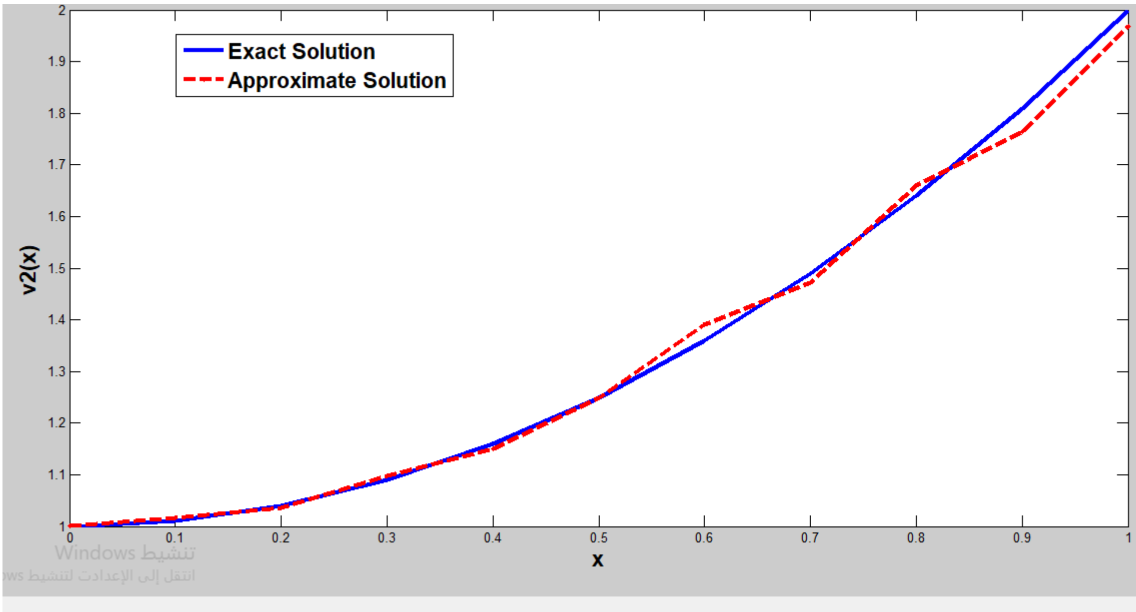

Figure 4. The exact and approximate solutions of $v_2(x)$ using Algorithm (5.3) for system (5.1) with $m = 3$.

for $p, q = 1, 2, \dots, s, i = 1, 2, \dots, m$ Define the vector $F_p = \left[f_{p1}, f_{p2}, \dots, f_{pm}\right]^T$ for $p = 1, 2, \dots, s$ Define the vectors $F_p^* = \left[f_{p0}, f_{p1}, \dots, f_{pm}\right]^T$ for $p, q = 1, 2, \dots, s$ Define the vectors $F^* = \left[F_1^*, F_2^*, \dots, F_s^*\right]^T$ for $p, q = 1, 2, \dots, s$ Calculate the matrices H_{pq}^* as defined in equation (**??**) for $p, q = 1, 2, \dots, s$ Define the matrix

$$P_{(m+1)\times(m+1)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Calculate the matrices $M_{pq}^* = H_{pq}^* \times P$ Define the matrix

$$C = \begin{pmatrix} I - M_{11}^* & -M_{12}^* & \dots & -M_{1s}^* \\ -M_{21}^* & I - M_{12}^* & \dots & -M_{2s}^* \\ \vdots & \vdots & \ddots & \vdots \\ -M_{s1}^* & -M_{s2}^* & \dots & I - M_{ss}^* \end{pmatrix}$$

Solve the algebraic system $CV^* = F^*$ to get the vector V_{sm1}^* Divide the vector V^* into vectors V_p^* for $p = 1, 2, \dots, s$ Compute the functions $V_p(x) = \phi^T(x) \times V_p^*$ for $p = 1, 2, \dots, s$ Set $v_{p(approx)}(x)$ for $p = 1, 2, \dots, s$ Input $v_{p(exact)}(x)$ for $p = 1, 2, \dots, s$ Plot $v_{p(approx)}(x), v_{p(exact)}(x)$ Define $error = |v_{p(exact)}(x) - v_{p(approx)}(x)|$ Plot error.

x	Exact solution $v_1(x) = x + 1$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^2 + 1$	Numerical solution $v_{2approx}$
0	1.0	1.01508538563714	1.0	1.000300615555048
0.1	1.1	1.124402273321139	1.01	1.015751278607047
0.2	1.2	1.186869066283425	1.04	1.035182907493903
0.3	1.3	1.311802652207996	1.09	1.097483665267616
0.4	1.4	1.374269445170281	1.16	1.150352794154472
0.5	1.5	1.499203031094852	1.25	1.249528551928186
0.6	1.6	1.624136617019424	1.36	1.389954309701899
0.7	1.7	1.686603409981709	1.49	1.471885938588756
0.8	1.8	1.811536995906281	1.64	1.659186696362469
0.9	1.9	1.874003788868565	1.81	1.764555825249326
1	2.0	1.983320676552566	2.00	1.967506488301325

Table 3. The exact and numerical solutions of applying Algorithm (5.4) for system (5.1) with $m = 16$.

x	Absolute error $\lvert v_1 - v_{1approx}\rvert$	Absolute error $\lvert v_2 - v_{2approx}\rvert$
0.0	$1.508538563714 \times 10^{-2}$	$3.0061555048 \times 10^{-4}$
0.1	$2.4402273321139 \times 10^{-2}$	$5.751278607047 \times 10^{-3}$
0.2	$1.3130933716575 \times 10^{-2}$	$4.817092506097 \times 10^{-3}$
0.3	$1.1802652207996 \times 10^{-2}$	$7.483665267616 \times 10^{-3}$
0.4	$2.5730554829719 \times 10^{-2}$	$1.9647205845528 \times 10^{-2}$
0.5	$7.96968905148 \times 10^{-4}$	$4.71448071814 \times 10^{-4}$
0.6	$2.4136617019424 \times 10^{-2}$	$2.9954309701899 \times 10^{-2}$
0.7	$1.3396590018291 \times 10^{-2}$	$1.81140614111244 \times 10^{-2}$
0.8	$1.1536995906281 \times 10^{-2}$	$1.9186696362469 \times 10^{-2}$
0.9	$2.5996211131435 \times 10^{-2}$	$4.5444174750674 \times 10^{-2}$
1.0	$1.667932344743 \times 10^{-2}$	$3.2493511698675 \times 10^{-2}$

Table 4. The absolute error using Algorithm (5.4) for system (5.1) with $m = 16$.

Example 5.5. Consider the system of linear Fredholm integral equations:

$$
\begin{aligned}
v_1(x) &= \frac{11}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3 + \int_0^1 \Big[(x-t)^3 v_1(t) + (x-t)^2 v_2(t)\Big]dt \\
v_2(x) &= -\frac{1}{30} - \frac{41}{60}x + \frac{3}{20}x^2 + \frac{23}{12}x^3 - \frac{1}{3}x^4 + \int_0^1 \Big[(x-t)^4 v_1(t) + (x-t)^3 v_2(t)\Big]dt
\end{aligned} \tag{5.5}
$$

The exact solution of system (5.5) is $v_1(x) = x^2$ and $v_2(x) = x^3 + x^2 - x$.

We start by implementing Algorithm (5.2) to solve system (5.5) using the Legendre wavelet method with $M = 5, k = 1$. We obtain the following equations:

$$\begin{aligned}
\psi_{10}(x) &= 1 \\
\psi_{11}(x) &= \sqrt{3}(2x-1) \\
\psi_{12}(x) &= \frac{\sqrt{5}}{2}\Big(12x^2 - 12x + 4\Big) \\
\psi_{13}(x) &= \sqrt{7}(20x^3 - 30x^2 + 12x - 1) \\
\psi_{14}(x) &= (210x^4 - 420x^3 + 270x^2 - 60x + 3)
\end{aligned} \tag{5.6}$$

Rewrite equations (5.6) in vector notation, we have

$$\psi(x) = \Big[1 \quad \sqrt{3}(-1+2x) \quad \frac{\sqrt{5}}{2}(6x^2 - 6x + 1) \quad \sqrt{7}(-1 + 12x - 30x^2 + 20x^3)\Big]^T$$

Moreover, we obtain:

$$H_{11} = \begin{bmatrix} 0 & \frac{-\sqrt{3}}{15} & 0 & \frac{-1}{20\sqrt{7}} & 0 \\ \frac{\sqrt{3}}{15} & 0 & \frac{1}{4\sqrt{15}} & 0 & 0 \\ 0 & \frac{-1}{4\sqrt{15}} & 0 & 0 & 0 \\ \frac{1}{20\sqrt{7}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6\sqrt{5}} & 0 & 0 \\ 0 & \frac{-1}{6} & 0 & 0 & 0 \\ \frac{1}{20\sqrt{7}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{21} = \begin{bmatrix} \frac{1}{15} & 0 & \frac{\sqrt{5}}{42} & 0 & \frac{1}{210} \\ 0 & \frac{-1}{6} & 0 & \frac{-1}{10\sqrt{21}} & 0 \\ \frac{\sqrt{5}}{42} & 0 & \frac{1}{30} & 0 & 0 \\ 0 & \frac{-1}{10\sqrt{21}} & 0 & 0 & 0 \\ \frac{1}{210} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 0 & \frac{-1}{5\sqrt{3}} & 0 & \frac{-1}{20\sqrt{7}} & 0 \\ \frac{1}{5\sqrt{3}} & 0 & \frac{1}{4\sqrt{15}} & 0 & 0 \\ 0 & \frac{-1}{4\sqrt{15}} & 0 & 0 & 0 \\ \frac{1}{20\sqrt{7}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
F_1(x) &= \Big[\frac{61}{180} \quad \frac{1}{2\sqrt{3}} \quad \frac{7}{36\sqrt{5}} \quad \frac{-1}{60\sqrt{7}} \quad 0\Big]^T \\
F_2(x) &= \Big[\frac{7}{80} \quad \frac{37}{80\sqrt{3}} \quad \frac{229}{560\sqrt{5}} \quad \frac{1}{16\sqrt{7}} \quad \frac{-1}{630}\Big]^T \\
\beta_1(x) &= \Big[\frac{1}{3} \quad \frac{1}{2\sqrt{3}} \quad \frac{1}{6\sqrt{5}} \quad 0 \quad 0\Big]^T \\
\beta_2(x) &= \Big[\frac{1}{12} \quad \frac{3\sqrt{3}}{20} \quad \frac{\sqrt{5}}{12} \quad \frac{1}{20\sqrt{7}} \quad 0\Big]^T
\end{aligned}$$

Then v_1 and v_2 becomes,

$$\begin{aligned}
v_1(x) &= \frac{1}{3}\psi_{10}(x) + \frac{1}{2\sqrt{3}}\psi_{11}(x) + \frac{1}{6\sqrt{5}}\psi_{12}(x) + 0\psi_{13}(x) + 0\psi_{14}(x) \\
&= x^2 \\
v_2(x) &= \frac{1}{12}\psi_{10}(x) + \frac{3\sqrt{3}}{20}\psi_{11}(x) + \frac{\sqrt{15}}{12}\psi_{12}(x) + \frac{1}{20\sqrt{7}}\psi_{13}(x) + 0\psi_{14}(x) \\
&= x^3 + x^2 - x
\end{aligned}$$

which is the exact solution.
Next, we implement Algorithm (5.3) to solve system (5.5) using the Taylor-series expansion

x	Exact solution $v_1(x) = x^2$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^3 + x^2 - x$	Numerical solution $v_{2approx}$
0	0.00	0.0000000000000	0.00	0.0000000000000
0.1	0.01	0.0096611287302334	-0.089	-0.088610336060839
0.2	0.04	0.038654967804856	-0.152	-0.148633606258219
0.3	0.09	0.090032942754002	-0.183	-0.18285596401785
0.4	0.16	0.160022861428563	-0.176	-0.17584305889143
0.5	0.25	0.2500876588758576	-0.125	-0.12493951204219
0.6	0.36	0.36005896586389	-0.024	-0.023918346963143
0.7	0.49	0.48854908732195	0.133	0.132860089442594
0.8	0.64	0.638382060054187	0.352	0.35201237111482
0.9	0.81	0.807776109967882	0.639	0.638629046286059
1	1.00	1.0046200188672	1.00	1.0011051000

Table 5. The exact and numerical solutions of applying Algorithm (5.3) for system (5.5) with $m = 5$.

x	Absolute error $\lvert v_1 - v_{1approx}\rvert$	Absolute error $\lvert v_2 - v_{2approx}\rvert$
0.0	0.0	0.0
0.1	$3.38876124866 \times 10^{-4}$	$3.896639392 \times 10^{-4}$
0.2	$1.345032129590 \times 10^{-3}$	$3.366393742 \times 10^{-3}$
0.3	$3.29427542401 \times 10^{-5}$	$1.440359822 \times 10^{-4}$
0.4	$2.28614286500 \times 10^{-5}$	$1.569411086 \times 10^{-4}$
0.5	$8.7658875867 \times 10^{-5}$	$6.048795781 \times 10^{-5}$
0.6	$5.8965863890 \times 10^{-5}$	$8.165303686 \times 10^{-5}$
0.7	$1.450912678 \times 10^{-3}$	$1.399105574 \times 10^{-4}$
0.8	$1.617939946 \times 10^{-3}$	$1.23111482 \times 10^{-5}$
0.9	$2.223890032 \times 10^{-3}$	$3.709537139 \times 10^{-4}$
1.0	$4.20018867 \times 10^{-3}$	$1.105100000 \times 10^{-3}$

Table 6. The resulting error for the numerical solution using Algorithm (5.3) for system (5.5) with $m = 5$.

method. Table (5) contains the exact and numerical solutions with $m = 5$ and Table (6) contains the resulting error. Figures (5) and (6) show a comparison between the exact and numerical solutions for system (5.5). The maximum error corresponding to $v_1(x)$ and $v_2(x)$ is $E_1 \simeq 4.2 \times 10^{-3}$ and $E_2 \simeq 3.4 \times 10^{-3}$ respectively.

Additionally, we use Algorithm (5.4) with $m = 16$ to solve system (5.5) using the improved block-Pulse function method with Matlab software. Table (11) contains both the exact and numerical solutions with $m = 16$ and Table (12) contains the resulting error. Figures (3) and

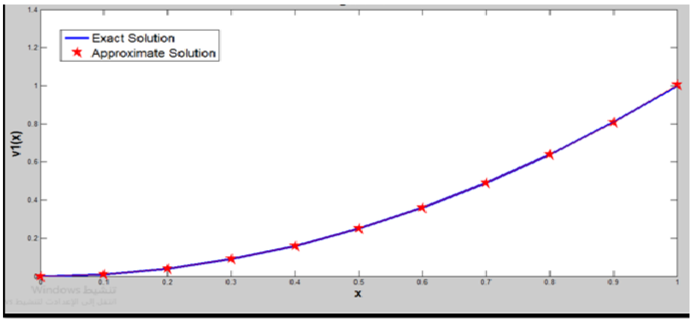

Figure 5. The exact and approximate solutions of $v_1(x)$ using Algorithm (5.3) for system (5.5) with $m = 5$.

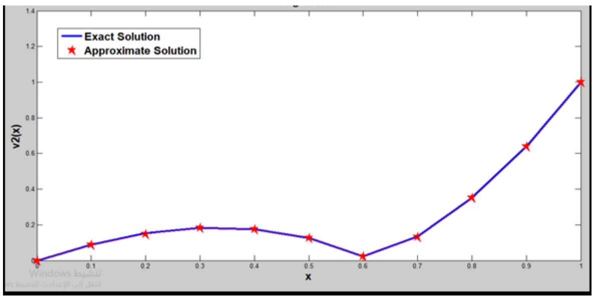

Figure 6. The exact and approximate solutions of $v_2(x)$ using Algorithm (5.3) for system (5.5) with $m = 5$.

x	Exact solution $v_1(x) = x^2$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^3 + x^2 - x$	Numerical solution $v_{2approx}$
0	0.00	0.00022203592009	0.00	-0.015085103359056
0.1	0.01	0.015793604848272	-0.089	-0.106772653428305
0.2	0.04	0.035305238634090	-0.152	-0.145056304467618
0.3	0.09	0.097789579986058	-0.183	-0.183554385594585
0.4	0.16	0.140762050516545	-0.176	-0.180835326942956
0.5	0.25	0.250166998697364	-0.125	-0.124127511582203
0.6	0.36	0.390851166252434	-0.024	-0.01717555014564
0.7	0.49	0.4729226610007333	0.133	0.11108066600742
0.8	0.64	0.660523879849701	0.352	0.385157805239512
0.9	0.81	0.766053366914708	0.639	0.651750046960059
1	1.00	0.984525660380627	1.00	0.938880589523453

Table 7. The exact and numerical solutions of applying Algorithm (5.3) for system (5.5) with $m = 16$.

(4) show a comparison between the exact and numerical solutions for system (5.5). The maximum error corresponding to $v_{1(approx)}$ and $v_{2(approx)}$ is $E_1 \simeq 4.4 \times 10^{-2}$ and $E_2 \simeq 7.7 \times 10^{-2}$ respectively.

x	Exact solution $v_1(x) = x^2$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^3 + x^2 - x$	Numerical solution $v_{2approx}$
0	0.00	0.00005510434145	0.00	-0.007675178148131
0.1	0.01	0.008832195581701	-0.089	-0.083977653976501
0.2	0.04	0.035190817851051	-0.152	-0.145574798705717
0.3	0.09	0.097686532557641	-0.183	-0.184130675512469
0.4	0.16	0.165071336403451	-0.176	-0.173956505640399
0.5	0.25	0.250038601853096	-0.125	-0.124780742733980
0.6	0.36	0.352588216320644	-0.024	-0.031657972854125
0.7	0.49	0.472720067220163	0.133	0.110357175718531
0.8	0.64	0.660245819387397	0.352	0.384312995453419
0.9	0.81	0.821402450794004	0.639	0.65942124946624
1	1.00	0.984513494524736	1.00	0.969095312995647

Table 8. The exact and numerical solutions of applying Algorithm (5.3) for system (5.5) with $m = 32$.

x	Absolute error $\lvert v_1 - v_{1approx}\rvert, m = 16$	Absolute error $\lvert v_1 - v_{2approx}\rvert, m = 32$
0.0	$2.22035920093 \times 10^{-3}$	$5.5104341450 \times 10^{-5}$
0.1	$5.793604848272 \times 10^{-3}$	$1.167804418299 \times 10^{-3}$
0.2	$4.694761365910 \times 10^{-3}$	$4.809182148949 \times 10^{-3}$
0.3	$7.789579986058 \times 10^{-3}$	$7.68653255761 \times 10^{-3}$
0.4	$1.9237949483455 \times 10^{-2}$	$5.071336403451 \times 10^{-3}$
0.5	$1.66998697364 \times 10^{-4}$	$3.86019536601 \times 10^{-5}$
0.6	$3.0851166252434 \times 10^{-2}$	$7.400783679356 \times 10^{-3}$
0.7	$1.7077338999267 \times 10^{-2}$	$1.7279932778937 \times 10^{-2}$
0.8	$2.0523879849701 \times 10^{-2}$	$2.0245819387397 \times 10^{-2}$
0.9	$4.3946633085292 \times 10^{-2}$	$1.1140245094004 \times 10^{-2}$
1.0	$1.5474339619373 \times 10^{-2}$	$1.5476505475264 \times 10^{-2}$

Table 9. The resulting error for the numerical solution using Algorithm (5.3) for $v_1(x)$ with $m = 16$ and $m = 32$.

x	Absolute error $\lvert v_2 - v_{1approx}\rvert, m = 16$	Absolute error $\lvert v_2 - v_{2approx}\rvert, m = 32$
0.0	$1.5085103359056 \times 10^{-2}$	$7.675178148131 \times 10^{-3}$
0.1	$1.7772653428305 \times 10^{-2}$	$5.022346023499 \times 10^{-3}$
0.2	$6.943695532382 \times 10^{-3}$	$6.245201294283 \times 10^{-3}$
0.3	$5.54385594296 \times 10^{-4}$	$1.1130675512469 \times 10^{-3}$
0.4	$4.835326942956 \times 10^{-3}$	$2.043494359601 \times 10^{-3}$
0.5	$8.72488417797 \times 10^{-4}$	$1.92572660200 \times 10^{-4}$
0.6	$1.3282444985436 \times 10^{-2}$	$7.657972854125 \times 10^{-3}$
0.7	$2.1919339925800 \times 10^{-2}$	$2.2642824281469 \times 10^{-2}$
0.8	$3.1578052395120 \times 10^{-2}$	$3.2312995453419 \times 10^{-2}$
0.9	$7.7249953039941 \times 10^{-2}$	$2.0421249466240 \times 10^{-2}$
1.0	$6.1119410476547 \times 10^{-2}$	$3.0904687004353 \times 10^{-2}$

Table 10. The resulting error for the numerical solution using Algorithm (5.3) for $v_1(x)$ with $m = 16$ and $m = 32$.

x	Exact solution $v_1(x) = x^2$	Numerical solution $v_{1approx}$	Exact solution $v_2(x) = x^3 + x^2 - x$	Numerical solution $v_{2approx}$
0	0.0	0.000222035920092587	0.0	-0.015085103359056
0.1	0.01	0.015793604848272	-0.089	-0.106772653428305
0.2	0.04	0.035305238634090	-0.152	-0.145056304467618
0.3	0.09	0.097789579986058	-0.183	-0.183554385594585
0.4	0.16	0.140762050516545	-0.176	-0.180835326942956
0.5	0.25	0.250166998697364	-0.125	-0.124127511582203
0.6	0.36	0.390851166252434	-0.024	-0.01717555014564
0.7	0.49	0.4729226610007333	0.133	0.11108066600742
0.8	0.64	0.660523879849701	0.352	0.385157805239512
0.9	0.81	0.766053366914708	0.639	0.651750046960059
1	1.00	0.984525660380627	1.0	0.938880589523453

Table 11. The exact and numerical solutions of applying Algorithm (5.4) for system (5.5) with $m = 16$.

x	Absolute error $\lvert v_1 - v_{1approx} \rvert$	Absolute error $\lvert v_2 - v_{2approx} \rvert$
0.0	$2.22035920093 \times 10^{-3}$	$1.5085103359056 \times 10^{-2}$
0.1	$5.793604848272 \times 10^{-3}$	$1.7772653428305 \times 10^{-2}$
0.2	$4.694761365910 \times 10^{-3}$	$6.943695532382 \times 10^{-3}$
0.3	$7.789579986058 \times 10^{-3}$	$5.54385594296 \times 10^{-4}$
0.4	$1.9237949483455 \times 10^{-2}$	$4.835326942956 \times 10^{-3}$
0.5	$1.66998697364 \times 10^{-4}$	$8.72488417797 \times 10^{-4}$
0.6	$3.0851166252434 \times 10^{-2}$	$1.3282444985436 \times 10^{-2}$
0.7	$1.7077338999267 \times 10^{-2}$	$2.1919339925800 \times 10^{-2}$
0.8	$2.0523879849701 \times 10^{-2}$	$3.1578052395120 \times 10^{-2}$
0.9	$4.3946633085292 \times 10^{-2}$	$7.7249953039941 \times 10^{-2}$
1.0	$1.5474339619373 \times 10^{-2}$	$6.1119410476547 \times 10^{-2}$

Table 12. The resulting error for the numerical solution using Algorithm (5.4) for system (5.5) with $m = 16$.

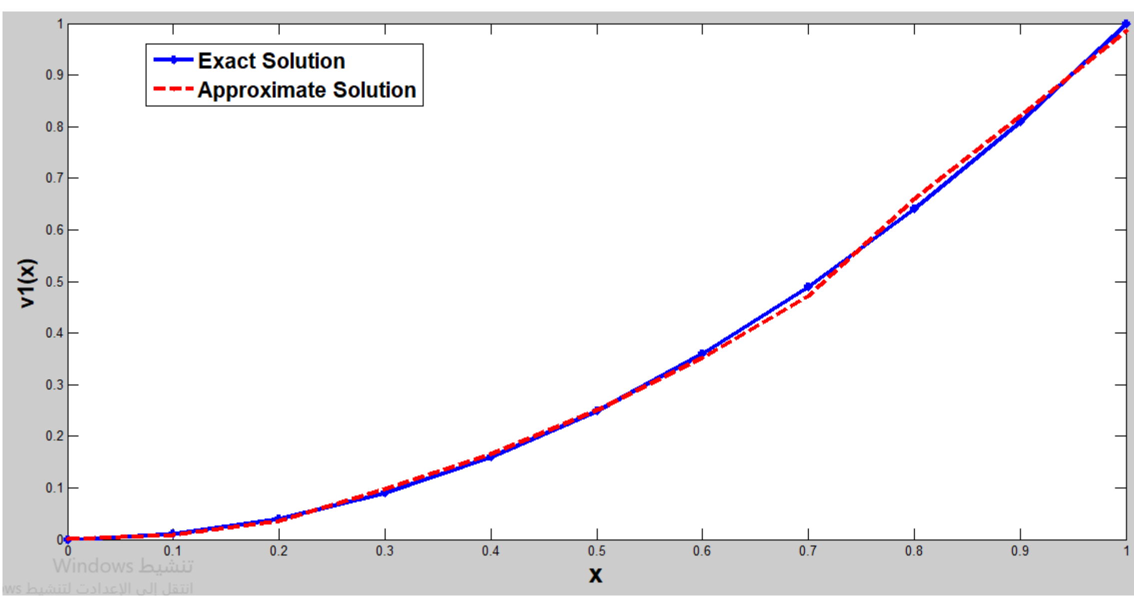

Figure 7. The exact and approximate solutions of $v_1(x)$ using Algorithm (5.4) for system (5.5) with $m = 16$.

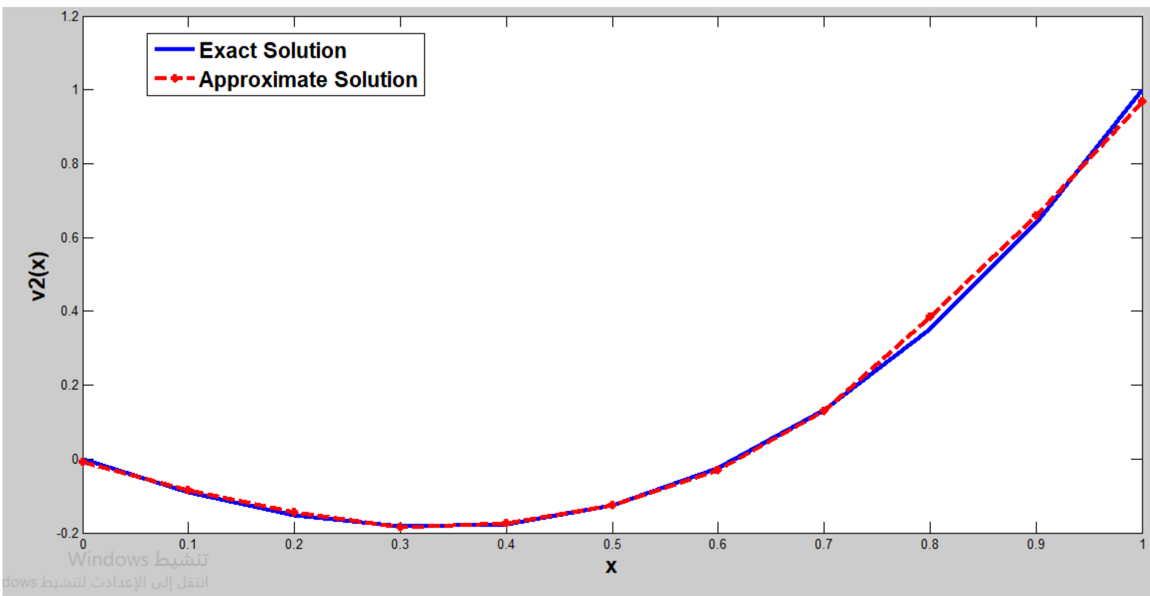

Figure 8. The exact and approximate solutions of $v_2(x)$ using Algorithm (5.4) for system (5.5) with $m = 16$.

6 Conclusion

In this article, a Legendre wavelets method, Taylor series expansion method and the improved block-pluse function method are proposed to solve linear system of Fredholm integral equations. The numerical results show that the convergence and accuracy of these methods were in good agreement with the analytical solution. Comparison of numerical results illustrated in tables and figures show clearly that the Legendre wavelets method provides more accurate results and therefore more effective than other methods for solving Fredholm integral equations.

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